Direct numerical solution of cell problems in homogenization of HJ equations via generalized Newton's method for inconsistent nonlinear systems

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Spoiler

- Ergodic problems for Hamilton-Jacobi equations
- Small-δ, large-t and theoretical formulas approximations
- The new approach: a Newton-like method for inconsistent systems
- Numerical results for:
  - Eikonal Hamiltonians
  - q-power Hamiltonians
  - Non-convex Hamiltonians
  - Second order Hamiltonians
  - Weakly coupled systems
  - Dislocation dynamics
  - Stationary MFG in Euclidean Spaces (single and multi-population)
  - Stationary MFG on Networks
  - Homogenization of Mean Field Games with Small Noise

Ergodic problems for Hamilton-Jacobi equations

Consider the problem

$$\begin{cases} v_t^\varepsilon + H(\frac{x}{\varepsilon}, Dv^\varepsilon) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ v^\varepsilon(\cdot, 0) = v_0 & \text{in } \mathbb{R}^n \end{cases}$$

where the Hamiltonian $H(x, p) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function, 1-periodic in x and coercive in p. The viscosity solution v^ε converges, as $\varepsilon \to 0$, to the viscosity solution v of the **effective problem**

$$\begin{cases} v_t + \overline{H}(Dv) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ v(\cdot, 0) = v_0 & \text{in } \mathbb{R}^n \end{cases}$$

where, for each $p \in \mathbb{R}^n$, the value $\lambda = \overline{H}(p)$ is the unique number such that the **cell problem**

$$H(x, Du + p) = \lambda \qquad \text{in } \mathbb{T}^n$$

admits a 1-periodic viscosity solution u in the torus $\mathbb{T}^n$.

The function $\overline{H} : \mathbb{R}^n \to \mathbb{R}$ is called the **effective Hamiltonian**.
The solution u is not unique in general, not even for addition of constants.

Computing $\overline{H}$ via regularization of the cell problem

Small-δ method

The viscosity solution of

$$\delta u^\delta + H(x, Du^\delta + p) = 0 \qquad \text{in } \mathbb{T}^n$$

satisfies

$$-\delta u^\delta \to \overline{H}(p) \quad \text{as } \delta \to 0\,, \quad \text{uniformly in } \mathbb{R}^n$$

Large-t method

The viscosity solution of

$$\begin{cases} u_t + H(x, Du + p) = 0 & \text{in } \mathbb{R}^n \times (0, +\infty) \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n \end{cases}$$

satisfies

$$-u(x,t)/t \to \overline{H}(p) \quad \text{as } t \to +\infty\,, \quad \text{uniformly in } \mathbb{R}^n$$

Computing $\overline{H}$ via theoretical formulas

inf-sup formula

$$\overline{H}(p) = \inf_{u \in C^\infty(\mathbb{T}^n)} \sup_{x \in \mathbb{T}^n} H(x, Du + p)$$

Variational approximation

The infimum is approximated for $k \to +\infty$ by the solution to

$$\operatorname{div}\left(e^{k\,H(x,Du+p)} H_p(x, Du + p)\right) = 0, \qquad x \in \mathbb{T}^n$$

Auxiliary boundary value problem for Homogeneous Hamiltonians

$$\overline{H}(x,p) = \max_{\|a\|=1} \{(p \cdot a)c(x,a)\} \quad \Longrightarrow \quad \overline{H}(p) = \max_{\|a\|=1} \{(p \cdot a)\bar{c}(a)\}$$

$$\begin{cases} H(\frac{x}{\varepsilon}, Du^\varepsilon(x)) = 1 \\ u^\varepsilon(0) = 0 \end{cases} \quad \xrightarrow{\varepsilon \to 0} \quad \begin{cases} \overline{H}(Du(x)) = 1 \\ u(0) = 0 \end{cases}$$

$$u(x) = u^\varepsilon(x) + \mathcal{O}(\varepsilon) \quad \text{and} \quad \frac{1}{\bar{c}(x/|x|)} = \frac{u^\varepsilon(x)}{|x|} + \mathcal{O}(\varepsilon)$$

A new approach: solving the cell problem directly

What is wrong with $H(x, Du + p) = \lambda$?

The problem is ill-posed, one equation in two unknowns: while the ergodic constant λ is unique, the viscosity solution u is in general not unique.

Nevertheless, we can perform in the torus $\mathbb{T}^n$ our favorite discretization (FD, FE, FV, DG, SL) getting a system of nonlinear equations of the form

$$\mathcal{S}(h, \mathbf{x}, \mathbf{U}, p) = \Lambda$$

where h is a discretization parameter (meant to go to zero), $\mathbf{x}$ is the vector of grid nodes, $\mathbf{U}$ is a grid function and Λ is a real number.

The operator $\mathcal{S}$ is a generic scheme, which is assumed to enjoy all the properties needed to ensure the convergence $(\mathbf{U}, \Lambda) \to (u, \lambda)$ as $h \to 0$.

In particular, $\mathcal{S}$ should employ a numerical Hamiltonian which is able to correctly select approximations of viscosity solutions (Lax-Friedrichs, Engquist-Osher, Godunov).

A Newton-like method for inconsistent nonlinear systems

The main assumption:

for each fixed h there exists a unique Λ for which
$\mathcal{S}(h, \mathbf{x}, \mathbf{U}, p) = \Lambda$ admits a solution $\mathbf{U}$ (in general not unique).

Collecting the unknowns $(\mathbf{U}, \Lambda)$ in a single vector $\mathbf{X}$ of length N and recasting the M equations (given by $\mathcal{S}$) as functions of $\mathbf{X}$, we get the nonlinear map $\mathbf{F} : \mathbb{R}^N \to \mathbb{R}^M$ defined by $\mathbf{F}(\mathbf{X}) = \mathcal{S}(h, \mathbf{x}, \mathbf{U}, p) - \Lambda$.

The discrete cell problem is equivalent to

find $\mathbf{X} \in \mathbb{R}^N$ such that $\mathbf{F}(\mathbf{X}) = \mathbf{0} \in \mathbb{R}^M$

Assuming that $\mathbf{F}$ is Fréchet differentiable with Jacobian $\mathbf{J_F} \in \mathbb{R}^{M \times N}$, we would like to approximate the zeros of $\mathbf{F}$ by using the Newton's method

$$\mathbf{J_F}(\mathbf{X}^{(k)})\delta = -\mathbf{F}(\mathbf{X}^{(k)}), \qquad \mathbf{X}^{(k+1)} = \mathbf{X}^{(k)} + \delta, \qquad k \geq 0,$$

but this system can be **inconsistent** for arbitrary M and N, i.e., **underdetermined** if $M < N$ and **overdetermined** if $M > N$.

The generalized *least-squares* solution

We denote by $\mathbf{J}_{\mathbf{F}}^{\dagger}$ the Moore-Penrose **pseudoinverse** of the Jacobian $\mathbf{J}_{\mathbf{F}}$, namely the unique $N \times M$ matrix such that

$$\mathbf{J}_{\mathbf{F}}\mathbf{J}_{\mathbf{F}}^{\dagger}\mathbf{J}_{\mathbf{F}} = \mathbf{J}_{\mathbf{F}}\,, \quad \mathbf{J}_{\mathbf{F}}^{\dagger}\mathbf{J}_{\mathbf{F}}\mathbf{J}_{\mathbf{F}}^{\dagger} = \mathbf{J}_{\mathbf{F}}^{\dagger}\,, \quad (\mathbf{J}_{\mathbf{F}}\mathbf{J}_{\mathbf{F}}^{\dagger})^T = \mathbf{J}_{\mathbf{F}}\mathbf{J}_{\mathbf{F}}^{\dagger}\,, \quad (\mathbf{J}_{\mathbf{F}}^{\dagger}\mathbf{J}_{\mathbf{F}})^T = \mathbf{J}_{\mathbf{F}}^{\dagger}\mathbf{J}_{\mathbf{F}}$$

It can be easily proved that

$$\delta^{\star} := -\mathbf{J}_{\mathbf{F}}^{\dagger}(\mathbf{X}^{(k)})\mathbf{F}(\mathbf{X}^{(k)})$$

is the unique vector of smallest Euclidean norm which minimizes the Euclidean norm of the residual $\mathbf{J}_{\mathbf{F}}(\mathbf{X}^{(k)})\delta + \mathbf{F}(\mathbf{X}^{(k)})$.

- In the overdetermined case ($M > N$), if $\mathbf{J}_{\mathbf{F}}$ has full column rank N

$$\mathbf{J}_{\mathbf{F}}^{\dagger} = (\mathbf{J}_{\mathbf{F}}^T\mathbf{J}_{\mathbf{F}})^{-1}\mathbf{J}_{\mathbf{F}}^T$$

- In the underdetermined case ($M < N$), if $\mathbf{J}_{\mathbf{F}}$ has full row rank M

$$\mathbf{J}_{\mathbf{F}}^{\dagger} = \mathbf{J}_{\mathbf{F}}^T(\mathbf{J}_{\mathbf{F}}\mathbf{J}_{\mathbf{F}}^T)^{-1}$$

Efficient implementation via QR factorization avoiding $\mathbf{J}_\mathbf{F}^\dagger$

- Overdetermined case ($M > N$), full column rank N: factoring $\mathbf{J}_\mathbf{F} = \mathbf{QR}$,

$$\mathbf{Q} = (\mathbf{Q_1}\ \mathbf{Q_2}) \in \mathbb{R}^{M\times M} \text{ orthogonal, } \mathbf{Q_1} \in \mathbb{R}^{M\times N} \text{ and } \mathbf{Q_2} \in \mathbb{R}^{M\times(M-N)},$$

$$\mathbf{R} = \begin{pmatrix}\mathbf{R_1}\\ \mathbf{0}\end{pmatrix} \in \mathbb{R}^{M\times N}, \mathbf{R_1} \in \mathbb{R}^{N\times N} \text{ upper triangular and } \mathbf{0} \in \mathbb{R}^{(M-N)\times N},$$

yields $\mathbf{J}_\mathbf{F}^\dagger = \mathbf{R}_1^{-1}\mathbf{Q}_1^T$, and $\delta^\star = -\mathbf{R}_1^{-1}\mathbf{Q}_1^T\mathbf{F}(\mathbf{X}^{(k)})$ via back-substitution.

- Underdetermined case ($M < N$), full row rank M: factoring $\mathbf{J}_\mathbf{F}^T = \mathbf{QR}$,

$$\mathbf{Q} = (\mathbf{Q_1}\ \mathbf{Q_2}) \in \mathbb{R}^{N\times N} \text{ orthogonal, } \mathbf{Q_1} \in \mathbb{R}^{N\times M} \text{ and } \mathbf{Q_2} \in \mathbb{R}^{N\times(N-M)},$$

$$\mathbf{R} = \begin{pmatrix}\mathbf{R_1}\\ \mathbf{0}\end{pmatrix} \in \mathbb{R}^{N\times M}, \mathbf{R_1} \in \mathbb{R}^{M\times M} \text{ upper triangular and } \mathbf{0} \in \mathbb{R}^{(N-M)\times M},$$

yields $\mathbf{J}_\mathbf{F}^\dagger = \mathbf{Q}_1\mathbf{R}_1^{-T}$, and $\delta^\star = -\mathbf{Q}_1\mathbf{R}_1^{-T}\mathbf{F}(\mathbf{X}^{(k)})$ via back-substitution.

The algorithm

GIVEN AN INITIAL GUESS $\mathbf{X} \in \mathbb{R}^N$ AND A TOLERANCE $\varepsilon > 0$,

REPEAT

- ASSEMBLE $\mathbf{F}(\mathbf{X}) \in \mathbb{R}^M$ AND $\mathbf{J}_\mathbf{F}(\mathbf{X}) \in \mathbb{R}^{M \times N}$
- SOLVE $\mathbf{J}_\mathbf{F}(\mathbf{X})\delta = -\mathbf{F}(\mathbf{X})$ IN THE LEAST-SQUARES SENSE, USING THE **QR** FACTORIZATION OF $\mathbf{J}_\mathbf{F}(\mathbf{X})$ IF $M > N$ OR $\mathbf{J}_\mathbf{F}(\mathbf{X})^T$ IF $M < N$
- UPDATE $\mathbf{X} \leftarrow \mathbf{X} + \delta$

UNTIL $\|\delta\|_2 < \varepsilon$ AND/OR $\|\mathbf{F}(\mathbf{X})\|_2 < \varepsilon$

Implementation in **C** employing the free library **SuiteSparseQR**, which is designed to efficiently compute in parallel the **QR** factorization and the least-squares solution to large and sparse linear systems.

Numerical tests performed on a Lenovo Ultrabook X1 Carbon, using 1 CPU Intel Quad-Core i5-4300U 1.90Ghz with 8 Gb Ram, running under the Linux Slackware 14.1 operating system.

Implementation tricks

• Sometimes Newton-like methods do not converge, due to oscillations around a minimum of the residual function $\|\mathbf{F}(\mathbf{X})\|_2$.
In this case we introduce a **dumping parameter** in the update step:
$\mathbf{X} \leftarrow \mathbf{X} + \mu\delta$ for some $0 < \mu < 1$ (usually a fixed value of μ works fine).
A more efficient (but costly) selection of the dumping parameter can be implemented using **line search** methods.

• It may happen that $\mathbf{J_F}(\mathbf{X})$ is nearly singular or rank deficient, so that the least-squares solution cannot be computed.
In the spirit of the **Levenberg-Marquardt** method, we can regularize $\mathbf{J_F}(\mathbf{X})$ with $\tau\mathbf{I} + \mathbf{J_F}(\mathbf{X})$, for some $\tau > 0$.

• Newton-like methods classically require that **F** is Fréchet differentiable.
In the spirit of **nonsmooth**-Newton methods, we can replace the usual differential with **any** element of the sub-differential.
For instance, $H(x,p) = \frac{1}{q}|p|^q - V(x)$ with $q \geq 1 \Longrightarrow H_p(x,p) = |p|^{q-2}p$, is singular at $p = 0$ for $1 \leq q < 2$. We typically choose $H_p(x,0) = 0$.

Eikonal Hamiltonians

$$\frac{1}{2}|Du+p|^2 - V(x) = \lambda \qquad \text{in } \mathbb{T}^n,$$

where $p \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and V is a 1-periodic potential.

A formula for the effective Hamiltonian is available in dimension $n = 1$:

$$\overline{H}(p) = \begin{cases} -\min V & \text{if } |p| \leq p_c \\ \lambda & \text{if } |p| > p_c \quad \text{s.t. } |p| = \int_0^1 \sqrt{2(V(s)+\lambda)}ds \end{cases}$$

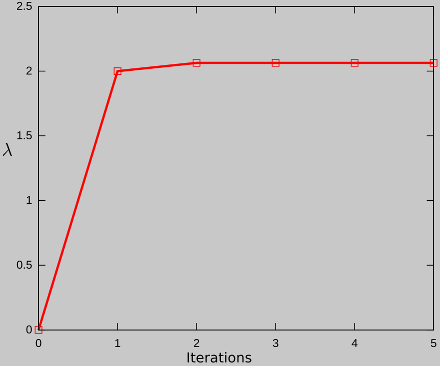
where $p_c = \int_0^1 \sqrt{2(V(s) - \min V)}ds$.

$\overline{H}$ has a plateau in the whole interval $\mathcal{P}_{\overline{H}} = [-p_c, p_c]$.

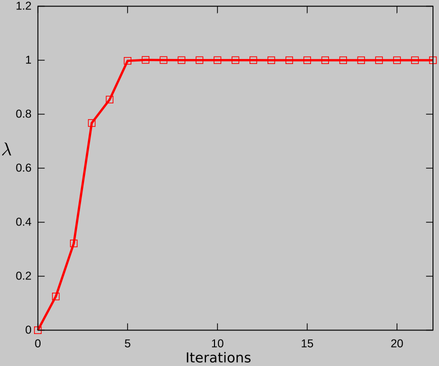
Eikonal Hamiltonians in 1D

$V(x) = \sin(2\pi x)$, $\min V = -1$ and $\mathcal{P}_{\overline{H}} = [-p_c, p_c]$, $p_c = 4/\pi \sim 1.2732$

Convergence: λ vs number of iterations



$p = 2 \notin \mathcal{P}_{\overline{H}}$

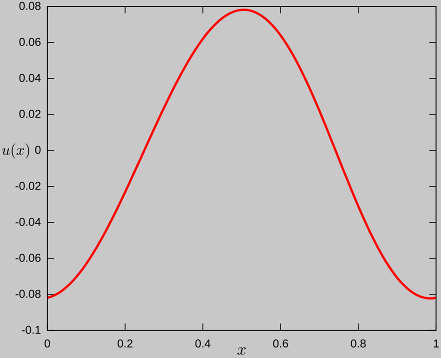


$p = 0.5 \in \mathcal{P}_{\overline{H}}$

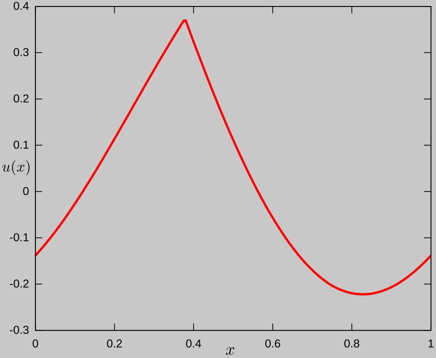
Eikonal Hamiltonians in 1D

$V(x) = \sin(2\pi x)$, $\min V = -1$ and $\mathcal{P}_{\overline{H}} = [-p_c, p_c]$, $p_c = 4/\pi \sim 1.2732$

Correctors



$p = 2 \notin \mathcal{P}_{\overline{H}}$



$p = 0.5 \in \mathcal{P}_{\overline{H}}$

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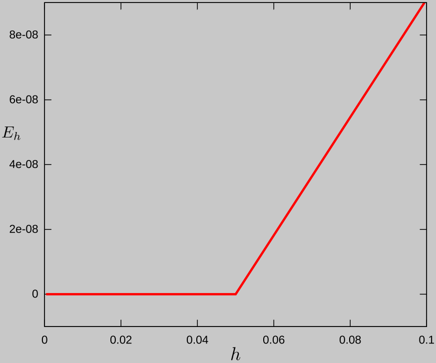
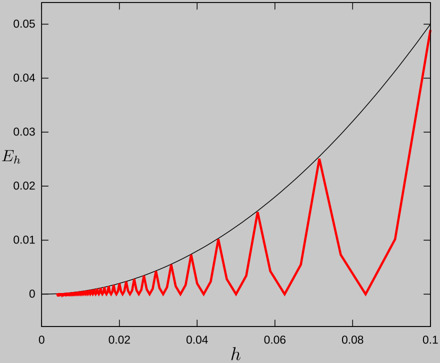
$V(x) = \sin(2\pi x)$, $\min V = -1$ and $\mathcal{P}_{\overline{H}} = [-p_c, p_c]$, $p_c = 4/\pi \sim 1.2732$

Convergence under grid refinement: error vs h



$p = 2 \notin \mathcal{P}_{\overline{H}}$

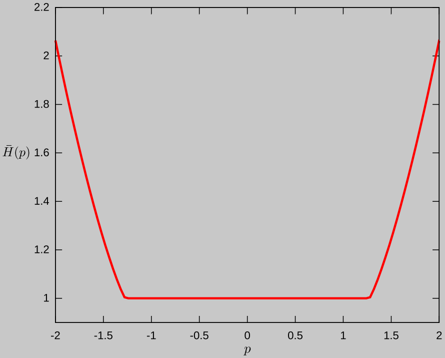


$p = 0.5 \in \mathcal{P}_{\overline{H}}$

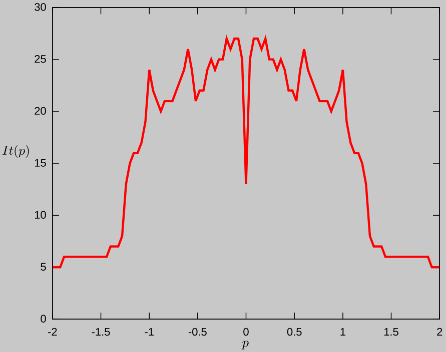
Eikonal Hamiltonians in 1D

$V(x) = \sin(2\pi x)$, $\min V = -1$ and $\mathcal{P}_{\overline{H}} = [-p_c, p_c]$, $p_c = 4/\pi \sim 1.2732$

Effective Hamiltonian and number of iterations for $p \in [-2, 2]$



$\overline{H}(p)$



Iterations(p)

N	N_p	Av.CPU/p (secs)	Av.Its/p	Tot.CPU (secs)
100	100	0.01	13	1.18

Direct Newton vs Small-δ and or Large-t

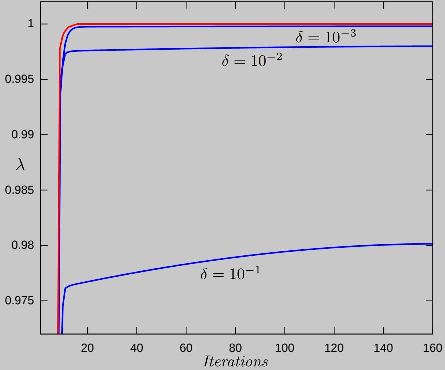
$$\delta w + H_p(x, Du^{(k)} + p) \cdot Dw = -\delta u^{(k)} - H(x, Du^{(k)} + p)$$

$$\frac{1}{\Delta t} w + H_p(x, Du^{(k)} + p) \cdot Dw = -\frac{1}{\Delta t}(u^{(k)} - u^n) - H(x, Du^{(k)} + p)$$

$$u^{(k+1)} = u^{(k)} + \mu w \quad \text{for each } k \geq 0\,, \quad 0 < \mu \leq 1$$

Small-δ: $u_\delta = \lim_{k\to\infty} u^{(k)}$ Large-t: $u = \lim_{n\to\infty} u^n$, where $u^{n+1} = \lim_{k\to\infty} u^{(k)}$

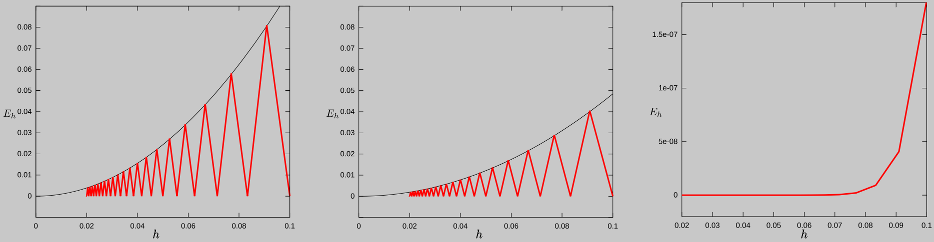
Coincide for $u^{(0)} \equiv 0$ and $u^{(0)} = u^n$ for each n, with $u^0 \equiv 0$ and $\delta = \frac{1}{\Delta t}$



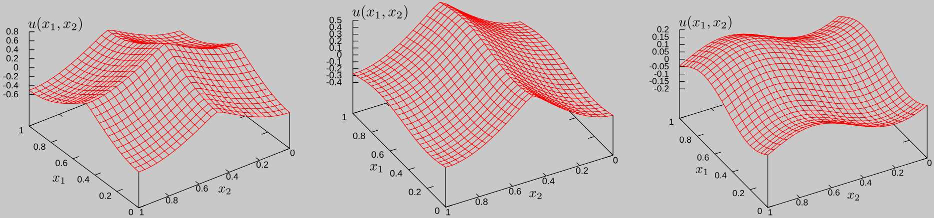
Eikonal Hamiltonians in 2D

$V_a(x_1, x_2) = \cos(2\pi x_1) + \cos(2\pi x_2)$

Convergence under grid refinement and correctors



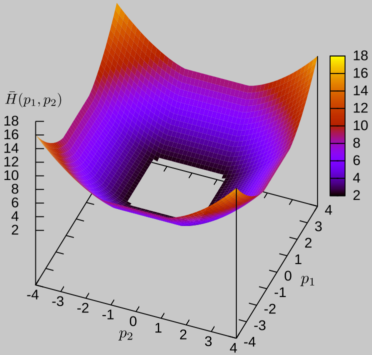
$p = (0, 0)$ $p = (2, 0)$ $p = (2, 2)$



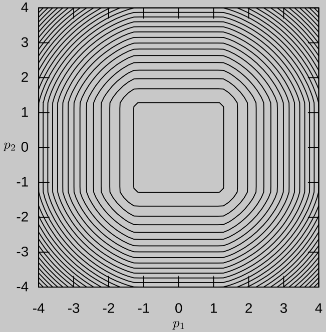
Eikonal Hamiltonians in 2D

$$V_a(x_1, x_2) = \cos(2\pi x_1) + \cos(2\pi x_2)$$

Effective Hamiltonian for $p \in [-4, 4]^2$



Surface



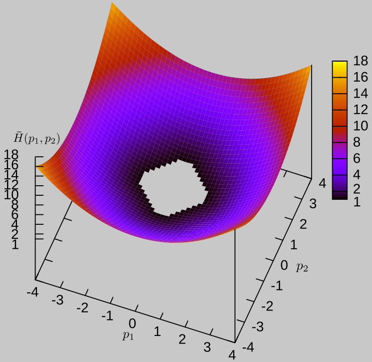
Level sets

N	N_p	Av.CPU/p (secs)	Av.Its/p	Tot.CPU (secs)
25^2	51^2	0.4	16	970.45

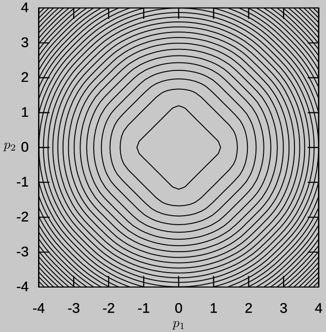
Eikonal Hamiltonians in 2D

$$V_b(x_1, x_2) = \sin(2\pi x_1)\sin(2\pi x_2)$$

Effective Hamiltonian for $p \in [-4, 4]^2$



Surface



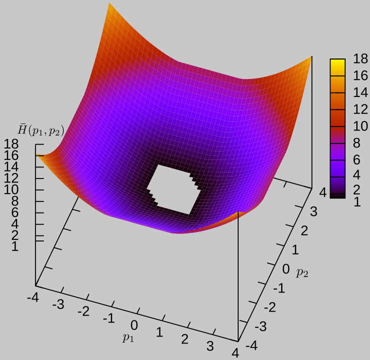
Level sets

N	N_p	Av.CPU/p (secs)	Av.Its/p	Tot.CPU (secs)
25^2	51^2	0.2	7	480.75

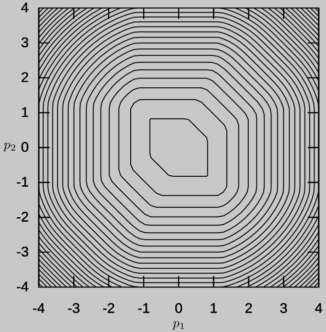
Eikonal Hamiltonians in 2D

$$V_c(x_1, x_2) = \cos(2\pi x_1) + \cos(2\pi x_2) + \cos(2\pi(x_1 - x_2))$$

Effective Hamiltonian for $p \in [-4, 4]^2$



Surface



Level sets

N	N_p	Av.CPU/p (secs)	Av.Its/p	Tot.CPU (secs)
25^2	51^2	0.24	10	630.77

q-power Hamiltonians

$$\frac{1}{q}|Du + p|^q - V(x) = \lambda \qquad \text{in } \mathbb{T}^n,$$

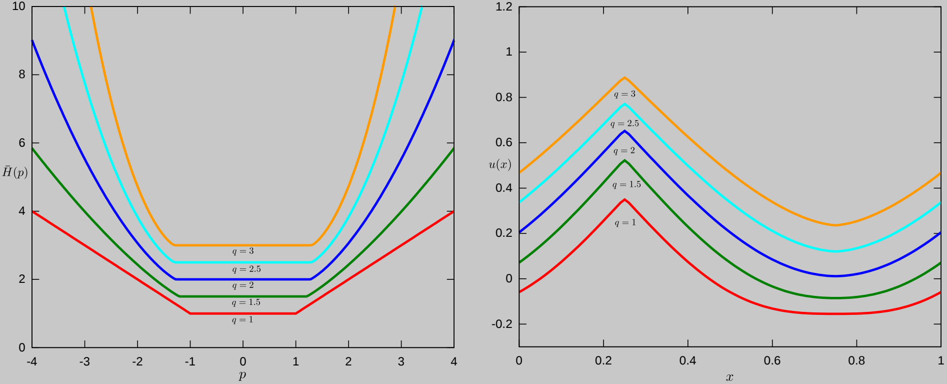
where $p \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, V is a 1-periodic potential and $q \geq 1$.

The singularity at the origin of the derivative of $|\cdot|^q$ for $1 \leq q < 2$ is handled by choosing, in a nonsmooth-Newton fashion, an element of the sub-differential. Here, we simply choose 0 if $Du + p = 0$ at some point.

q-power Hamiltonians in 1D

$$V(x) = \sin(2\pi x)$$

Effective Hamiltonians for $p \in [-4, 4]$ and correctors for $p = 0$

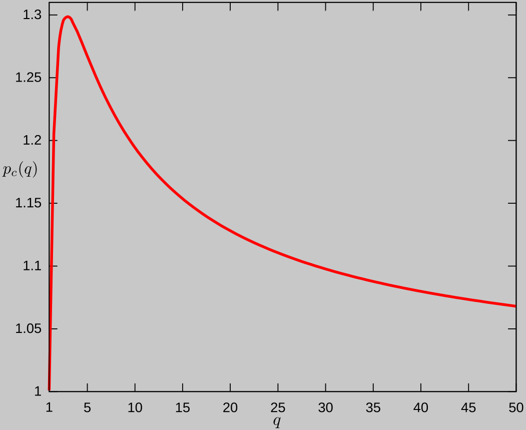


q	N	N_p	Av.CPU/q (secs)
1, 3/2, 2, 5/2, 3	100	200	2.5

q-power Hamiltonians in 1D

$V(x) = \sin(2\pi x)$

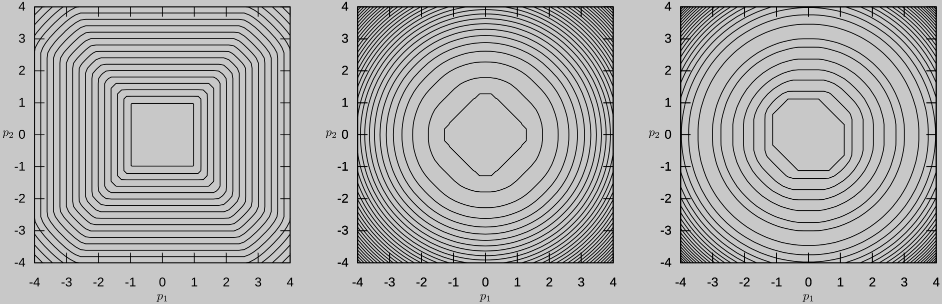
Extension of the plateau $\mathcal{P}_{\overline{H}}$: p_c vs q



max at $q^* = 2.865$ with $p_c(q^*) = 1.298$

q-power Hamiltonians in 2D

Effective Hamiltonian level sets for $p \in [-4, 4]^2$



(a) $q = 1$ and V_a (b) $q = 3$ and V_b (c) $q = 5$ and V_c

Test	N	N_p	Av.CPU/p (secs)	Av.Its/p	Tot.CPU (secs)
(a)	25^2	51^2	0.6	28	1633.12
(b)	25^2	51^2	0.2	9	522.37
(c)	25^2	51^2	0.4	18	1042.18

Non-convex Hamiltonians

$$\frac{1}{2}(|Du+p|^2-1)^2 - V(x) = \lambda \qquad \text{in } \mathbb{T}^n,$$

where $p \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and V is a 1-periodic potential.

A formula for the effective Hamiltonian is still available in dimension $n = 1$:

$$\overline{H}(p) = \begin{cases} -\min V & \text{if } |p| \le p_c \\ \lambda & \text{if } |p| > p_c \quad \text{s.t. } |p| = \int_0^1 \sqrt{1+\sqrt{2(V(s)+\lambda)}}ds \end{cases}$$

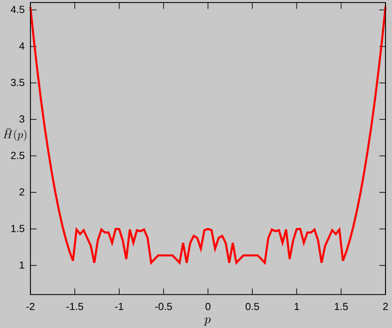
where $p_c = \int_0^1 \sqrt{1+\sqrt{2(V(s)-\min V)}}ds$.

$\overline{H}$ has a plateau in the whole interval $\mathcal{P}_{\overline{H}} = [-p_c, p_c]$.

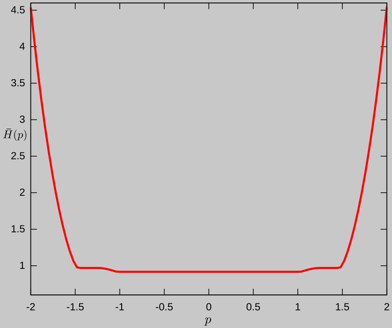
Non-convex Hamiltonians in 1D

$V(x) = \sin(2\pi x)$

Effective Hamiltonian for $p \in [-2, 2]$



(a) Engquist-Osher



(b) Lax-Friedrichs

Test	N	N_p	Av.Its/p	Tot.CPU (secs)
(a)	100	100	38	3.15
(b)	100	100	126	8.97

Second order Hamiltonians

$$H(x, p, D^2u + s) = \lambda \qquad \text{in } \mathbb{T}^n,$$

where $p \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $s \in \mathcal{S}^n$ (symmetric $n \times n$ matrices).

Assuming H continuous and uniformly elliptic, there exists a unique $\lambda = \bar{H}(p, s)$ and a unique (up to a constant) u such that the cell problem admits a viscosity solution.

A simple case in dimension one:

$$-\alpha|D^2u + s|(D^2u + s) + \frac{1}{2}|p|^2 - V(x) = \lambda \qquad \text{in } \mathbb{T}^n,$$

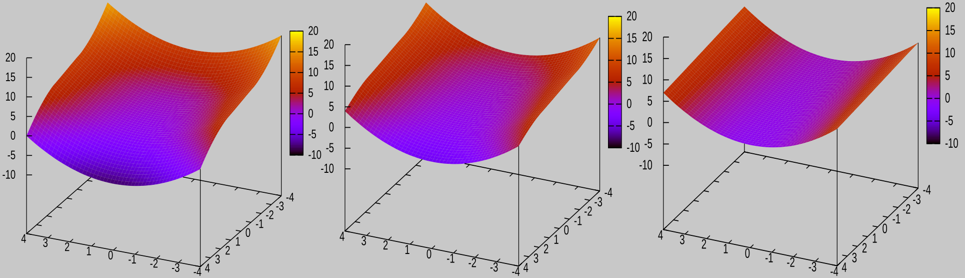
where $p, s \in \mathbb{R}$, $\alpha > 0$ and V is a 1-periodic potential.

Again, the singularity of the derivative of $|\cdot|$ is handled in a nonsmooth-Newton fashion.

Second order Hamiltonians in 1D

$V(x) = \sin(2\pi x)$

Effective Hamiltonian surface for $(p, s) \in [-4, 4]^2$



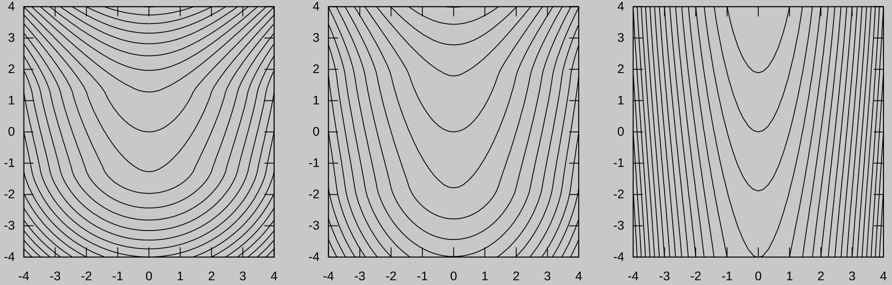
(*a*) $\alpha = 1$ (*b*) $\alpha = 1/2$ (*c*) $\alpha = 1/10$

Test	N	$N_{p,s}$	Av.CPU/(p,s) (secs)	Av.Its/(p,s)	Tot.CPU (secs)
(a)	100	51^2	0.009	7	25.29
(b)	100	51^2	0.009	8	25.26
(c)	100	51^2	0.011	10	30.78

Second order Hamiltonians in 1D

$V(x) = \sin(2\pi x)$

Effective Hamiltonian level sets for $(p, s) \in [-4, 4]^2$



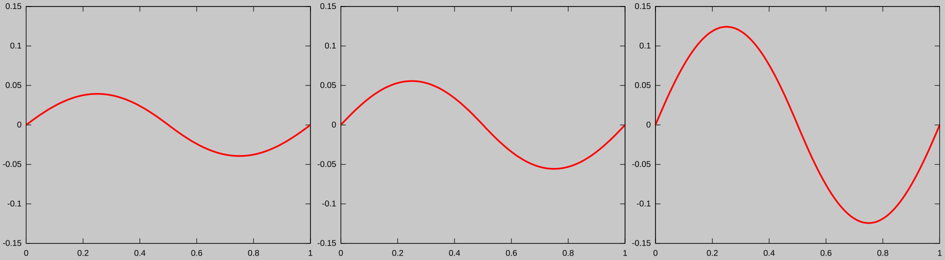
$(a)\ \alpha = 1$ $(b)\ \alpha = 1/2$ $(c)\ \alpha = 1/10$

Test	N	$N_{p,s}$	Av.CPU/(p,s) (secs)	Av.Its/(p,s)	Tot.CPU (secs)
(a)	100	51^2	0.009	7	25.29
(b)	100	51^2	0.009	8	25.26
(c)	100	51^2	0.011	10	30.78

Second order Hamiltonians in 1D

$V(x) = \sin(2\pi x)$

Correctors for $(p, s) = (0, 0)$



$(a)\ \alpha = 1$ $(b)\ \alpha = 1/2$ $(c)\ \alpha = 1/10$

Test	N	$N_{p,s}$	Av.CPU/(p, s) (secs)	Av.Its/(p, s)	Tot.CPU (secs)
(a)	100	51^2	0.009	7	25.29
(b)	100	51^2	0.009	8	25.26
(c)	100	51^2	0.011	10	30.78

Weakly coupled systems

$$H_i(x, Du_i + p) + C(x)u = \lambda \qquad \text{in } \mathbb{T}^n, \quad i = 1 \dots, M,$$

where $p \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$, $u = (u_1, \dots, u_M)$ and $C(x) = \{C_{ij}(x)\}_{i,j} \in \mathbb{R}^{M \times M}$.

Assuming the Hamiltonians H_i continuous and coercive, and the coupling matrix C continuous, irreducible and such that

$$C_{ij}(x) \leq 0 \text{ for } j \neq i, \qquad \sum_{j=1}^{M} C_{ij}(x) = 0, \quad i = 1, \dots, M,$$

there exists a unique λ such that the system admits a viscosity solution.

A simple case of two weakly coupled Eikonal Hamiltonians in $\mathbb{T}^n$ ($n = 1, 2$)

$$\begin{cases} \frac{1}{2}|Du_1 + p|^2 - V_1(x) + c_1(x)(u_1 - u_2) = \lambda \\ \frac{1}{2}|Du_2 + p|^2 - V_2(x) + c_2(x)(u_2 - u_1) = \lambda \end{cases}$$

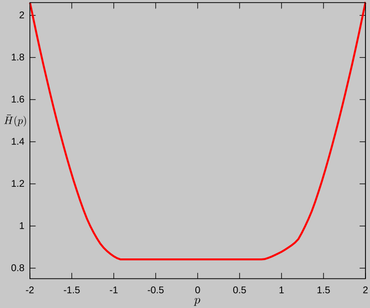
with V_1, V_2 1-periodic and c_1, c_2 nonnegative 1-periodic.

Weakly coupled systems in 1D

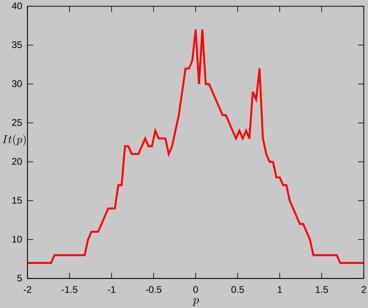
$$V_1(x) = \sin(2\pi x) \qquad V_2(x) = \cos(2\pi x)$$
$$c_1(x) = 1 - \cos(4\pi x) \qquad c_2(x) = 1 + \sin(4\pi x)$$

$$\mathcal{P}_{\overline{H}} = \{\overline{H}(p) = 0.8417\} = [-0.925, 0.788] \subset [-1.29, 1.36] =: \mathcal{I}$$

Effective Hamiltonian and number of iterations for $p \in [-2, 2]$



$\overline{H}(p)$



Iterations(p)

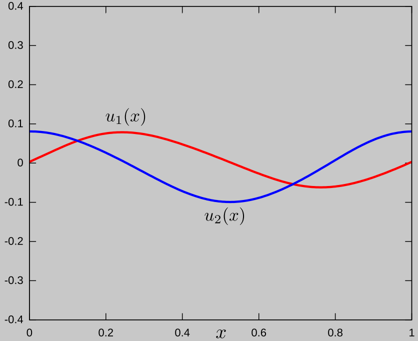
N	N_p	Av.CPU/p (secs)	Av.Its/p	Tot.CPU (secs)
2×100	100	0.03	17	3.19

Weakly coupled systems in 1D

$$V_1(x) = \sin(2\pi x) \qquad V_2(x) = \cos(2\pi x)$$
$$c_1(x) = 1 - \cos(4\pi x) \qquad c_2(x) = 1 + \sin(4\pi x)$$

$$\mathcal{P}_{\overline{H}} = \{\overline{H}(p) = 0.8417\} = [-0.925, 0.788] \subset [-1.29, 1.36] =: \mathcal{I}$$

Correctors for $p = -2$



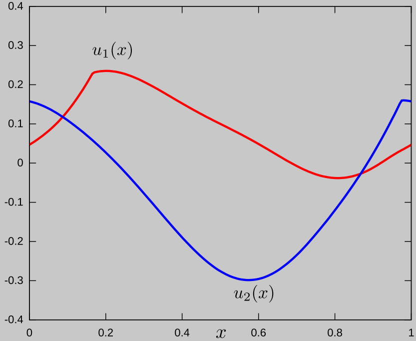
N	N_p	Av.CPU/p (secs)	Av.Its/p	Tot.CPU (secs)
2×100	100	0.03	17	3.19

Weakly coupled systems in 1D

$$V_1(x) = \sin(2\pi x) \qquad V_2(x) = \cos(2\pi x)$$
$$c_1(x) = 1 - \cos(4\pi x) \qquad c_2(x) = 1 + \sin(4\pi x)$$

$$\mathcal{P}_{\overline{H}} = \{\overline{H}(p) = 0.8417\} = [-0.925, 0.788] \subset [-1.29, 1.36] =: \mathcal{I}$$

Correctors for $p = -1$



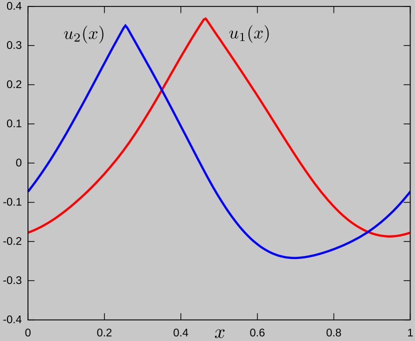
N	N_p	Av.CPU/p (secs)	Av.Its/p	Tot.CPU (secs)
2×100	100	0.03	17	3.19

Weakly coupled systems in 1D

$$V_1(x) = \sin(2\pi x) \qquad V_2(x) = \cos(2\pi x)$$
$$c_1(x) = 1 - \cos(4\pi x) \qquad c_2(x) = 1 + \sin(4\pi x)$$

$$\mathcal{P}_{\overline{H}} = \{\overline{H}(p) = 0.8417\} = [-0.925, 0.788] \subset [-1.29, 1.36] =: \mathcal{I}$$

Correctors for $p = 0$



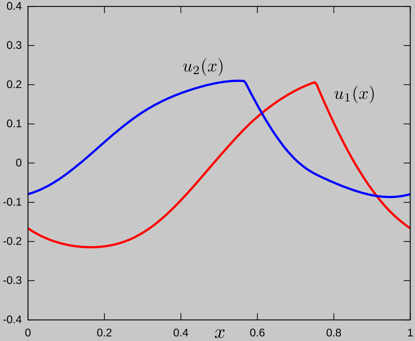
N	N_p	Av.CPU/p (secs)	Av.Its/p	Tot.CPU (secs)
2×100	100	0.03	17	3.19

Weakly coupled systems in 1D

$$V_1(x) = \sin(2\pi x) \qquad V_2(x) = \cos(2\pi x)$$
$$c_1(x) = 1 - \cos(4\pi x) \qquad c_2(x) = 1 + \sin(4\pi x)$$

$$\mathcal{P}_{\overline{H}} = \{\overline{H}(p) = 0.8417\} = [-0.925, 0.788] \subset [-1.29, 1.36] =: \mathcal{I}$$

Correctors for $p = 1$



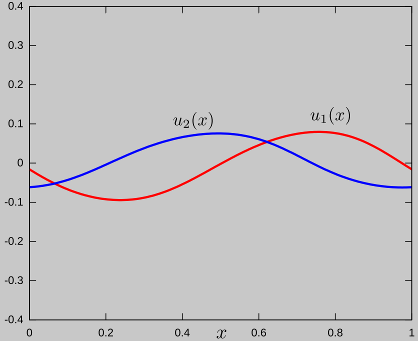
N	N_p	Av.CPU/p (secs)	Av.Its/p	Tot.CPU (secs)
2×100	100	0.03	17	3.19

Weakly coupled systems in 1D

$$V_1(x) = \sin(2\pi x) \qquad V_2(x) = \cos(2\pi x)$$
$$c_1(x) = 1 - \cos(4\pi x) \qquad c_2(x) = 1 + \sin(4\pi x)$$

$$\mathcal{P}_{\overline{H}} = \{\overline{H}(p) = 0.8417\} = [-0.925, 0.788] \subset [-1.29, 1.36] =: \mathcal{I}$$

Correctors for $p = 2$

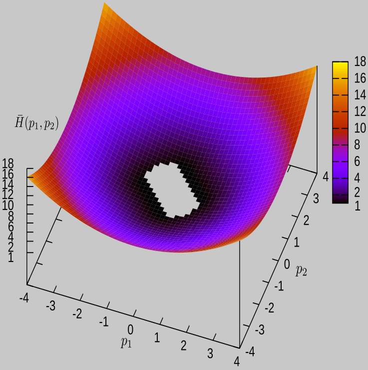


N	N_p	Av.CPU/p (secs)	Av.Its/p	Tot.CPU (secs)
2×100	100	0.03	17	3.19

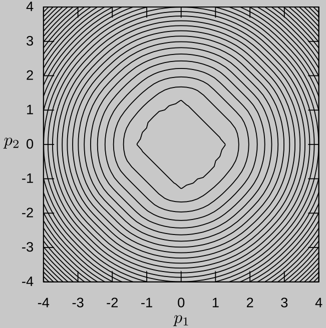
Weakly coupled systems in 2D

$$V_1(x_1,x_2) = \sin(2\pi x_1)\sin(2\pi x_2) \quad V_2(x_1,x_2) = \cos(2\pi x_1)\cos(2\pi x_2)$$
$$c_1(x_1,x_2) = 1 - \cos(4\pi x_1)\cos(4\pi x_2) \quad c_2(x_1,x_2) = 1 + \sin(4\pi x_1)\sin(4\pi x_2)$$

Effective Hamiltonian for $p \in [-4,4]^2$



Surface



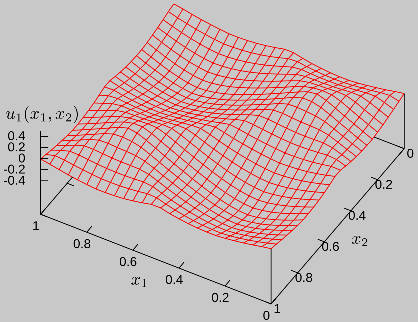
Level sets

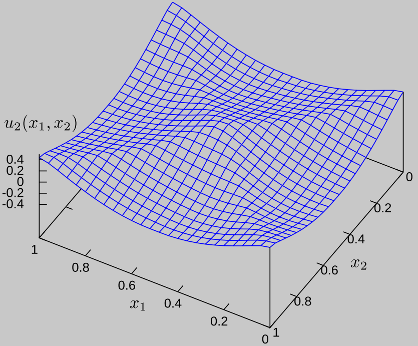
N	N_p	Av.CPU/p (secs)	Av.Its/p	Tot.CPU (secs)
2×25^2	51^2	1.39	12	3640.38

Weakly coupled systems in 2D

$$V_1(x_1,x_2) = \sin(2\pi x_1)\sin(2\pi x_2) \quad V_2(x_1,x_2) = \cos(2\pi x_1)\cos(2\pi x_2)$$
$$c_1(x_1,x_2) = 1 - \cos(4\pi x_1)\cos(4\pi x_2) \quad c_2(x_1,x_2) = 1 + \sin(4\pi x_1)\sin(4\pi x_2)$$

Correctors for $p = (0,0)$



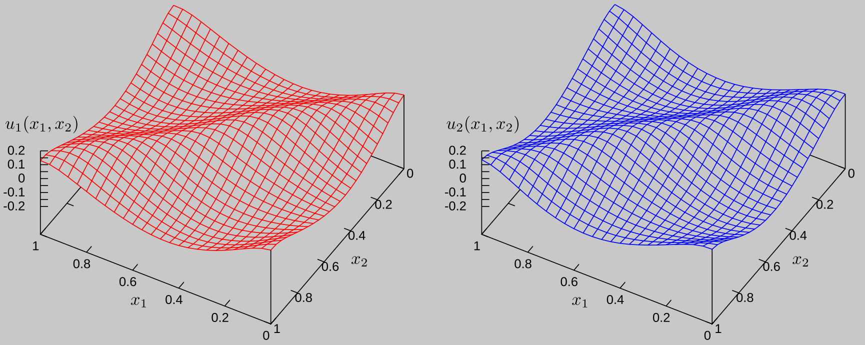


N	N_p	Av.CPU/p (secs)	Av.Its/p	Tot.CPU (secs)
2×25^2	51^2	1.39	12	3640.38

Weakly coupled systems in 2D

$$V_1(x_1, x_2) = \sin(2\pi x_1)\sin(2\pi x_2) \quad V_2(x_1, x_2) = \cos(2\pi x_1)\cos(2\pi x_2)$$
$$c_1(x_1, x_2) = 1 - \cos(4\pi x_1)\cos(4\pi x_2) \quad c_2(x_1, x_2) = 1 + \sin(4\pi x_1)\sin(4\pi x_2)$$

Correctors for $p = (2, 2)$



N	N_p	Av.CPU/p (secs)	Av.Its/p	Tot.CPU (secs)
2×25^2	51^2	1.39	12	3640.38

Dislocation dynamics

Dislocations: line defects in the lattice structure of crystals, responsible for the plastic properties of the materials.

Cell problem for a nonlocal Hamilton-Jacobi equation in dimension one: find $\lambda \in \mathbb{R}$ such that

$$c_p[u]|Du + p| = \lambda \qquad \text{in } \mathbb{T}^1$$

admits a bounded and 1-periodic viscosity solution u, where

- $c_p[u] = (c(x) + L + M_p[u])$
- $p \in \mathbb{R}$ is the density of dislocations, represented by the integer level sets of $u(x) + px$ (particle points, looking at a cross section of a slip plane).
- c is a 1-periodic potential acting as an obstacle to the motion.
- $L \in \mathbb{R}$ is a constant external stress.
- $M_p[u]$ is a nonlocal operator describing interactions between dislocations.

Dislocation dynamics

The nonlocal interaction operator is given by

$$M_p[u](x) = \int_{\mathbb{R}} \mathcal{J}(z) \left\{ E\left(u(x+z) - u(x) + pz\right) - pz \right\} dz\,,$$

where $\mathcal{J} : \mathbb{R} \to \mathbb{R}^+$ is a nonnegative kernel satisfying

$$\mathcal{J}(-z) = \mathcal{J}(z) \quad \forall z \in \mathbb{R}\,, \qquad \mathcal{J}(z) \sim \frac{1}{z^2} \quad \text{for } |z| \gg 1$$

and $E : \mathbb{R} \to \mathbb{R}$ is the (odd) integer part

$$E(\alpha) = \begin{cases} k & \text{if } \alpha = k \in \mathbb{Z}\,, \\ k + 1/2 & \text{if } k < \alpha < k+1\,, \quad k \in \mathbb{Z}\,. \end{cases}$$

Numerical approximation is simplified considering rational densities $p = P/Q$, for $P \in \mathbb{Z}$ and $Q \in \mathbb{N}$.

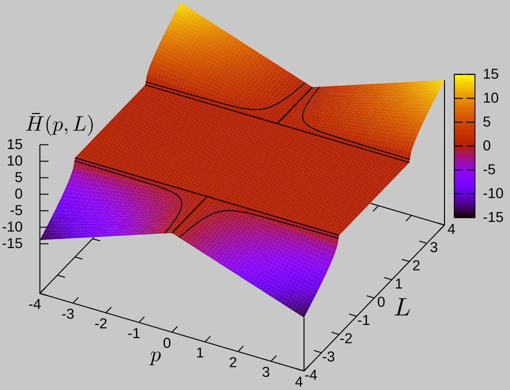
The integer part E is mollified around the jumps.

Engquist-Osher discretization of Du according to the sign of $c_p[u]$.

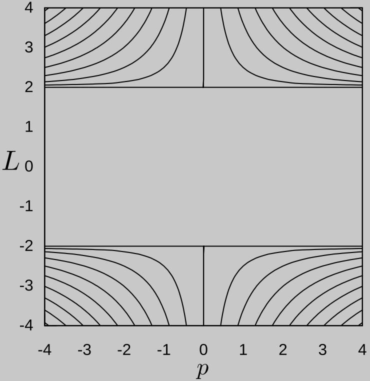
Dislocation dynamics in 1D

No interactions: $\mathcal{J} \equiv 0\,,\ c(x) = 2\sin(2\pi x)$

Effective Hamiltonian for $(p, L) \in [-4, 4]^2$



Surface



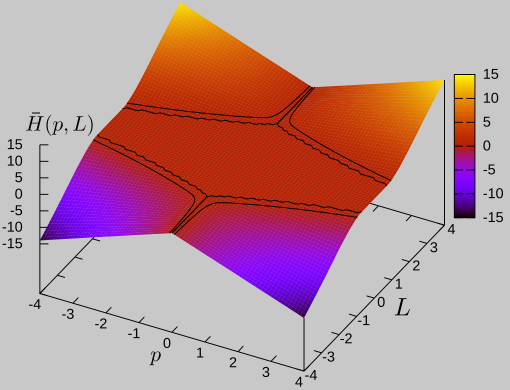
Level sets

Q	N	$N_{p,L}$	Av.CPU/(p, L) (secs)	Av.Its/(p, L)	Tot.CPU (secs)
10	100	81^2	0.017	5	115.71

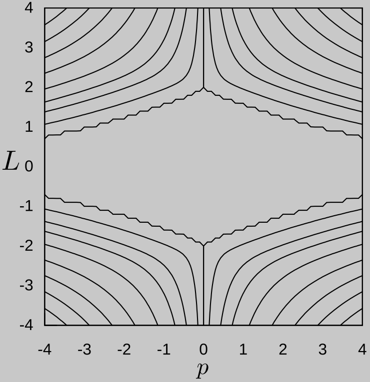
Dislocation dynamics in 1D

Regularization: $\mathcal{J}$ smooth, $E(\alpha) = \alpha$ (no jumps), $c(x) = 2\sin(2\pi x)$

Effective Hamiltonian for $(p, L) \in [-4, 4]^2$



Surface



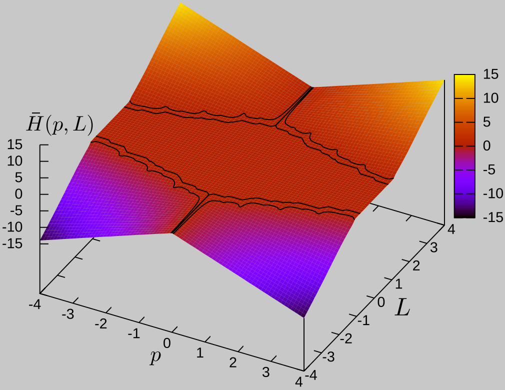
Level sets

Q	N	$N_{p,L}$	Av.CPU/(p, L) (secs)	Av.Its/(p, L)	Tot.CPU (secs)
10	100	81^2	0.032	7	215.74

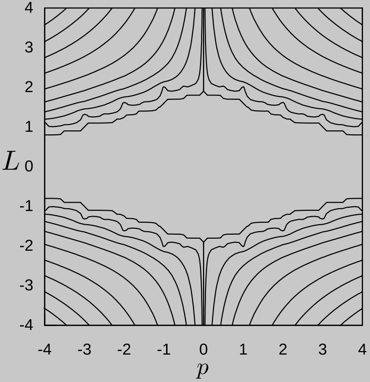
Dislocation dynamics in 1D

Complete: $\mathcal{J}(z) = C/z^2\,,\ c(x) = 2\sin(2\pi x)$

Effective Hamiltonian for $(p, L) \in [-4, 4]^2$



Surface



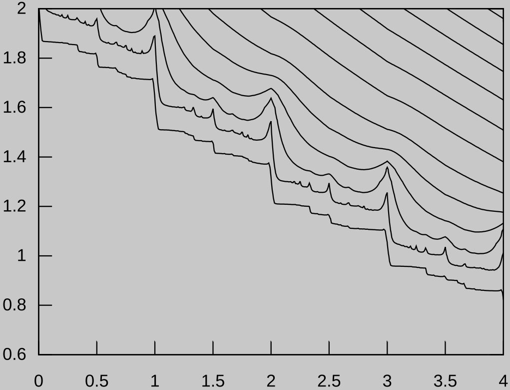
Level sets

Q	N	$N_{p,L}$	Av.CPU/(p, L) (secs)	Av.Its/(p, L)	Tot.CPU (secs)
10	100	81^2	0.115	16	759.31

Dislocation dynamics in 1D

Complete: $\mathcal{J}(z) = C/z^2$, $c(x) = 2\sin(2\pi x)$

Effective Hamiltonian for $(p, L) \in [0, 4] \times [0.6, 2]$



Level sets

Q	N	$N_{p,L}$	Av.CPU/(p, L) (secs)	Tot.CPU (secs)
100	100	401×28	1.02	10612

Stationary Mean Field Games

$$\begin{cases} -\nu\Delta u + H(x, Du) + \lambda = V[m] & x \in \mathbb{T}^n \\ \nu\Delta m + \operatorname{div}(m\, H_p(x, Du)) = 0 & x \in \mathbb{T}^n \\ \int_{\mathbb{T}^n} u(x)dx = 0, \int_{\mathbb{T}^n} m(x)dx = 1,\ m \geq 0\,. \end{cases}$$

Assuming $\nu > 0$, H smooth and convex, there exists a unique classical solution (u, m, λ).

A simple case for an Eikonal Hamiltonian in dimension two, with a cost function f and a local potential V:

$$\begin{cases} -\nu\Delta u + |Du|^2 + f(x) + \lambda = V(m) & x \in \mathbb{T}^2 \\ \nu\Delta m + 2\operatorname{div}(m\, Du) = 0 & x \in \mathbb{T}^2 \\ \int_{\mathbb{T}^2} u(x)dx = 0, \int_{\mathbb{T}^2} m(x)dx = 1,\ m \geq 0\,. \end{cases}$$

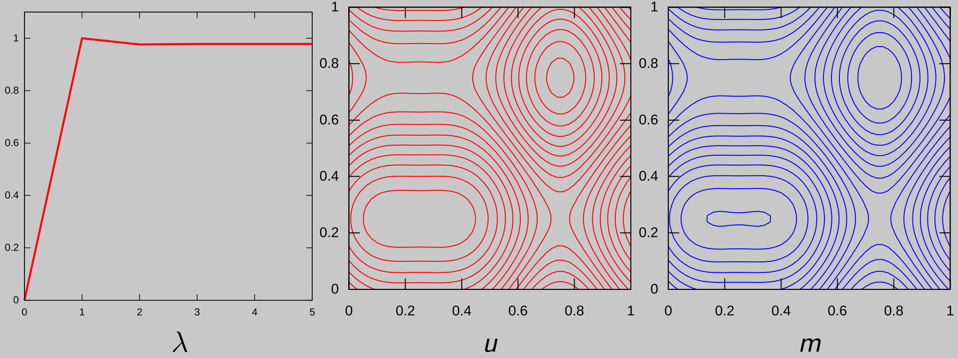
Overdetermined problem: $2N + 2$ equations in $2N + 1$ unknowns.

We do not impose the constraint $m \geq 0$: the normalization condition on m seems enough to force numerically its nonnegativity.

Stationary Mean Field Games in 2D

$$\nu = 1\,, \quad V(m) = m^2\,, \quad f(x) = \sin(2\pi x_1) + \cos(4\pi x_1) + \sin(2\pi x_2)$$

λ vs number of iterations and level sets of (u, m)

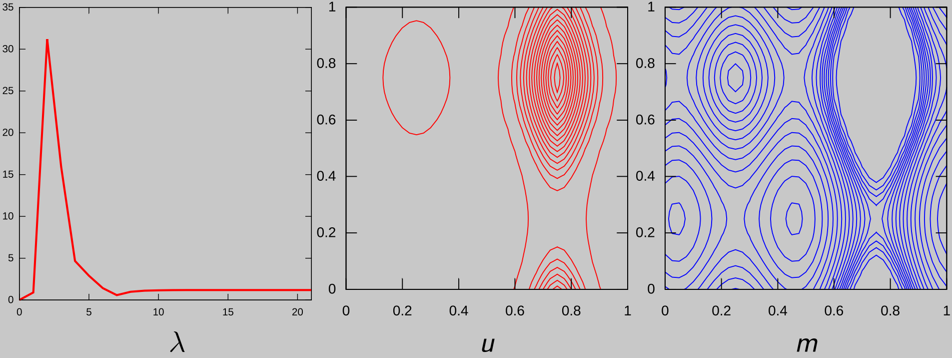


N	λ	Its	Tot.CPU (secs)
50^2	0.9784	5	8.06

Stationary Mean Field Games in 2D

$$\nu = 0.01\,, \quad V(m) = m^2\,, \quad f(x) = \sin(2\pi x_1) + \cos(4\pi x_1) + \sin(2\pi x_2)$$

λ vs number of iterations and level sets of (u, m)

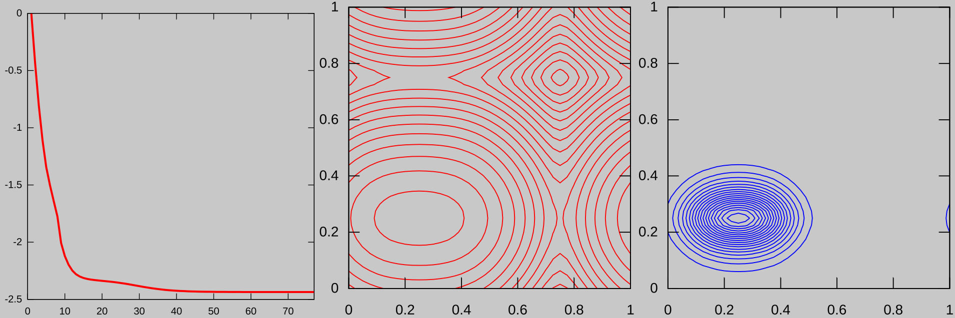


N	λ	Its	Tot.CPU (secs)
50^2	1.1878	21	10.72

Stationary Mean Field Games in 2D

$$\nu = 0.1\,, \quad V(m) = -\log(m)\,, \quad f(x) = \sin(2\pi x_1) + \cos(4\pi x_1) + \sin(2\pi x_2)$$

λ vs number of iterations and level sets of (u, m)



λ $\qquad$ u $\qquad$ m

N	λ	Its	Tot.CPU (secs)
50^2	-2.4358	77	42.33

Stationary multi-population Mean Field Games

A system of P Eikonal Hamiltonians in $\Omega = [0,1]^n$ for $n = 1, 2$ with a linear local potential V and Neumann boundary conditions:

$$\begin{cases} -\nu\Delta u_i + |Du_i|^2 + \lambda_i = V_i(m) & \text{in } \Omega\,, \quad i = 1, ..., P \\ \nu\Delta m_i + 2\mathrm{div}(m_i\, Du_i) = 0 & \text{in } \Omega\,, \quad i = 1, ..., P \\ \partial_n u_i = 0\,, \quad \partial_n m_i = 0 & \text{on } \partial\Omega\,, \quad i = 1, ..., P \\ \int_\Omega u_i(x)dx = 0\,, \quad \int_\Omega m_i(x)dx = 1\,, \quad m_i \geq 0 & i = 1, ..., P\,, \end{cases}$$

where $u = (u_1, ..., u_P)$, $m = (m_1, ..., m_P)$, $\lambda = (\lambda_1, ..., \lambda_P)$ and $V(m) = (V_1, ..., V_P)(m) = \Theta m$ with a weight matrix $\Theta = (\theta_{ij})_{i,j=1,...,P}$.

Existence and uniqueness of the trivial solution $u_i \equiv 0$, $m_i \equiv 1$, $\lambda_i = \displaystyle\sum_{j=1}^{P} \theta_{ij}$ (for $i = 1, ..., P$) can be proved assuming Θ positive semi-definite. We drop it and look for nontrivial solutions, choosing $\theta_{ij} = 1 - \delta_{ij}$.

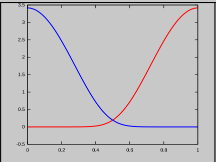
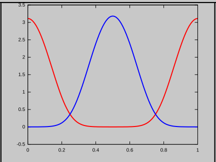
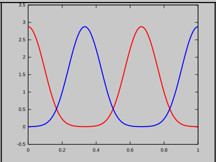
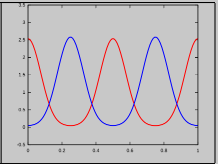
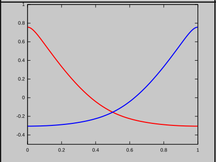
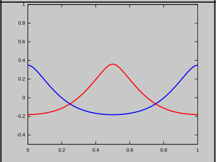
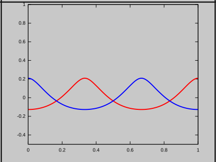
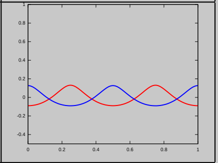
Overdetermined problem: $P(2N+2)$ equations in $P(2N+1)$ unknowns.

Again, we do not impose the constraint $m \geq 0$: the normalization condition on m seems enough to force numerically its nonnegativity.

Stationary multi-population Mean Field Games in 1D

$$P = 2\,, \quad \nu = 0.05\,, \quad V(m_1, m_2) = (m_2, m_1)$$

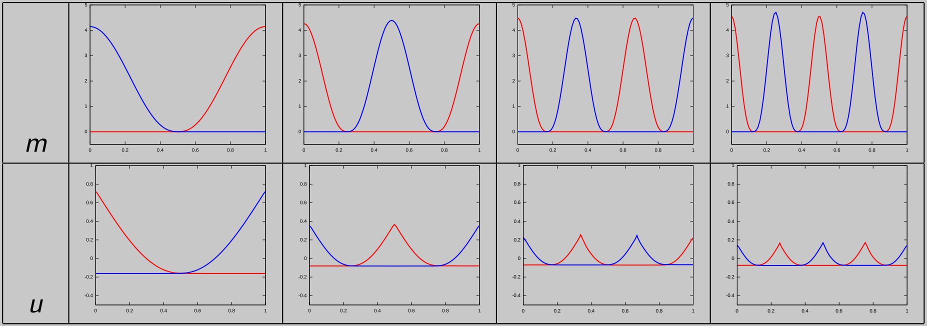
Two-population MFG solutions (u, m, λ)

m				
u				
λ	$(0.039, 0.039)$	$(0.143, 0.140)$	$(0.295, 0.295)$	$(0.496, 0.485)$
ITs	19	48	28	31
CPU	0.14	0.33	0.19	0.21

Stationary multi-population Mean Field Games in 1D

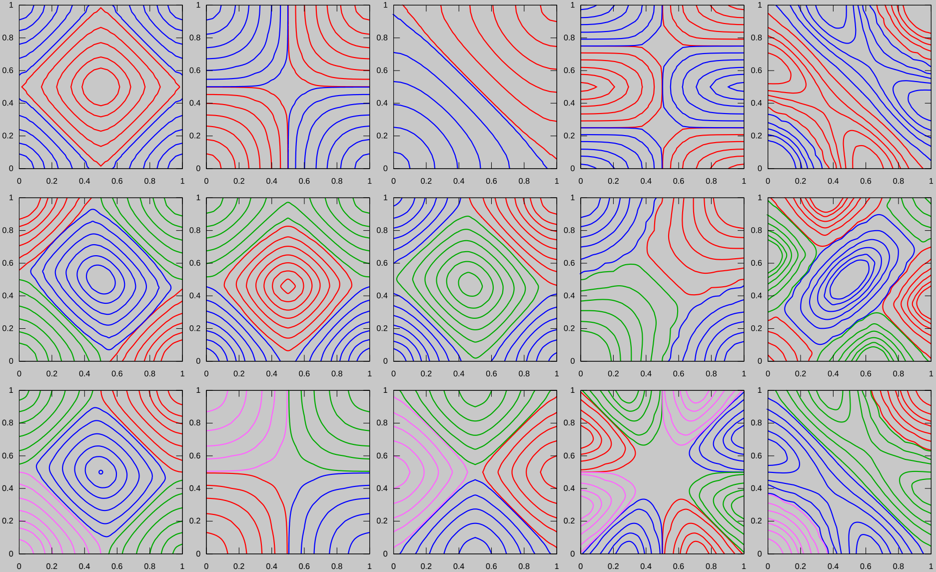
$$P = 2\,, \quad \nu = 10^{-4}\,, \quad V(m_1, m_2) = (m_2, m_1)$$

Two-population MFG solutions (u, m, λ)



Stationary multi-population Mean Field Games in 2D

P-population MFG solutions m for $\nu = 10^{-4}$ and $P = 2, 3, 4$



Stationary Mean Field Games on Networks

Network: a connected set $\Gamma = (\mathcal{V}, \mathcal{E})$ with

- Vertices $\mathcal{V} := \{v_i\}_{i \in I}$
- Edges $\mathcal{E} := \{e_j\}_{j \in J}$
- Incident edges to vertex $Inc_i := \{j \in J : v_i \in e_j\}$

$$
\begin{cases}
-\nu_j \partial^2 u + H_j(x, \partial u) + \lambda = V[m] & x \in e_j \qquad (HJ) \\
\nu_j \partial^2 m + \partial(m\, \partial_p H_j(x, \partial u)) = 0 & x \in e_j \qquad (FP) \\
\displaystyle\sum_{j \in Inc_i} \nu_j \partial_j u(v_i) = 0 & v_i \in \mathcal{V} \qquad (K) \\
\displaystyle\sum_{j \in Inc_i} [\nu_j \partial_j m(v_i) + \partial_p H_j(v_i, \partial_j u) m_j(v_i)] = 0 & v_i \in \mathcal{V} \\
\displaystyle\int_\Gamma u(x)dx = 0, \quad \int_\Gamma m(x)dx = 1, \qquad m \geq 0 &
\end{cases}
$$

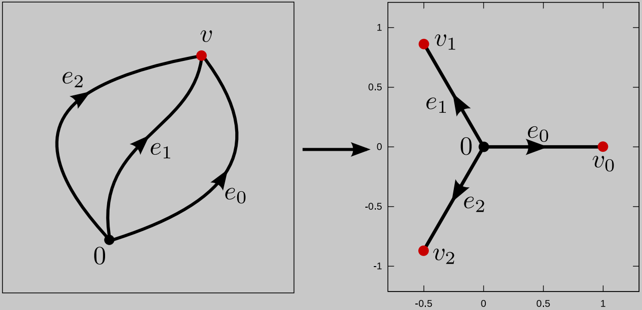
Kirchhoff transition condition and total flux conservation.

Assuming $\nu_j > 0$, H_j smooth and convex, V suitably monotone, there exists a unique classical solution (u, m, λ).

Stationary Mean Field Games on Networks

A network with 2 vertices and 3 edges mapped in an equivalent network with boundary vertices identified.

Each edge has unit length and connects $(0, 0)$ to $(cos(2\pi j/3), sin(2\pi j/3))$ with $j = 0, 1, 2$.

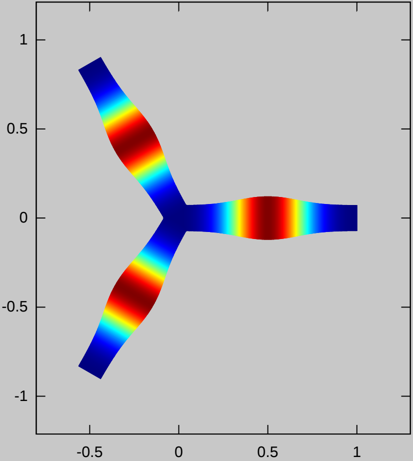


$$H_j(x, p) = \frac{1}{2}|p|^2 + f(x), \qquad f(x) = s_j\left(1 + \cos(2\pi\left(x + \tfrac{1}{2}\right))\right),\ s_j \in \{0, 1\}$$

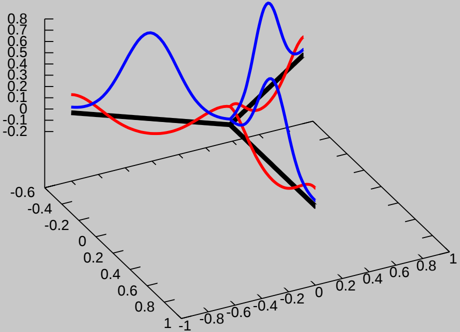
$$V[m] = m^2, \qquad \nu_j \equiv \nu$$

Stationary Mean Field Games on Networks

$\nu = 0.1, \quad s_0 = 1,\ s_1 = 1,\ s_2 = 1, \quad \lambda \sim -1.066667$



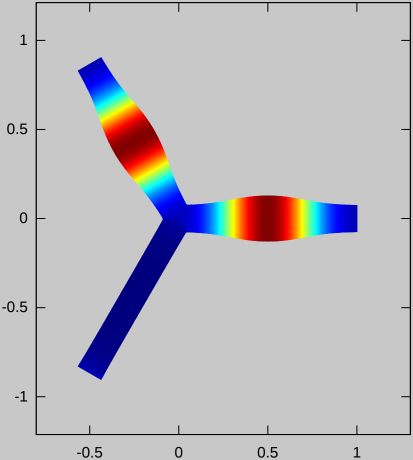
m



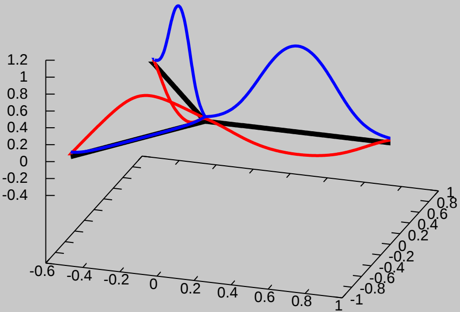
m (blue) and u (red)

Stationary Mean Field Games on Networks

$\nu = 0.1, \quad s_0 = 1,\ s_1 = 1,\ s_2 = 0, \quad \lambda \sim -0.741639$



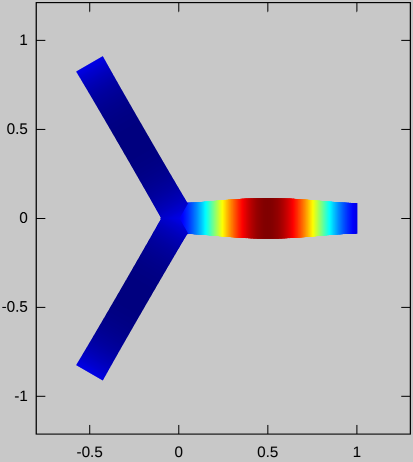
m



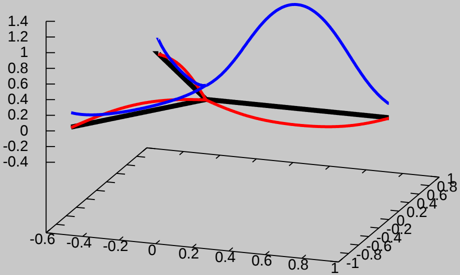
m (blue) and *u* (red)

Stationary Mean Field Games on Networks

$\nu = 0.1, \quad s_0 = 1,\ s_1 = 0,\ s_2 = 0, \quad \lambda \sim -0.116733$



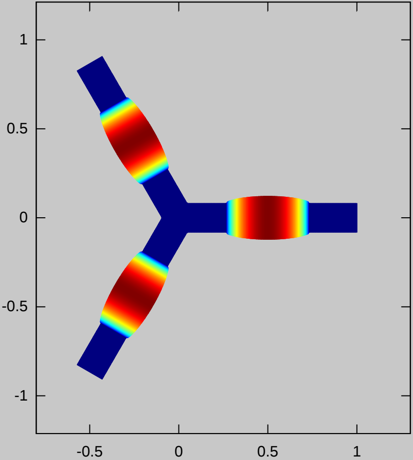
m



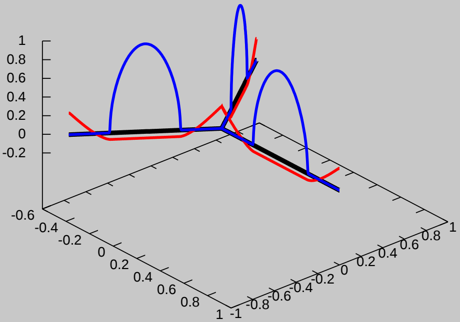
m (blue) and *u* (red)

Stationary Mean Field Games on Networks

$\nu = 10^{-4}, \quad s_0 = 1,\ s_1 = 1,\ s_2 = 1, \quad \lambda \sim -1.116603$



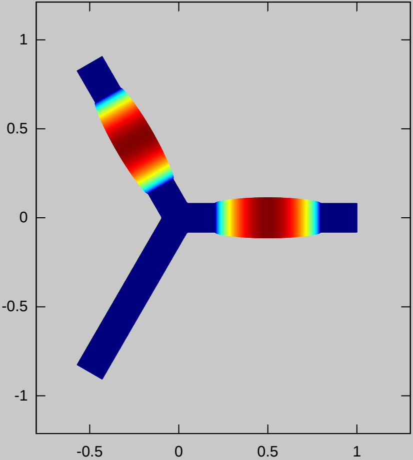
m



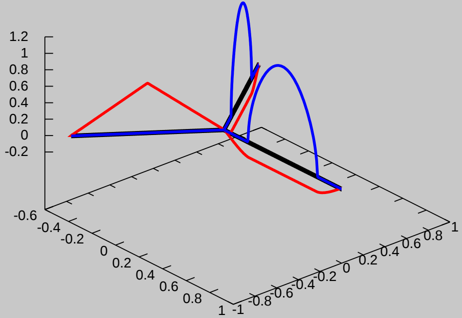
m (blue) and *u* (red)

Stationary Mean Field Games on Networks

$\nu = 10^{-4}, \quad s_0 = 1,\ s_1 = 1,\ s_2 = 0, \quad \lambda \sim -0.725463$



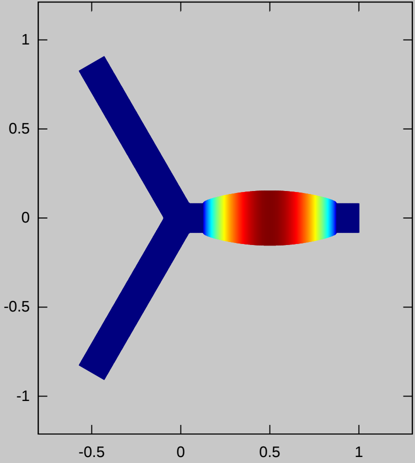
m



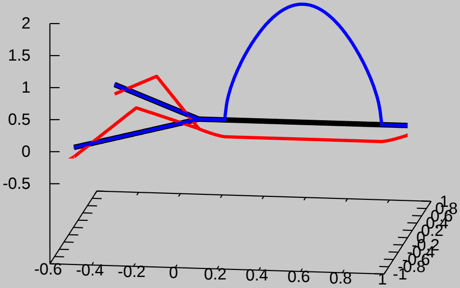
m (blue) and u (red)

Stationary Mean Field Games on Networks

$\nu = 10^{-4}, \quad s_0 = 1,\ s_1 = 0,\ s_2 = 0, \quad \lambda \sim -0.002345$

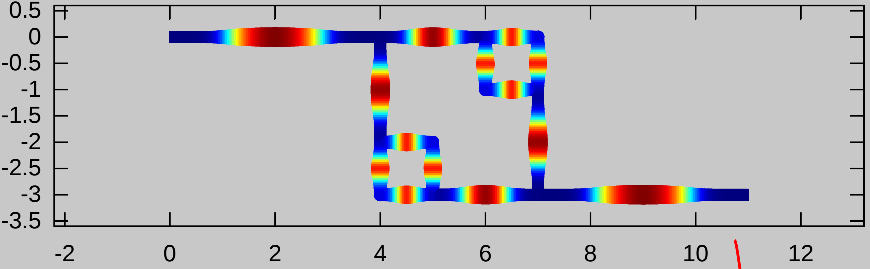


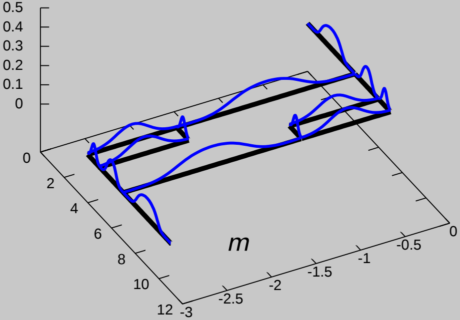
m

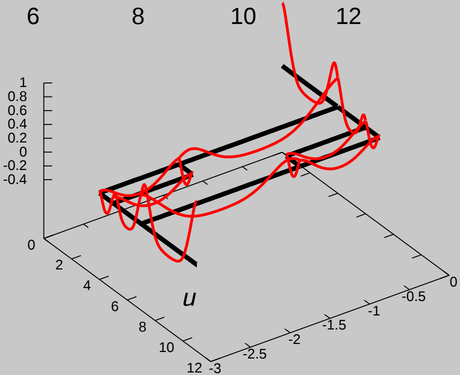


m (blue) and u (red)

Stationary Mean Field Games on Networks







Homogenization of Mean Field Games with Small Noise

$$\begin{cases} -u_t^\varepsilon - \varepsilon\Delta u^\varepsilon + \frac{1}{2}a(\frac{x}{\varepsilon})|Du^\varepsilon|^2 = V(\frac{x}{\varepsilon}, m^\varepsilon) & x \in \mathbb{R}^n \times (0, T) \\ m_t^\varepsilon - \varepsilon\Delta m^\varepsilon - \operatorname{div}(a(\frac{x}{\varepsilon})m^\varepsilon Du^\varepsilon) = 0 & x \in \mathbb{R}^n \times (0, T) \\ u^\varepsilon(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n \\ m^\varepsilon(\cdot, 0) = m_0 & \text{in } \mathbb{R}^n \\ \int_{\mathbb{R}^n} u^\varepsilon(x, \cdot)dx = 0, \int_{\mathbb{R}^n} m^\varepsilon(x, \cdot)dx = 1,\ m^\varepsilon \geq 0 & t \in [0, T] \end{cases}$$

$a : \mathbb{R}^n \to (0, +\infty)$ is 1-periodic Lipschitz and $V : \mathbb{R}^n \times [0, +\infty) \to \mathbb{R}$ is 1-periodic Lipschitz with $V(y, \cdot)$ nondecreasing for each y

e.g. $V(y, m) = v(y) + m^q$ or $V(y, m) = v(y) + \log m$

The viscosity solution $(u^\varepsilon, m^\varepsilon)$ converges, as $\varepsilon \to 0$, to the viscosity solution (u, m) of the **Effective Mean Field Game (?)**

$$\begin{cases} -u_t + \overline{H}(Du, m) = 0 & x \in \mathbb{R}^n \times (0, T) \\ m_t - \operatorname{div}(m\overline{b}(Du, m) = 0 & x \in \mathbb{R}^n \times (0, T) \\ u(\cdot, 0) = u_0 & \text{in } \mathbb{R}^n \\ m(\cdot, 0) = m_0 & \text{in } \mathbb{R}^n \\ \int_{\mathbb{R}^n} u(x, \cdot)dx = 0, \int_{\mathbb{R}^n} m(x, \cdot)dx = 1,\ m \geq 0 & t \in [0, T] \end{cases}$$

Homogenization of Mean Field Games with Small Noise

For every $p \in \mathbb{R}^n$ and $\alpha \geq 0$ there exists a unique value $\bar{H}$ for which there exists a solution on $\mathbb{T}^n$ to the

Ergodic Mean Field Game: Effective Hamiltonian

$$\begin{cases} -\Delta u + \frac{1}{2}a(y)|\nabla u + p|^2 - V(y,\alpha m) = \bar{H}(P,\alpha) & x \in \mathbb{T}^n \\ -\Delta m - \operatorname{div}(a(y)m\nabla u) = 0 & x \in \mathbb{T}^n \\ \int_{\mathbb{T}^n} u(x)dx = 0, \int_{\mathbb{T}^n} m(x)dx = 1, \; m \geq 0 & \end{cases}$$

Effective Drift

$$\bar{b}(P,\alpha) := \int_{\mathbb{T}^n} a(y)m(\nabla u + P)dy$$

Mean Field Game structure is lost due to a

Strange term coming from nowhere!

$$D_p\bar{H}(p,\alpha) = \bar{b}(p,\alpha) - \alpha \int_{\mathbb{T}^n} V_m(y,\alpha m)\tilde{m}\, m dy$$

Homogenization of Mean Field Games with Small Noise

For $i = 1, \ldots, n$ the triplet $\big(\tilde{u}_i, \tilde{m}_i, D_{p_i}\overline{H}(p,\alpha)\big)$ is the solution of the

Auxiliary Ergodic Linear Problem in p

$$\begin{cases} -\Delta \tilde{u}_i + a\nabla \tilde{u}_i \cdot (\nabla u + p) + a(\nabla u + p)\cdot e_i - V_m(y,\alpha m)\alpha \tilde{m}_i = D_{p_i}\overline{H}(p,\alpha) \\ -\Delta \tilde{m}_i - \operatorname{div}\big(a(p+\nabla u)\tilde{m}_i\big) = \operatorname{div}(am(\nabla \tilde{u}_i + e_i)) \\ \int_{\mathbb{T}^n} \tilde{m}_i = \int_{\mathbb{T}^n} \tilde{u}_i = 0 \end{cases}$$

Similarly $\big(\bar{u}, \bar{m}, D_\alpha \overline{H}(p,\alpha)\big)$ is the solution of the

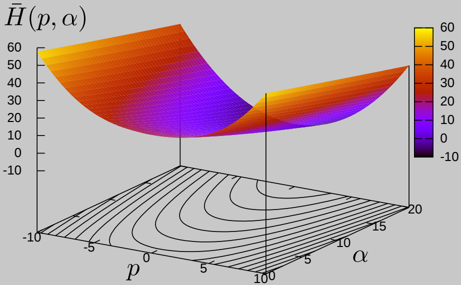
Auxiliary Ergodic Linear Problem in α

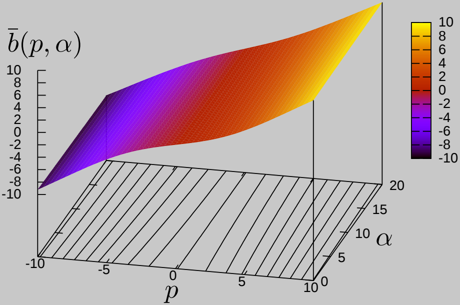
$$\begin{cases} -\Delta \bar{u} + a(y)\nabla \bar{u}\cdot(\nabla u + p) - V_m(y,\alpha m)\alpha \bar{m} - V_m(y,\alpha m)m = D_\alpha \overline{H}(p,\alpha) \\ -\Delta \bar{m} - \operatorname{div}\big(a(y)(p+\nabla u)\bar{m}\big) - \operatorname{div}(a(y)m\nabla \bar{u}) = 0 \\ \int_{\mathbb{T}^n} \bar{m} = \int_{\mathbb{T}^n} \bar{u} = 0 \end{cases}$$

$$D_\alpha \overline{H}(p,\alpha) = -\int_{\mathbb{T}^n} \left[V_m(y,\alpha m)(m+\alpha\bar{m})^2 + \alpha a(y) m |\nabla \bar{u}|^2\right] dy$$

Homogenization of Mean Field Games with Small Noise

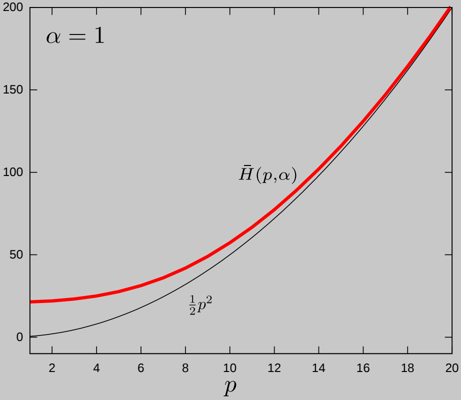
The 1D case $a \equiv 1$ and $V(x, m) = 1 + \sin(2\pi x) + m$

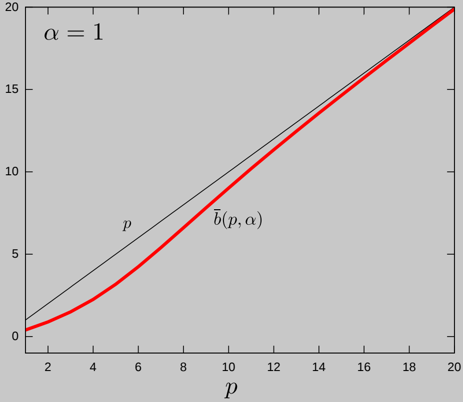




Homogenization of Mean Field Games with Small Noise

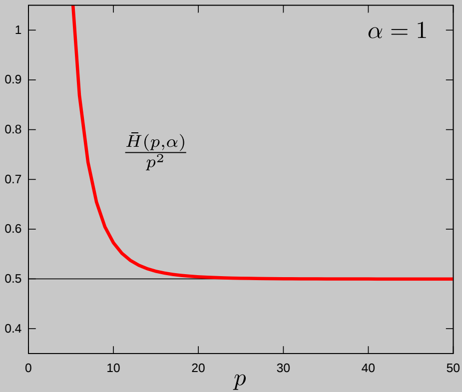
The 1D case $a \equiv 1$ and $V(x, m) = 1 + \sin(2\pi x) + m$

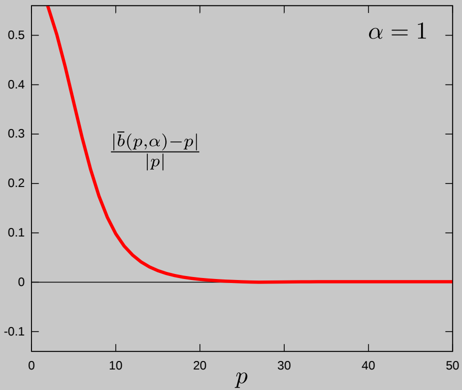




Homogenization of Mean Field Games with Small Noise

The 1D case $a \equiv 1$ and $V(x, m) = 1 + \sin(2\pi x) + m$





THANKYOU