Probabilistic max-plus schemes for solving Hamilton-Jacobi-Bellman equations

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A finite horizon diffusion control problem involving "discrete" and "continuum" controls

Maximize

$$J(t,x,\mu,u) := \mathbb{E}\Big[\int_t^T e^{-\int_t^s \delta^{\mu_\tau}(\xi_\tau,u_\tau)d\tau}\ell^{\mu_s}(\xi_s,u_s)ds + e^{-\int_t^T \delta^{\mu_\tau}(\xi_\tau,u_\tau)d\tau}\psi(\xi_T) \mid \xi_t = x\Big] \ ,$$

- $\xi_s \in \mathbb{R}^d$, the state process, satisfies the stochastic differential equation

$$d\xi_s = f^{\mu_s}(\xi_s,u_s)ds + \sigma^{\mu_s}(\xi_s,u_s)dW_s \ ,$$

- $\mu := (\mu_s)_{0\le s\le T}$, and $u := (u_s)_{0\le s\le T}$ are admissible control processes, $\mu_s \in \mathcal{M}$ a finite set and $u_s \in \mathcal{U} \subset \mathbb{R}^p$,
- $(W_s)_{s\ge 0}$ is a d-dimensional Brownian motion,
- $\delta^m(x,u) \ge 0$ is the discount rate.

Define $v : [0,T]\times\mathbb{R}^d \to \mathbb{R}$, $v(t,x) = \sup_{\mu,u} J(t,x,\mu,u)$.

The Hamilton-Jacobi-Bellman (HJB) equation

Theorem

Under suitable assumptions, the value function v is the unique (continuous) viscosity solution of the HJB equation

$$-\frac{\partial v}{\partial t} - \mathcal{H}(x, v(t,x), Dv(t,x), D^2 v(t,x)) = 0, \quad x \in \mathbb{R}^d,\ pt \in [0,T),$$
$$v(T,x) = \psi(x), \quad x \in \mathbb{R}^d,$$

satisfying also some growth condition at infinity (in space).

With the Hamiltonian:

$$\mathcal{H}(x,r,p,\Gamma) := \max_{m \in \mathcal{M}} \mathcal{H}^m(x,r,p,\Gamma) \ ,$$
$$\mathcal{H}^m(x,r,p,\Gamma) := \max_{u \in \mathcal{U}} \mathcal{H}^{m,u}(x,r,p,\Gamma) \ ,$$
$$\mathcal{H}^{m,u}(x,r,p,\Gamma) := \frac{1}{2} \operatorname{tr}\left(\sigma^m(x,u)\sigma^m(x,u)^{\mathsf{T}}\Gamma\right) + f^m(x,u)\cdot p - \delta^m(x,u) r + \ell^m(x,u) \ .$$

Standard grid based discretizations solving HJB equations suffer the curse of dimensionality malediction:
for an error of ϵ, the computing time of finite difference or finite element methods is at least in the order of $(1/\epsilon)^{d/2}$.

Some possible curse of dimensionality-free methods:

- Idempotent methods introduced by McEneaney (2007) in the deterministic case, and by McEneaney, Kaise and Han (2011) in the stochastic case.
- Probabilistic numerical methods based on a backward stochastic differential equation interpretation of the HJB equation, simulations and regressions:
  - Quantization Bally, Pagès (2003) for stopping time problems.
  - Introduction of a new process without control: Bouchard, Touzi (2004) when σ does not depend on control; Cheridito, Soner, Touzi and Victoir (2007) and Fahim, Touzi and Warin (2011) in the fully-nonlinear case.
  - Control randomization: Kharroubi, Langrené, Pham (2013).
  - Fixed point iterations: Bender, Zhang (2008) for semilinear PDE (which are not HJB equations).

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A Finance example

Problem: *pricing and hedging an option with uncertain volatility and several underlying stock processes.*

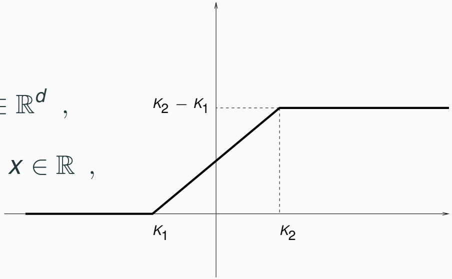
- *The dynamics:* $d\xi_i = \sigma_i \xi_i dB_i$, where the Brownians B_i have uncertain correlations: $\langle dB_i, dB_j \rangle = \mu_{i,j} ds$.
- We know: $\mu \in \mathrm{cvx}(\mathcal{M})$ with $\mathcal{M}$ a finite set.
- *Maximize*

$$J(t, x, \mu) := \mathbb{E}\left[\psi(\xi(T)) \mid \xi(t) = x\right] \quad , \quad \text{with}$$

$$\psi(x) = \phi(\max_{i \text{ odd}} x_i - \min_{j \text{ even}} x_j), \quad x \in \mathbb{R}^d \quad ,$$

$$\phi(x) = (x - K_1)^+ - (x - K_2)^+, \quad x \in \mathbb{R} \quad ,$$

$$x^+ = \max(x, 0), \quad K_1 < K_2 \quad .$$



A Finance example

- Since the dynamics is linear, we can reduce to $\mu_s \in \mathcal{M}$.
- The parameters with respect to the previous model: $\mathcal{M}$ is a finite subset of the set of positive definite symmetric matrices with 1 on the diagonal and

$$\begin{aligned} f^m &= 0 \\ \delta^m &= 0 \\ \ell^m &= 0 \\ [\sigma^m(\xi)\sigma^m(\xi)^\mathsf{T}]_{i,j} &= \sigma_i \xi_i \sigma_j \xi_j \mu_{i,j} \ . \end{aligned}$$

- Proposed with 2 stocks in Kharroubi, Langrené, Pham (2013) and solved using randomized control+regression.
- Solved in dimension 2 in A., Fodjo (CDC 2016) with a probabilistic max-plus method.
- In both cases: $\sigma_1 = 0.4,\ \sigma_2 = 0.3,\ K_1 = -5,\ K_2 = 5,\ T = 0.25$, and

$$\mathcal{M} = \{m = \begin{bmatrix} 1 & m_{12} \\ m_{12} & 1 \end{bmatrix} \mid m_{12} = \pm\rho\} \qquad \rho = 0.8 \ .$$

The algorithm of Fahim, Touzi and Warin

Decompose the Hamiltonian $\mathcal{H}$ of HJB as $\mathcal{H} = \mathcal{L} + \mathcal{G}$ with

$$\mathcal{L}(x, r, p, \Gamma) := \frac{1}{2}\operatorname{tr}(a(x)\Gamma) + \underline{f}(x) \cdot p \ , \quad a(x) = \underline{\sigma}(x)\underline{\sigma}(x)^{\mathsf{T}} > 0 \ ,$$

and $\partial_\Gamma \mathcal{G} \geq 0$, for all $x \in \mathbb{R}^d$, $r \in \mathbb{R}, p \in \mathbb{R}^d, \Gamma \in \mathbb{S}_d$.

Theorem (Cheridito, Soner, Touzi and Victoir, 2007)

If v is the viscosity solution of HJB, X_t is the diffusion with generator $\mathcal{L}$:

$$dX_t = \underline{f}(X_t)dt + \underline{\sigma}(X_t)dW_t \ , \quad X_0 = x$$

then $Y_t = v(t, X_t)$, $Z_t = Dv(t, X_t)$ and $\Gamma_t = D^2 v(t, X_t)$ satisfy the second-order backward stochastic diff. eq.:

$$\begin{aligned} dY_t &= -\mathcal{G}(X_t, Y_t, Z_t, \Gamma_t)dt + Z_t^{\mathsf{T}} \underline{\sigma}(X_t)dW_t \\ dZ_t &= A_t dt + \Gamma_t dX_t \\ Y_T &= \psi(X_T) \ . \end{aligned}$$

Idea of the algorithm of Fahim, Touzi and Warin: after time discretization, simulate X_t, then apply a regression estimator to compute Y_t

Denote by $\hat{X}$ the Euler discretization of X_t:

$$\hat{X}(t+h) = \hat{X}(t) + \underline{f}(\hat{X}(t))h + \underline{\sigma}(\hat{X}(t))(W_{t+h} - W_t) \ .$$

The following is a time discretization of HJB:

$$v^h(t,x) = T_{t,h}(v^h(t+h,\cdot))(x), \quad t \in \mathcal{T}_h := \{0, h, 2h, \dots, T-h\} \ ,$$

with

$$T_{t,h}(\phi)(x) = \mathcal{D}^0_{t,h}(\phi)(x) + h\mathcal{G}(x, \mathcal{D}^0_{t,h}(\phi)(x), \mathcal{D}^1_{t,h}(\phi)(x), \mathcal{D}^2_{t,h}(\phi)(x)) \ ,$$

and $\mathcal{D}^i_{t,h}(\phi)$ the approximation of the ith differential of $e^{h\mathcal{L}}\phi$ given by:

$$\begin{aligned}
\mathcal{D}^i_{t,h}(\phi)(x) :=& \mathbb{E}(D^i\phi(\hat{X}(t+h)) \mid \hat{X}(t) = x) \\
=& \mathbb{E}(\phi(\hat{X}(t+h))\mathcal{P}^i_{t,x,h}(W_{t+h} - W_t) \mid \hat{X}(t) = x), \quad i = 0,1,2 \ , \\
\mathcal{P}^0_{t,x,h}(w) =& 1 \ , \\
\mathcal{P}^1_{t,x,h}(w) =& (\underline{\sigma}(x)^{\mathsf{T}})^{-1} h^{-1} w \ , \\
\mathcal{P}^2_{t,x,h}(w) =& (\underline{\sigma}(x)^{\mathsf{T}})^{-1} h^{-2}(ww^{\mathsf{T}} - hI)(\underline{\sigma}(x))^{-1} \ .
\end{aligned}$$

Lemma (Fahim, Touzi and Warin, 2011)

When $\mathrm{tr}(a(x)^{-1}\partial_\Gamma \mathcal{G}) \leq 1$, $\partial_\Gamma \mathcal{G}$ *is lower bounded by some* > 0 *matrix and* $\mathcal{G}$ *is Lipschitz continuous,* $T_{t,h}$ *is L-almost monotone on the set* $\mathcal{F}$ *of Lipschitz continuous functions* $\mathbb{R}^d \to \mathbb{R}$*, for some constant* $L = O(h)$*:*

$$\phi, \psi \in \mathcal{F},\ \phi \leq \psi \implies T(\phi) \leq T(\psi) + L \sup(\psi - \phi) \ .$$

- Then Barles and Souganidis (90) $\Rightarrow$ convergence and error estimation of the time discretization scheme.
- Under these conditions, and given the convergence of the regression estimator approximating the $\mathcal{D}^i_{t,h}(\phi)$, the full Fahim, Touzi and Warin algorithm converges.
- Note that theoretically, the sample size necessary to obtain the convergence of the estimator is at least in the order of $1/h^{d/2}$. Also the dimension of the linear regression space should be in this order.

- The critical constraint $\mathrm{tr}(a(x)^{-1}\partial_\Gamma \mathcal{G}) \leq 1$ does not allow in general to handle the case of the Hamiltonian $\mathcal{H}$ directly, since it may be nonsmooth and with *noncomparable* diffusion coefficients.
- In particular, it fails for the finance example with $|m_{12}| = \rho \geq 0.5$.
- Guo, Zhang and Zhuo (2015) proposed a monotone scheme which combine a usual finite difference scheme to the above scheme. This allows one to relax the critical constraint, but still fails for the above finance example.
- In A. and Fodjo (2016), we only assumed that the Hamiltonians $\mathcal{H}^m$ satisfy the critical constraint, and applied the above scheme to the Hamiltonians $\mathcal{H}^m$, that is

$$v^h(t,x) = T_{t,h}(v^h(t+h,\cdot))(x), \quad t \in \mathcal{T}_h \ ,$$
$$T_{t,h}(\phi)(x) = \max_{m \in \mathcal{M}} T^m_{t,h}(\phi)(x) \ ,$$

with $T^m_{t,h}$ constructed as above but with respect to a decomposition $\mathcal{H}^m = \mathcal{L}^m + \mathcal{G}^m$ depending on m.
- Here, we propose another approximation of $\mathbb{E}(D^2\phi(\hat{X}(t+h)) \mid \hat{X}(t) = x)$ or $D^2\phi(x)$ depending on the point x via $\sigma^m(x,u)$ and leading to a monotone operator $T_{t,h}$.

A monotone probabilistic scheme for fully nonlinear PDEs

Theorem

Let $\Sigma \in \mathbb{R}^{d\times\ell}$ and denote $A = \Sigma\Sigma^\mathsf{T}$. For a nonnegative integer k, consider the polynomial

$$\mathcal{P}_{\Sigma,k}(w) = c_k \sum_{j=1}^{\ell} ([\Sigma^\mathsf{T} w]_j)^{4k+2} \|\Sigma_{.j}\|_2^{-4k} - K \ , \qquad w \in \mathbb{R}^d$$

$$c_k = \frac{1}{\mathbb{E}\left[N^{4k+4} - N^{4k+2}\right]} \ , \qquad K := \frac{\operatorname{tr}(A)}{4k+2} = \frac{\sum_{j=1}^{\ell} \|\Sigma_{.j}\|_2^2}{4k+2} \ , \quad N = N(0,1).$$

For $v \in \mathcal{C}_b^4$, and $\hat{X}$ as before, we have

$$\mathbb{E}\left[\mathcal{P}_{\Sigma,k}(h^{-1/2}(W_{t+h} - W_t)) \mid \hat{X}(t) = x\right] = 0$$

$$h^{-1}\mathbb{E}\left[v(t+h, \hat{X}(t+h))\mathcal{P}_{\Sigma,k}(h^{-1/2}(W_{t+h} - W_t)) \mid \hat{X}(t) = x\right] =$$

$$\frac{1}{2}\operatorname{tr}(\underline{\sigma}(x) A \underline{\sigma}^\mathsf{T}(x)\, D^2 v(t,x)) + O(h) \ ,$$

where the error $O(h)$ is uniform in $(t,x) \in [0,T] \times \mathbb{R}^d$.

Let $\Sigma^m(x,u) \in \mathbb{R}^{d\times\ell}$ be such that

$$\sigma^m(x,u)\sigma^m(x,u)^\mathsf{T} - a(x) = \underline{\sigma}(x)\Sigma^m(x,u)\Sigma^m(x,u)^\mathsf{T}\,\underline{\sigma}(x)^\mathsf{T} \quad .$$

Corollary (Consistency)

Define

$$\mathcal{G}_1^{m,u}(x,r,p) = \mathcal{G}^{m,u}(x,r,p,\Gamma) - \frac{1}{2}\operatorname{tr}\left(\underline{\sigma}(x)\Sigma^m(x,u)\Sigma^m(x,u)^\mathsf{T}\,\underline{\sigma}(x)^\mathsf{T}\,\Gamma\right)$$

$$\mathcal{D}^2_{t,h,\Sigma,k}(\phi)(x) := h^{-1}\mathbb{E}\left[\phi(\hat{X}(t+h))\mathcal{P}_{\Sigma,k}(h^{-1/2}(W_{t+h}-W_t)) \mid \hat{X}(t) = x\right]$$

$$\begin{aligned} T_{t,h}(\phi)(x) &:= \mathcal{D}^0_{t,h}(\phi)(x) \\ &\quad + h \max_{m\in\mathcal{M},u\in\mathcal{U}} \left(\mathcal{G}_1^{m,u}(x,\mathcal{D}^0_{t,h}(\phi)(x),\mathcal{D}^1_{t,h}(\phi)(x)) + \mathcal{D}^2_{t,h,\Sigma^m(x,u),k}(\phi)(x)\right) \quad . \end{aligned}$$

Then, for $v \in \mathcal{C}^4_b$, $t \in \mathcal{T}_h$, and $x \in \mathbb{R}^d$, we have

$$\frac{T_{t,h}(v(t+h,\cdot))(x) - v(t,x)}{h} = \frac{\partial v}{\partial t} + \mathcal{H}(x,v(t,x),Dv(t,x),D^2v(t,x)) + O(h) \quad .$$

Theorem (Monotonicity)

Let $T_{t,h}$ be as before.

Assume that $\mathrm{tr}(\Sigma^m(x,u)\Sigma^m(x,u)^\top) \leq \bar{a}$ for all x, m, u.

Assume also that δ^m is upper bounded, and that there exists a bounded map g^m (in x and u) such that $f^m(x,u) - \underline{f}^m(x) = \underline{\sigma}(x)\Sigma^m(x,u)g^m(x,u)$.

Then, for k such that $\bar{a} < 4k+2$, there exists h_0 such that
$T_{t,h}$ is monotone for $h \leq h_0$
over the set of bounded continuous functions $\mathbb{R}^d \to \mathbb{R}$,
and there exists $C > 0$ such that
$T_{t,h}$ is Ch-almost monotone for all $h > 0$.

Note that when $k = 0$, $T_{t,h}$ is as in (Fahim, Touzi and Warin (2011)).

T is *additively α-subhomogeneous* if

$$\lambda \in \mathbb{R}, \lambda \geq 0,\ \phi \in \mathcal{F} \implies T(\phi + \lambda) \leq T(\phi) + \alpha\lambda\ .$$

Lemma (Sub-homogeneity)

Assume that δ^m is lower bounded in x and u.

Then, $T_{t,h}$ is additively α_h-subhomogeneous over the set of bounded continuous functions $\mathbb{R}^d \to \mathbb{R}$, for some constant $\alpha_h = 1 + Ch$ with $C \geq 0$.

Corollary (Stability)

Under the previous assumptions and if ψ and ℓ^m are bounded, and $v^h(T, x) = \psi(x)$ for all $x \in \mathbb{R}^d$, then, v^h is bounded.

Corollary

Assume that HJB has a strong uniqueness property for viscosity solutions and let v be its unique viscosity solution.
Then, when $h \to 0^+$, v^h converges to v locally uniformely in $t \in [0, T]$ and $x \in \mathbb{R}^d$.

The idempotent method of McEneaney, Kaise and Han

Given m and u, denote by $\hat{\xi}^{m,u}$ the Euler discretization of the process ξ:

$$\hat{\xi}^{m,u}(t+h) = \hat{\xi}^{m,u}(t) + f^m(\hat{\xi}^{m,u}(t), u)h + \sigma^m(\hat{\xi}^{m,u}(t), u)(W_{t+h} - W_t) \ .$$

The following is a time discretization of HJB:

$$v^h(t,x) = T_{t,h}(v^h(t+h,\cdot))(x), \quad t \in \mathcal{T}_h = \{0, h, 2h, \dots, T-h\} \ ,$$

with

$$T^m_{t,h}(\phi)(x) = \sup_{m \in \mathcal{M},\ u \in \mathcal{U}} \left\{ h\ell^m(x,u) + e^{-h\delta^m(x,u)} \mathbb{E}\left[\phi(\hat{\xi}^{m,u}(t+h)) \mid \hat{\xi}^{m,u}(t) = x\right] \right\} \ .$$

Under appropriate assumptions, v^h converges to the solution of HJB when h goes to zero.

The deterministic case

If $\sigma^m \equiv 0$, then $T_{t,h}$ is max-additive:

$$T_{t,h}(\phi \vee \phi') = T_{t,h}(\phi) \vee T_{t,h}(\phi') \ .$$

Moreover, if $\delta^m \equiv 0$, then $T_{t,h}$ is max-plus linear:

$$T_{t,h}(\lambda + \phi) = \lambda + T_{t,h}(\phi) \ .$$

Let q_i^{t+h} be "max-plus basis" functions, then

$$v^h(t+h, x) = \max_{i=1,\dots,N} (\lambda_i + q_i^{t+h}(x)) \implies v^h(t, x) = \max_{i=1,\dots,N} (\lambda_i + q_i^t(x)) \ ,$$

with $q_i^t = T_{t,h}(q_i^{t+h})$ and

we only need to compute the effect of the dynamic programming operator $T_{t,h}$ on the finite basis $q_i^T, \ i = 1, \dots, N$.

The deterministic case

If $\sigma^m \equiv 0$, then $T_{t,h}$ is max-additive:

$$T_{t,h}(\phi \oplus \phi') = T_{t,h}(\phi) \oplus T_{t,h}(\phi') \ .$$

Moreover, if $\delta^m \equiv 0$, then $T_{t,h}$ is max-plus linear:

$$T_{t,h}(\lambda \otimes \phi) = \lambda \otimes T_{t,h}(\phi) \ .$$

Let q_i^{t+h} be "max-plus basis" functions, then

$$v^h(t+h,x) = \max_{i=1,\dots,N}(\lambda_i + q_i^{t+h}(x)) \implies v^h(t,x) = \max_{i=1,\dots,N}(\lambda_i + q_i^t(x)) \ ,$$

with $q_i^t = T_{t,h}(q_i^{t+h})$ and

we only need to compute the effect of the dynamic programming operator $T_{t,h}$ on the finite basis $q_i^T, \ i = 1, \dots, N$.

The deterministic case

- First type of max-plus methods: project the operator $T_{t,h}$ or the q_i^t on a fixed basis, see Fleming and McEneaney (2000) and A.,Gaubert,Lakoua (2008) $\Longrightarrow$ **same difficulty as grid based methods.**
- Second type of max-plus methods (McEneaney, 2007): Assume that the $\mathcal{H}^m$ correspond to LQ problems, then

$$T_{t,h}(\phi)(x) = \max_{m \in \mathcal{M}} T^m_{t,h}(\phi)(x)$$

  with

$$q \text{ quadratic} \Longrightarrow T^m_{t,h}(q) \text{ quadratic.}$$

  So,

$$v^h(T,\cdot) \text{ finite sup of quad. forms} \Longrightarrow v^h(t,\cdot) \text{ finite sup of quad. forms.}$$

- The number of quadratic forms for $v^h(0,\cdot)$ is exponential in the number of time step only. So the method is **curse of dimensionality-free**.
- It can be reduced by pruning.

The stochastic case

Theorem (McEneaney, Kaise and Han, 2011)

Assume $\delta^m = 0$, σ^m is constant, f^m is affine, ℓ^m is concave quadratic (with respect to (x, u)), and ψ is the supremum of a finite number of concave quadratic forms. Then, for all $t \in \mathcal{T}_h$, there exists a set Z_t and a map $g_t : \mathbb{R}^d \times Z_t \to \mathbb{R}$ such that for all $z \in Z_t$, $g_t(\cdot, z)$ is a concave quadratic form and

$$v^h(t, x) = \sup_{z \in Z_t} g_t(x, z) \ .$$

Moreover, the sets Z_t satisfy

$$Z_t = \mathcal{M} \times \{\bar{z}_{t+h} : \mathcal{W} \to Z_{t+h} \mid \text{Borel measurable}\} \ ,$$

where $\mathcal{W} = \mathbb{R}^d$ is the space of values of the Brownian process.

The proof uses the max-plus (infinite) distributivity property.

- In the deterministic case, the sets Z_t are finite, and their cardinality is exponential in time: $\#Z_t = M \times \#Z_{t+h} = \cdots = M^{N_t} \times \#Z_T$ with $M = \#\mathcal{M}$ and $N_t = (T-t)/h$.
- In the stochastic case, the sets Z_t are infinite as soon as $t < T$.
- If the Brownian process is discretized in space, then $\mathcal{W}$ can be replaced by the finite subset with fixed cardinality p, and the sets Z_t become finite.
- Nevertheless, their cardinality increases doubly exponentially in time: $\#Z_t = M \times (\#Z_{t+h})^p = \cdots = M^{\frac{p^{N_t}-1}{p-1}} \times (\#Z_T)^{p^{N_t}}$ where $p \geq 2$ ($p = 2$ for the Bernouilli discretization).
- Then, McEneaney, Kaise and Han proposed to apply a pruning method to reduce at each time step $t \in \mathcal{T}_h$ the cardinality of Z_t.
- **Here, we shall replace the above time discretization scheme by the monotone probabilistic scheme and pruning by a random sampling based on a fixed process.**

The probabilistic max-plus method

Let $\mathcal{W} = \mathbb{R}^d$. The operator $T_{t,h}$ of the monotone probabilistic scheme can be written as

$$T_{t,h}(\phi)(x) = G_{t,h,x}(\tilde{\phi}_{t,h,x}) \qquad x \in \mathbb{R}^d \ ,$$

where

$$\tilde{\phi}_{t,h,x} = \phi(S_{t,h}(x,\cdot)) \ ,$$

$$S_{t,h} : \mathbb{R}^d \times \mathcal{W} \to \mathbb{R}^d, \ (x, W) \mapsto S_{t,h}(x, W) = x + \underline{f}(x)h + \underline{\sigma}(x)W \ ,$$

$$G_{t,h,x}(\tilde{\phi}) = D^0_{t,h,x}(\tilde{\phi}) + h \max_{m\in\mathcal{M},\ u\in\mathcal{U}} \left[\mathcal{G}^{m,u}_1(x, D^0_{t,h,x}(\tilde{\phi}), D^1_{t,h,x}(\tilde{\phi})) + D^2_{t,h,\Sigma(x,u),k}(\tilde{\phi})\right] \ ,$$

$$D^0_{t,h,x}(\tilde{\phi}) = \mathbb{E}(\tilde{\phi}(W_{t+h} - W_t)) \ ,$$

$$D^1_{t,h,x}(\tilde{\phi}) = \mathbb{E}(\tilde{\phi}(W_{t+h} - W_t)(\underline{\sigma}(x)^{\mathsf{T}})^{-1}h^{-1}(W_{t+h} - W_t)) \ ,$$

$$D^2_{t,h,\Sigma,k}(\tilde{\phi})(x) := h^{-1}\mathbb{E}\left[\tilde{\phi}(W_{t+h} - W_t)\mathcal{P}_{\Sigma,k}(h^{-1/2}(W_{t+h} - W_t))\right] \ .$$

Let $\mathcal{D}$ be the set of measurable functions from $\mathcal{W}$ to $\mathbb{R}$ with at most some given growth or growth rate. One can observe that

- $G_{t,h,x}$ is an operator from $\mathcal{D}$ to $\mathbb{R}$ and $\tilde{\phi}_{t,h,x} \in \mathcal{D}$ if $\phi \in \mathcal{D}$;
- Using the same arguments as for $T_{t,h}$, one obtain the stronger property that the operator $G_{t,h,x}$ is monotone additively α_h-subhomogeneous from $\mathcal{D}$ to $\mathbb{R}$, for $h \leq h_0$.
- Assume that $\mathcal{L}$ corresponds to a linear dynamics, then $x \mapsto \tilde{\phi}_{t,h,x}$ is a *random quadratic form* if ϕ is a quadratic form;
- Assume that $\mathcal{H}$ corresponds to a LQ problem, then

$$x \mapsto \tilde{\phi}_x \text{ random quadratic} \Longrightarrow G_{t,h,x}(\tilde{\phi}_x) \text{ quadratic.}$$

- Assume that $\mathcal{H}^m$ corresponds to a LQ problem, then

$$G_{t,h,x}(\tilde{\phi}) = \max_{m \in \mathcal{M}} G^m_{t,h,x}(\tilde{\phi})$$

  with

$$x \mapsto \tilde{\phi}_x \text{ random quadratic} \Longrightarrow G^m_{t,h,x}(\tilde{\phi}_x) \text{ quadratic.}$$

Theorem (A., Fodjo, 2016)

Let G be a monotone additively α-subhomogeneous operator from $\mathcal{D} \to \mathbb{R}$, for some constant $\alpha > 0$. Let $(Z, \mathfrak{A})$ be a measurable space, and let $\mathcal{W}$ be endowed with its Borel σ-algebra. Let $\phi : \mathcal{W} \times Z \to \mathbb{R}$ be a measurable map such that for all $z \in Z$, $\phi(\cdot, z)$ is continuous and belongs to $\mathcal{D}$. Let $v \in \mathcal{D}$ be such that $v(W) = \sup_{z \in Z} \phi(W, z)$. Assume that v is continuous and bounded. Then,

$$G(v) = \sup_{\bar{z} \in \overline{Z}} G(\bar{\phi}^{\bar{z}})$$

where $\bar{\phi}^{\bar{z}} : \mathcal{W} \to \mathbb{R}, \; W \mapsto \phi(W, \bar{z}(W))$, and

$$\overline{Z} = \{\bar{z} : \mathcal{W} \to Z, \; \textit{measurable and such that } \bar{\phi}^{\bar{z}} \in \mathcal{D}\}.$$

This says that any monotone continuous map distributes over max and generalizes the max-plus distributivity.

Formally, we have $G(v) = G(\bar{\phi}^{\bar{z}})$, when $v(W) = \phi(W, \bar{z}(W))$.

Theorem (A., Fodjo, 2016, compare with McEneaney, Kaise and Han, 2011)

Assume that, for each $m \in \mathcal{M}$, δ^m and σ^m are constant, f^m is affine with respect to (x, u), ℓ^m is concave quadratic with respect to (x, u), and that ψ is the supremum of a finite number of concave quadratic forms.
Consider the monotone probabilistic scheme with $T_{t,h}$ as above.
Assume that the operators $G^m_{t,h,x}$ are monotone additively α_h-subhomogeneous from $\mathcal{D}$ to $\mathbb{R}$, for some constant $\alpha_h = 1 + Ch$ with $C \geq 0$.
Assume also that the value function v^h belongs to $\mathcal{D}$ and is locally Lipschitz continuous with respect to x.
Then, for all $t \in \mathcal{T}_h$, there exists a set Z_t and a map $g_t : \mathbb{R}^d \times Z_t \to \mathbb{R}$ such that for all $z \in Z_t$, $g_t(\cdot, z)$ is a concave quadratic form and

$$v^h(t, x) = \sup_{z \in Z_t} g_t(x, z) \ .$$

Moreover, the sets Z_t satisfy

$$Z_t = \mathcal{M} \times \{\bar{z}_{t+h} : \mathcal{W} \to Z_{t+h} \mid \text{Borel measurable}\} \ .$$

The probabilistic max-plus method: the sampling algorithm

Denote $q(x,z) := \frac{1}{2}x^{\mathsf{T}}Qx + b\cdot x + c$ for $z = (Q,b,c) \in \mathcal{Q}_d = \mathbb{S}_d^- \times \mathbb{R}^d \times \mathbb{R}$.
Input: $M = \#\mathcal{M}$, $\epsilon > 0$, $Z_T \subset \mathcal{Q}_d$ such that $|\psi(x) - \max_{z\in Z_T} q(x,z)| \leq \epsilon$ and $\#Z_T \leq N_{\mathrm{in}}$,
$N = (N_{\mathrm{in}}, N_x, N_w)$ (the numbers of samples with $N_x \leq N_{\mathrm{in}}$).
Output: $Z_t \subset \mathcal{Q}_d$, $t \in \mathcal{T}_h \cup \{T\}$, and $v^{h,N}$.
Initialization: Define $v^{h,N}(T,x) = \max_{z\in Z_T} q(x,z)$. Construct a sample of $(\hat{X}(0), (W_{t+h} - W_t)_{t\in\mathcal{T}_h})$ of size N_{in} indexed by $\omega \in \Omega_{N_{\mathrm{in}}}$, and deduce $\hat{X}(t,\omega)$.

For $t = T-h, T-2h, \dots, 0$ *do*

1. Construct independent subsamples of sizes N_x and N_w of $\Omega_{N_{\mathrm{in}}}$, then take the product of samplings,
   leading to $(\omega_\ell, \omega'_\ell)$ for $\ell \in \Omega_{N_{\mathrm{rg}}} := [N_x] \times [N_w]$.
   Induce the sample $\hat{X}(t,\omega_\ell)$ (resp. $(W_{t+h} - W_t)(\omega'_\ell)$) for $\ell \in \Omega_{N_{\mathrm{rg}}}$ of $\hat{X}(t)$ (resp. $W_{t+h} - W_t$).
   Denote by $\mathcal{W}_t^N \subset \mathcal{W}$ the set of $(W_{t+h} - W_t)(\omega'_\ell)$ for $\ell \in \Omega_{N_{\mathrm{rg}}}$.

The probabilistic max-plus method: the sampling algorithm cont.

2. For each $\omega \in \Omega_{N_{\text{in}}}$ denote $x_t = \hat{X}(t, \omega)$.
   (a) Choose $\bar{z}_{t+h} : \mathcal{W}_t^N \to Z_{t+h}$ such that, for all $\ell \in \Omega_{N_{\text{rg}}}$, we have

   $$\bar{z}_{t+h}((W_{t+h} - W_t)(\omega'_\ell)) \in \operatorname*{Argmax}_{z \in Z_{t+h}} q\big(S_{t,h}(x_t, (W_{t+h} - W_t)(\omega'_\ell)), z\big) \ .$$

   Let $\tilde{q}_{t,h,x}$ be the element of $\mathcal{D}$ given by $W \in \mathcal{W} \mapsto q(S_{t,h}(x, W), \bar{z}_{t+h}(W))$.
   (b) For each m, approximate $x \mapsto G^m_{t,h,x}(\tilde{q}_{t,h,x})$ by a linear regression estimation on the set of quadratic forms using the sample $(\hat{X}(t, \omega_\ell), (W_{t+h} - W_t)(\omega'_\ell))$, with $\ell \in \Omega_{N_{\text{rg}}}$, and denote by $z_t^m \in \mathcal{Q}_d$ the parameter of the resulting quadratic form.
   (c) Choose $z_t \in \mathcal{Q}_d$ optimal among the $z_t^m \in \mathcal{Q}_d$ at the point x_t, that is such that $q(x_t, z_t) = \max_{m \in \mathcal{M}} q(x_t, z_t^m)$.
3. Denote by Z_t the set of the parameters $z_t \in \mathcal{Q}_d$ obtained in this way, and define

   $$v^{h,N}(t, x) = \max_{z \in Z_t} q(x, z) \quad \forall x \in \mathbb{R}^d \ .$$

The probabilistic max-plus method: the sampling algorithm cont.

Computational time:

$$O(d^2 N_{\mathrm{in}}^2 \times N_w + d^3 M \times N_{\mathrm{in}} \times N_x \times N_w)$$

where the first term corresponds to step (a) and the second one to steps (b) and (c).

Note also that N_x can be choosen to be in the order of a polynomial in d since the regression is done on the set of quadratic forms,
so in general the second term is negligeable.

The pricing and hedging an option example

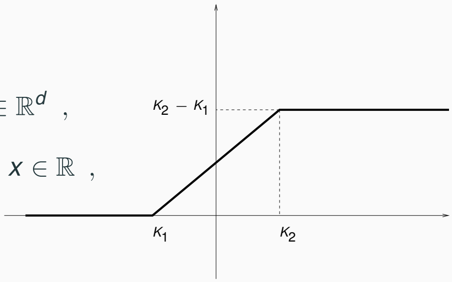
- *The dynamics:* $d\xi_i = \sigma_i \xi_i dB_i$, where the Brownians B_i have uncertain correlations: $\langle dB_i, dB_j \rangle = \mu_{i,j} ds$.
- We know: $\mu \in \mathrm{cvx}(\mathcal{M})$ with $\mathcal{M}$ a finite set.
- *Maximize*

$$J(t,x,\mu) := \quad \mathbb{E}\left[\psi(\xi(T)) \mid \xi(t) = x\right] \quad , \quad \text{with}$$

$$\psi(x) = \phi(\max_{i \text{ odd}} x_i - \min_{j \text{ even}} x_j), \quad x \in \mathbb{R}^d \quad ,$$

$$\phi(x) = (x - K_1)^+ - (x - K_2)^+, \quad x \in \mathbb{R} \quad ,$$

$$x^+ = \max(x, 0), \quad K_1 < K_2 \quad .$$



- $\mathcal{M}$ is a finite subset of the set of positive definite symmetric matrices with 1 on the diagonal and

$$[\sigma^m(\xi)\sigma^m(\xi)^\mathsf{T}]_{i,j} = \sigma_i \xi_i \sigma_j \xi_j \mu_{i,j} \ .$$

- We take $K_1 = -5,\ K_2 = 5,\ T = 0.25$, and $h = 0.01$.
- In dimension 2, we take $\sigma = (0.4, 0.3)$, and

$$\mathcal{M} = \{m = \begin{bmatrix} 1 & m_{12} \\ m_{12} & 1 \end{bmatrix} \mid m_{12} = \pm\rho\} \ .$$

- In dimension 5, we take $\sigma = (0.4, 0.3, 0.2, 0.3, 0.4)$ and

$$\mathcal{M} = \{m = \begin{bmatrix} 1 & m_{12} & 0 & 0 & 0 \\ m_{12} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & m_{45} \\ 0 & 0 & 0 & m_{45} & 1 \end{bmatrix} \mid m_{12} = \pm\rho,\ m_{45} = \pm\rho\} \ .$$

- We tested the cases $\rho = 0$, $\rho = 0.4$ and 0.8.

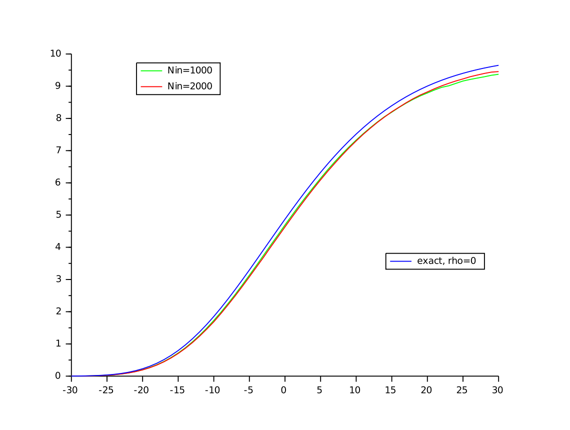


Figure 1: Value function obtained at $t = 0$, and $x_2 = 50$ as a function of $x_1 - x_2 \in [-30, 30]$. Here $\rho = 0$, $N_{\text{in}} = 1000$, or 2000, $N_x = 10$, $N_w = 1000$.

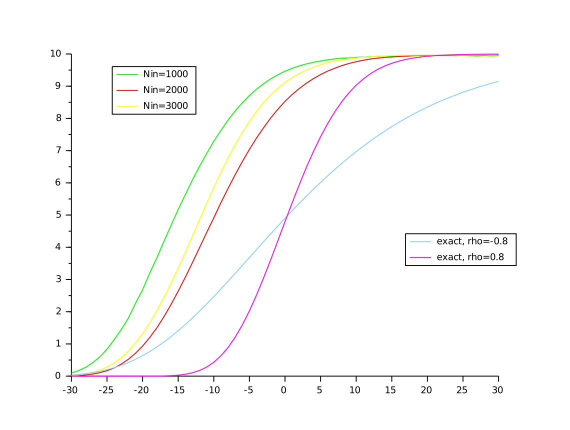


Figure 2: Value function obtained at $t = 0$, and $x_2 = 50$ as a function of $x_1 - x_2 \in [-30, 30]$. Here $\rho = 0.8$, $N_{\text{in}} = 1000$, or 2000 or 3000, $N_x = 10$, $N_w = 1000$.

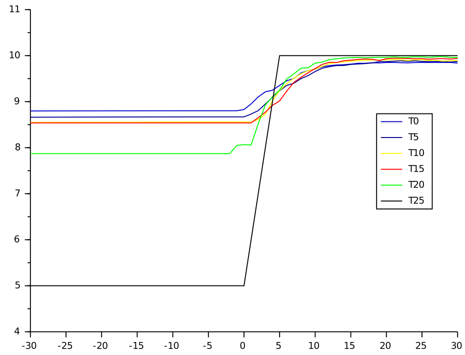


Figure 3: Value function obtained in dimension 5 at $x_2 = x_3 = x_4 = x_5 = 50$ as a function of $x_1 - x_2 \in [-30, 30]$. Here $\rho = 0.8$, $N_{\text{in}} = 3000$, $N_x = 50$, $N_w = 1000$. The time by time iteration is $\simeq 2500s$ and the total time is $\simeq 19h$ on a 12 core.

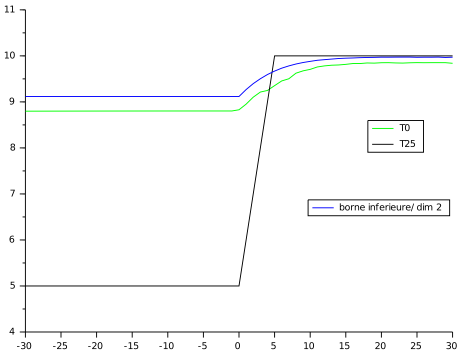


Figure 4: Comparizon between the value function obtained in dimension 5 at $t = 0$, and $x_2 = x_3 = x_4 = x_5 = 50$ as a function of $x_1 - x_2 \in [-30, 30]$, and a lower bound from the dimension 2. Here $\rho = 0.8$, $N_{\text{in}} = 3000$, $N_x = 50$, $N_w = 1000$.

Conclusion

- We proposed an algorithm to solve HJB equations, combining ideas included in the idempotent algorithm of McEneaney, Kaise and Han (2011) and in the probabilistic numerical scheme of Fahim, Touzi and Warin (2011).
- The advantages with respect to the pure probabilistic scheme are that the regression estimation is over a linear space of small dimension.
- The advantages with respect to the pure idempotent scheme is that one may avoid the pruning step: the number of quadratic forms generated by the algorithm is linear with respect to the sampling size times the number of discrete controls.
- We improved the probabilistic numerical scheme of Fahim, Touzi and Warin (2011) to obtain a monotone scheme and so apply the probabilistic max-plus method in general situations.
- The theoretical results suggest that it can also be applied to Isaacs equations of zero-sum games.
- *Open:* improve the optimization step to decrease the complexity.