The principal eigenvalue for non-variational operators

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Introduction

- the notion of **principal eigenvalue** for linear second order partial differential operators in general (i.e. non-variational form) and the min-max representation formula [according to Donsker-Varadhan, Berestycki-Nirenberg-Varadhan]
- the general **fully nonlinear degenerate elliptic** case
- the **positivity of a generalized principal eigenvalue** is equivalent to the validity of the **weak Maximum Principle**
- **approximation of the principal eigenvalue** in the non-variational case

I will report in particular on :

H. Berestycki, A. Porretta, L. Rossi, ICD, Maximum Principle and generalized principal eigenvalue for degenerate elliptic operators, JMPA 2014

I. Birindelli, F. Camilli, ICD, On the approximation of the principal eigenvalue for a class of nonlinear elliptic operators, Comm. in Math. Sciences 2016

One motivating example: principal eigenvalue and ergodic control

Consider the viscous Hamilton-Jacobi equation, i.e. the Bellman equation satisfied by **the value function** of an infinite horizon stochastic **discounted** optimal control problem with running cost V

$$-\frac{1}{2}\triangle u_\alpha + \frac{1}{2}|\nabla u_\alpha|^2 - V(x) + \alpha u_\alpha = 0$$

where $\alpha > 0$ is the **discount parameter** and the **eigenvalue problem** for the **linear** Schrödinger type equation

$$-\frac{1}{2}\triangle \Phi + V(x)\Phi = \lambda \Phi$$

One motivating example: principal eigenvalue and ergodic control

If $(\lambda_1, \Phi > 0)$ is the **principal eigenvalue-eigenfunction** pair it is easy to check that the function $w = -\log \Phi + \int \Phi^2 \log \Phi \, dx$ satisfies

$$-\frac{1}{2}\triangle w + \frac{1}{2}|\nabla w|^2 - V(x) + \lambda_1 = 0 \quad , \quad \int w \, \Phi^2 = 0$$

This is the Bellman equation of **ergodic control**.

It can be proved, under some conditions on V, that as $\alpha \to 0$

$$\alpha u_\alpha \to \lambda_1 \quad , \quad u_\alpha - \int u_\alpha \, \Phi^2 \, dx \to w$$

This approach to ergodic optimal control introduced by Lasry (1974), developed later by P.L. Lions (1985), Bensoussan-Frehse (1987), Bensoussan-Nagai (1991) and many other authors in various different and more general settings, e.g when

$$-\frac{1}{2}\triangle w + \frac{1}{2}|\nabla w|^2 - V(x)$$

is replaced by

$$F(D^2(w)) + H(x, Dw)$$

One motivating example: principal eigenvalue and ergodic control

Above discussion shows the relevance of the **principal eigenvalue** of second order operators in the Dynamic Programming approach to stochastic ergodic control.

Interesting issues:

- what if Δ is replaced by the infinitesimal generator of a more general diffusion process, i.e.

$$L[u] = \mathrm{Tr}(A(x)D^2u) + b(x) \cdot Du + c(x)u,$$

  with either $A(x) \geq cI$ with $c > 0$ or $A(x) \geq 0$?

- what if $L[u]$ is replaced by a degenerate elliptic fully nonlinear operator $F[u]$ of Bellman or Isaacs type?
- as recalled above, in some cases, the representation formula $\lambda_1 = \lim_{\alpha \to 0+} \alpha u_\alpha$ holds

  Different representations formulas for λ_1, perhaps more suitable to a **computational** approach?

The principal eigenvalue

The Berestycki,Nirenberg and Varadhan notion [Comm. Pure Appl. Math. 47 (1994)]:
associate to the **uniformly elliptic** operator in non-divergence form

$$L[u] = \mathrm{Tr}(A(x)D^2 u) + b(x)\cdot Du + c(x)u, \;\; A(x) \geq \alpha I$$

in a bounded domain Ω the number λ_1 defined by

$$\lambda_1 := \sup\{\lambda \in \mathbb{R} : \; \exists\, \phi > 0 \text{ in } \Omega \;\text{ such that } L[\phi] + \lambda\phi \leq 0 \quad \text{in } \Omega\}$$

In the definition of λ_1, $\phi \in W^{2,p}_{loc}(\Omega)$.

Remark.
Note that:

- $A(x)$ **uniformly positive definite**, *not necessarily* **symmetric**
- *presence of a* 1^{st}*-order drift term* b
- $A(x)$ *have just bounded and measurable entries*

The principal eigenvalue

Note that even for symmetric A the operator L is not in general **self-adjoint** due to the presence of the drift term b.

BNV proved that the number λ_1 in the previous slide shares some of the properties of the classical **principal eigenvalue** for the Dirichlet problem, namely

- there exists a **principal eigenfunction** $w_1 > 0 \quad in\ \Omega$ such that $L[w_1] + \lambda_1 w_1 = 0 \quad in\ \Omega, \quad w_1 = 0 \quad on\ \partial\Omega$
- w_1 is simple
- $Re\lambda \geq \lambda_1$ for any other eigenvalue λ of L

The existence of an associated **positive, simple eigenfunction** follows from compactness estimates guaranteed by the Krein-Rutman theorem thanks to **uniform ellipticity** of L and **boundedness** of Ω

The principal eigenvalue

The Berestycki-Nirenberg-Varadhan definition above can be expressed by the equivalent **pointwise min-max** formula

$$\lambda_1 = - \inf_{\phi(x)>0} \sup_{x \in \Omega} \frac{L\phi(x)}{\phi(x)}$$

where $\phi \in W^{2,p}_{loc}(\Omega)$.

The same formula, under more restrictive conditions, was considered before by M.D. Donsker and S.R.S. Varadhan in their seminal paper "On the principal eigenvalue of second-order elliptic differential operators", Comm. Pure Appl . Math. 29, 1976.

In that same paper different equivalent representation formulas for λ_1 were also proposed in terms of the **average long run behavior** of the positive semigroup generated by L. More precisely,

$$\lambda_1 = - \lim_{t \to +\infty} \frac{1}{t} \log \sup_{x \in \Omega} \int_{\Omega} p(t,x,y)dy$$

where $p(t,x,y)$ is the positive density function associated to the semigroup generated by $-L$

Two classical references

A much older reference on the formula for the case of Laplacian:
Barta, J., Sur la vibration fondamentale d' une membrane, C. R. Acad. Sci. Paris 204, 1937, cited in Protter-Weinberger 1965.

Recall also the Collatz-Wielandt (Mathematische Zeitschrift 48 (1942)) **min-max** representation formula for the Perron-Frobenius eigenvalue of irreducible stochastic matrices:

$$\rho(A) = \max_{x \in C} \min_{1 \le i \le n\,,\, x_i \neq 0} \frac{(Ax)_i}{x_i}$$

where C is the cone $\{x \geq 0\ ,\ x \neq 0\}$.

This number and the corresponding eigenvector play important role in **turnpike theory** for macroeconomic exogenous growth model in economics, see Neumann, J. V. (1946). "A Model of General Economic Equilibrium". Review of Economic Studies. 13

The principal eigenvalue and the weak maximum principle

The **weak Maximum Principle** for

$$L[u] = \mathrm{Tr}(A(x)D^2u) + b(x) \cdot Du + c(x)u$$

in $\Omega \subseteq \mathbb{R}^n$ is the following **sign propagation property** :

wMP

any u such that

$$\begin{array}{ll} L[u] \geq 0 & \text{in } \Omega \\ u \leq 0 & \text{on } \partial\Omega \end{array}$$

satisfies

$$u \leq 0 \ \text{ on } \Omega$$

Several **sufficient** conditions of different nature known to imply the validity of **wMP** in a **bounded** domain Ω, e.g.

- $c(x) \leq 0$
- exists $\phi > 0$ in $\overline{\Omega}$ such that $L[\phi] \leq 0$
- Ω is **narrow** (i.e. contained in a suitably small strip)

Examples show that none of these conditions is however **necessary** for the validity of the Maximum Principle.

The principal eigenvalue and the weak maximum principle

An important characterization result due to Berestycki,Nirenberg and Varadhan:

wMP holds for **uniformly elliptic** operators

$$L[u] = \mathrm{Tr}(A(x)D^2u) + b(x) \cdot Du + c(x)u, \ A(x) \geq \alpha I$$

in a bounded domain Ω **if and only if** the number λ_1 defined by

$$\lambda_1 := \sup\{\lambda \in \mathbb{R} : \exists\, \phi > 0 \text{ in } \Omega \text{ such that } L[\phi] + \lambda\phi \leq 0 \quad \text{in } \Omega\}$$

is **strictly positive**. In the definition of λ_1, $\phi \in W^{2,p}_{loc}(\Omega)$.

Notably, this very nice result, which extends previous ones in the classical setting, was proved to hold under mild conditions on the coefficients and on $\partial\Omega$, see Comm. Pure Appl. Math. 47 (1994).

Degenerate fully nonlinear elliptic operators

Is this characterization valid for general partial differential inequalities of the form

$$F(x, u, Du, D^2u) \geq 0$$

involving a nonlinear mapping F ?

This question answered in Berestycki-CD-Porretta-Rossi (2014)

Let Ω be a domain in $\mathbb{R}^n$ and $\mathcal{S}^n$ be the space of $n \times n$ symmetric matrices endowed with the usual partial order:

$Y \geq 0$ means $Yp \cdot p \geq 0$ for all $p \in \mathbb{R}^n$ (i.e. Y is non negative semidefinite)

A mapping $F : \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ is **degenerate elliptic** if F is **non decreasing** in the matrix entry, i.e.

$$F(x, r, p, X + Y) \geq F(x, r, p, X) \quad \forall (x, r, p, X, Y) \,, \; Y \geq 0$$

Degenerate fully nonlinear elliptic operators

A basic example to keep in mind is of course that of linear operators in **non divergence form**

$$F(x,u,Du,D^2u) = \mathrm{Tr}(A(x)D^2u) + b(x)\cdot Du + c(x)u \quad , \quad x \in \Omega$$

where $A(x)$ is **nonnegative definite**, even in the case $A \equiv 0$ corresponding to the **transport operator**

$$b(x)\cdot Du + c(x)u$$

or in the very extreme one $F(u(x))$ (no derivatives !).

The weak Maximum Principle wMP

The **weak Maximum Principle** for F in $\Omega \subseteq \mathbb{R}^n$ is the following **sign propagation property** :

wMP

any $u \in C(\overline{\Omega})$ such that

$$\begin{array}{ll} F(x,u,Du,D^2u) \geq 0 & \text{in } \Omega \\ u \leq 0 & \text{on } \partial\Omega \end{array}$$

[in the **viscosity sense**] satisfies also

$$u \leq 0 \text{ in } \overline{\Omega}$$

Of course in our degenerate elliptic framework the set of functions u satisfying the above conditions may be empty if the boundary inequality is **not compatible** with the subsolution condition in the interior of the domain.

Just think at

$$u' \geq 0 \quad \text{in } (a,b) \quad , \quad u(a) > u(b)$$

A "principal eigenvalue" for the degenerate case?

Question:
does the Berestycki-Nirenberg-Varadhan characterization holds true as it is, or may be with suitable modifications, in the case of **degenerate** elliptic operators

$$\mathrm{Tr}(A(x)D^2u) + b(x) \cdot Du + c(x)u$$

with $A(x)$ **non-negative definite** and, more generally, for **fully nonlinear degenerate elliptic** operators?

That is, is there a number associated to F and Ω whose **positivity** enforces the validity of **wMP** and conversely?

The starting point of the joint research with Berestycki, Porretta and Rossi was the observation that the B-N-V definition of λ_1 **does not work** at this purpose in the case of **degenerate** ellipticity as shown by very simple examples.

A "principal eigenvalue" for the degenerate case?

An extremely simple one-dimensional example:

the function $u(x) = x(1-x)$ satisfies $L[u] = -\frac{x}{2}u' + u > 0$ in $(0,1)$, $u(0) = u(1) = 0$ and $u(x) > 0$ for all $x \in (0,1)$.
So, **wMP** does not hold in this case.

On the other hand, looking at functions $\phi(x) = x^k$ it is easy to check that $\lambda_1 > 0$, indeed $\lambda_1 = +\infty$.

A new notion of "generalized" principal eigenvalue

Definition.
Given a domain Ω in $\mathbb{R}^N$ and an open set $\mathcal{O}$ such that $\overline{\Omega} \subset \mathcal{O}$ and an operator F **positively homogeneous** of degree $\alpha > 0$ in $\mathcal{O}$, we define

$$\mu_1(F,\Omega) := \sup\{\lambda \in \mathbb{R} : \exists \Omega' \supset \overline{\Omega},\ \exists \phi \in C(\Omega'),\ \phi > 0, F[\phi] + \lambda\phi^\alpha \leq 0 \text{ in } \Omega'\}$$

A new notion of "generalized" principal eigenvalue

One cannot expect, in the general case, that $\mu_1(F,\Omega)$ is a **genuine principal eigenvalue**.

A simple example is given by $F[u] = x^2 u'$ in $\Omega = (-1,1)$.
It can be checked that $\mu_1(F,\Omega) = 0$. [Since the indicator function of the singleton 0 violates **wMP** , then $\mu_1(F,\Omega) \leq 0$ by our characterization result.
On the other hand, $\mu_1(F,\Omega) \geq 0$, as it is seen by taking $\phi \equiv 1$ in the definition but the unique solution of the eigenvalue equation

$$F[u] + \mu_1(F,\Omega)u^\alpha = 0$$

wit $u = 0$ on $\partial\Omega$ is $U \equiv 0$.

• Under **uniform ellipticity** for F, $\mu_1(F,\Omega)$ is indeed a genuine **principal eigenvalue** with **positive eigenfunction**, as proved by Birindelli-Demengel (2006).

$\mu_1(F,\Omega) > 0$ if and only if **wMP** holds

Our result concerning the characterization of the validity of **wMP** in the simplified setting where

$$F(x, u, Du, D^2u) = F(D^2u) - f(x)$$

is as follows:

Theorem
Let Ω be a bounded domain in $\mathbb{R}^n$ and $\mathcal{O}$ an open set such that $\overline{\Omega} \subset \mathcal{O} \subset \mathbb{R}^n$. Assume that F is continuous, degenerate elliptic, positively homogeneous of degree $\alpha > 0$. Assume also that $f \in C(\overline{\Omega})$.
Then,

F *satisfies* **wMP** *in* $\Omega \subset\subset \mathcal{O}$ **if and only if** $\mu_1(F,\Omega) > 0$

H. Berestycki, A. Porretta, L. Rossi, ICD, *Maximum Principle and generalized principal eigenvalue for degenerate elliptic operators*, JMPA 2014

A few remarks:

- for more general F depending on $x \in \Omega$ some extra continuity conditions are required.
- as far as we know the above result is new even for **smooth subsolutions of degenerate elliptic linear operators**

A few examples

- **zero order operators** $F(u) \geq 0,\ x \in \Omega\ ,\ u \leq 0,\ x \in \partial\Omega$
  If F decreasing and $F(0) = 0$ then, trivially, $\mu_1 > 0$ and $u \leq F^{-1}(0) = 0$,
  [think, for example, to $c(x)u \geq 0$ with $c(x) < 0$]
- **transport operators** $b(x) \cdot \nabla u \geq 0,\ x \in \Omega\ ,\ u \leq 0,\ x \in \partial\Omega$
  Not difficult to check that if b vanishes somewher in Ω then $\mu_1 = 0$
  On the other hand, if there exists a Lyapunov function L such that
  $\nabla L \neq 0$ and $b \cdot \nabla L > 0$ then $\mu_1 > 0$
- **proper operators** If $\max_{x\in\overline{\Omega}} F(x, r, 0, 0) < 0$ for all $r > 0$ (think about $\Delta u + c(x)u$ with $c(x) < 0$), then it is well-known that **wMP** holds for F.
  On the other hand, as an easy consequence of the definition of viscosity subsolution, one checks that $\mu_1(F, \Omega) > 0$.
- **subelliptic operators** If the ellipticity of F is not degenerate in some direction ν, that is

$$F(x, r, p, X + \nu \otimes \nu) - F(x, r, p, X) \geq \beta > 0$$

  and if the positive constants are supersolutions of $F = 0$ in $\mathcal{O}$, i.e., $F(x, 1, 0, 0) \geq 0$ in $\mathcal{O}$, then $\mu_1(F, \Omega) > 0$.
  This is seen by taking $\phi(x) = 1 - \varepsilon e^{\sigma\nu\cdot x}$, with σ large and ε small.
  Above conditions satisfied for instance by the 2-dimensional Grushin operator: $\partial_{xx} + |x|^k \partial_{yy}$ with k an even positive integer.

A few examples

- **Harvey-Lawson Hessian operators**

$$\mathcal{H}_k(D^2u) := \eta_{n-k+1}(D^2u) + \ldots + \eta_n(D^2u),$$

k an integer between 1 and n, $\eta_1(D^2u) \leq \eta_2(D^2u) \leq \ldots \leq \eta_N(D^2u)$ the ordered eigenvalues of the matrix D^2u.

These are 1-homogeneous degenerate Hessian operators introduced by F. R. Harvey and H. B. Lawson (2013) to characterize the validity of the Maximum Principle for operators on Riemannian manifolds depending only on the eigenvalues of the Hessian matrix.

A test with quadratic polynomials shows that $\mu_1(H_k) > 0$.

A few examples

- **Pucci operators**
  The Pucci maximal operator $\mathcal{P}_{\gamma,\Gamma}$ where $0 < \gamma < \Gamma$ is the 1-homogeneous uniformly elliptic Hessian operator

$$\mathcal{P}_{\gamma,\Gamma}(D^2u) = \Gamma\,\Sigma_{i\in I_+}\eta_i(D^2u) + \gamma\,\Sigma_{i\in I_-}\eta_i(D^2u)$$

  Here I_+, I_- correspond, respectively, to positive and negative eigenvalues of D^2u.
  It is known that **wMP** holds for the Pucci maximal operator: this can be proved as a consequence of the (deep and difficult to prove) ABP estimate in Caffarelli-Cabré book.

  On the other hand, one can also check directly that $\mu_1(\mathcal{P}_{\gamma,\Gamma}) > 0$
- Bellman-Isaacs operators

$$\inf_{\gamma\in\mathcal{G}}\sup_{\beta\in\mathcal{B}}\left[-\mathrm{tr}\,(A^{\beta,\gamma}(x)D^2u) + b^{\beta,\gamma}(x)\cdot Du + c^{\beta,\gamma}(x)u\right] \qquad (1)$$

Approximation of the principal eigenvalue

Consider the elliptic self-adjoint operator

$$Lu(x) = \partial_i \left(a_{ij}(x) \partial_j u(x) \right), \tag{2}$$

where $a_{ij} = a_{ji}$ are smooth functions in Ω, a smooth bounded open subset of $\mathbb{R}^n$, satisfying $a_{ij}\xi_i\xi_j \geq \alpha|\xi|^2$ for some $\alpha > 0$.

It is well-known that the minimum value λ_1 in the Rayleigh-Ritz variational formula

$$\lambda_1 = \inf_{\phi \in H_0^1(\Omega), \phi \not\equiv 0} \frac{-\int_\Omega \phi(x)\, L\phi(x)\ dx}{\|\phi\|^2_{L^2(\Omega)}} = \inf_{\phi \in H_0^1(\Omega), \phi \not\equiv 0} \frac{\int_\Omega a_{ij}(x) \partial_j \phi(x) \partial_i \phi(x)\ dx}{\|\phi\|^2_{L^2(\Omega)}}$$

is attained at some function w_1 and that

$$\begin{cases} Lw_1(x) + \lambda_1 w_1(x) = 0 & x \in \Omega, \\ w_1(x) = 0 & x \in \partial\Omega. \end{cases}$$

It can be proved moreover that λ_1 is the **principal eigenvalue** of L in Ω and w_1 is the corresponding **principal eigenfunction**.

Approximation of the principal eigenvalue

For linear operators in **divergence form** there is a vast literature on computational methods for the principal eigenvalue., see for example

I. Babuska, J.E. Osborn, *Finite element Galerkin approximation of the eigenvalues and eigenvectors of self-adjoint problems*, Math. Comp. 52, 275-297, 1989.
D. Boffi, *Finite element approximation of eigenvalue problems*, Acta Numer. 19, 1-120, 2010.

General non-divergence type elliptic operators, namely

$$Lu(x) = a_{ij}(x)\partial_{ij}u(x) + b_i(x)\partial_i u(x) + c(x)u \tag{3}$$

are **not self-adjoint** and the spectral theory is then much more involved: in particular, the Rayleigh-Ritz variational formula **is not available** anymore.

Approximation of the principal eigenvalue

Together with I. Birindelli and F. Camilli Comm. in Math. Sciences 2016 we developed a **finite difference scheme** for the computation of the principal eigenvalue and the principal eigenfunction of **fully nonlinear uniformly elliptic** operators based on the min-max formula discussed above:

$$\lambda_1 = -\inf_{\phi(x)>0} \sup_{x\in\Omega} \frac{F[\phi(x)]}{\phi(x)}$$

That formula can be seen as a **pointwise** alternative to the Rayleigh-Ritz L^2 formula.
Our approach applies in particular to **linear operators in non-divergence form**. Few references found even in the linear case, see e.g.

V. Heuveline, C. Bertsch, On multigrid methods for the eigenvalue computation of nonselfadjoint elliptic operators,East-West J. Numer. Math. 8 (2000), no. 4

A class of difference operators

Let $h\mathbb{Z}^n$ be the orthogonal lattice in $\mathbb{R}^n$ where $h > 0$ is a discretization parameter and $\mathcal{C}_h$ the space of the mesh functions defined on $\Omega_h = \Omega \subset \mathbb{Z}_h^n$. Consider a discrete operator F_h defined by

$$F_h[u](x) := F_h(x, u(x), [u]_x)$$

where

- $h > 0$ is the discretization parameter (h is meant to tend to 0),
- $x \in \Omega_h$ is a grid point
- $u \in \mathcal{C}_h$
- $[\cdot]_x$ represents the stencil of the scheme, i.e. the points in $\Omega_h \backslash \{x\}$ where the value of u are computed for writing the scheme at the point x (we assume that $[w]_x$ is independent of $w(y)$ for $|x - y| > Mh$ for some fixed $M \in \mathbb{N}$).

A class of difference operators

Following Kuo-Trudinger (Siam J.Num.Analysis 1992) we introduce some basic structure assumptions which are to be satisfied by the difference operator F_h:

(i) The operator F_h is of **positive type**, i.e. for all $x \in \Omega_h$, $z, \tau \in \mathbb{R}$, $u, \eta \in \mathcal{C}_h$ satisfying $0 \leq \eta(y) \leq \tau$ for each $y \in \Omega_h$, then

$$F_h(x, z, [u + \eta]_x) \geq F_h(x, z, [u]_x) \geq F_h(x, z + \tau, [u + \eta]_x)$$

(ii) The operator F_h is **positively homogeneous**, i.e. for all $x \in \Omega_h$, $z \in \mathbb{R}$, $u \in \mathcal{C}_h$ and $t \geq 0$, then

$$F_h(x, tz, [tu]_x) = tF_h(x, z, [u]_x)$$

(iii) The family of operator $\{F_h, 0 < h \leq h_0\}$, where h_0 is a positive constant, is **consistent** with operator F on the domain $\Omega \subset \mathbb{R}^n$, i.e. for each $u \in C^2(\Omega)$

$$\sup_{\Omega_h} \left| F(x, u(x), Du(x), D^2u(x)) - F_h(x, u(x), [u]_x) \right| \to 0 \quad \text{as } h \to 0,$$

uniformly on compact subsets of Ω.

A class of difference operators

The discretized equations for this kind of approximate operators satisfy some crucial pointwise estimates which are the discrete analogues of those valid for fully nonlinear, uniformly elliptic equations.

If F is **uniformly elliptic**, it is always possible to find a scheme of the previous type which is of **positive type and consistent** with F.

We don't know how to deal with this issue in the case of **degenerate** ellipticity.

As in the continuous case, we define a **principal eigenvalue** for F_h by means of the formula

$$\lambda_1^h := \sup\{\lambda \in \mathbb{R} : \quad \exists\, \phi > 0 \ \text{ in } \Omega_h, \ F_h[\phi] + \lambda\phi \leq 0 \ \}$$

Facts:

- There is a **positive solution** ϕ_1^h of

$$\begin{cases} F[\phi] + \lambda_1^h \phi = 0 & \text{in } \Omega_h, \\ \phi = 0 & \text{on } \Omega_h^C, \end{cases}$$

- For any $\lambda < \lambda_1$ the **Maximum Principle** holds for $F_h + \lambda$, i.e.

  If u is such that $F_h[u] + \lambda u \geq 0$ in Ω_h and $u \leq 0$ on Ω_h^C, then $u \leq 0$ in Ω_h
- λ_1^h is given by the **finite dimensional optimization problem**

$$\lambda_1^h = -\inf_{\phi \in \mathcal{C}_h,\, \phi > 0} \sup_{x \in \Omega_h} \frac{F_h[\phi](x)}{\phi(x)}$$

Convergence of $\lambda_1^h \longrightarrow \lambda_1$ as $h \to 0$

Theorem

Let (λ_1^h, ϕ_1^h) be the sequence of the discrete eigenvalues and of the corresponding eigenfunctions associated to F_h.
Then,

$$\lambda_1^h \to \lambda_1, \qquad \phi_1^h \to \phi_1$$

uniformly in $\overline{\Omega}$ as $h \to 0$, where λ_1 and ϕ_1 are respectively the principal eigenvalue and the corresponding eigenfunction associated to F.

The proof of the convergence result cannot rely on **standard stability results** in viscosity solution theory (Barles-Souganidis' method) since the limit problem

$$\begin{cases} F[\phi] + \lambda_1 \phi = 0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases}$$

does not satisfy a Strong Comparison Principle, implying **uniqueness** of viscosity solutions for problem above.

Indeed, the principal eigenfunction $\phi_1 > 0$ and $\phi \equiv 0$ (we are assuming $F[0] = 0$) are **two distinct solutions** of the problem.

Convergence of $\lambda_1^h \longrightarrow \lambda_1$ as $h \to 0$

Different techniques are needed, the main ingredients of the proof are:

- the **semi-relaxed limits** in viscosity solution sense;
- a **Maximum Principle** for the limit problem (rather than the Comparison Principle);
- the following **local Hölder estimate** proved by Kuo-Trudinger:
  If u_h is a solution of $F_h[u] = f$, then for any $x, y \in \Omega_h$

$$|u_h(x) - u_h(y)| \le C \frac{|x-y|^\delta}{R} \left(\max_{B_R^h} u_h + \frac{R}{\alpha_0} \left\{ \sum_{x \in \Omega_h} h^n |f(x)|^n \right\}^{\frac{1}{n}} \right),$$

  where $R = \min \operatorname{dist}(x, \partial\Omega_h)$, $B_R^h = B(0,R) \cap \Omega_h$, δ, α_0 and C are positive constants independent of h.

The algorithm for convex operators: the Hamilton-Jacobi-Bellman case

In the case of **convex** operators F such as those arising in the optimal control theory of degenerate diffusion processes, that is F is the supremum of a family of linear operators :

$$F(x, u, Du, D^2u) = \sup_{i \in I} Tr(A^i(x)D^2u) + b^i(x) \cdot Du + c^i(x)u$$

our numerical approach leads to a finite dimensional convex optimization problem.

Simulation can be easily performed with the Optimization Toolbox of MATLAB

Some examples

Example 1. To validate the algorithm we begin by studying the eigenvalue problem:

$$\begin{cases} w'' + \lambda_1 w = 0 & x \in (0,1), \\ w(x) = 0 & x = 0,\ 1 \end{cases}$$

In this case the eigenvalue and the corresponding eigenfunction are given by

$$\lambda_1 = \pi^2, \qquad w_1(x) = \sin(\pi x)$$

Note that since the eigenfunctions are defined up to multiplicative constant, we normalize the value by taking $\|w_1\|_\infty = \|w_{1,h}\|_\infty = 1$

Given a discretization step h and the corresponding grid points $x_i = ih$, $i = 0, \dots, N_h + 1$, the optimization problem is

$$\lambda_{1,h} = -\min_{U \in \mathbb{R}^{N_h}} \left[\max_{i=1,\dots,N_h} \frac{U_{i+1} + U_{i-1} - 2U_i}{h^2 U_i} \right]$$

(with $U_0 = U_{N_h+1} = 0$).

Some examples

h	$Err(\lambda_1)$	$Order(\lambda_1)$	$Err_\infty(w_1)$	$Err_2(w_1)$
$1.00 \cdot 10^{-1}$	$8.0908 \cdot 10^{-2}$		$3.3662 \cdot 10^{-11}$	$5.7732 \cdot 10^{-11}$
$5.00 \cdot 10^{-2}$	$2.0277 \cdot 10^{-2}$	1.9964	$1.4786 \cdot 10^{-10}$	$3.8119 \cdot 10^{-10}$
$2.50 \cdot 10^{-2}$	$5.0723 \cdot 10^{-3}$	1.9991	$6.6613 \cdot 10^{-16}$	$1.8731 \cdot 10^{-15}$
$1.25 \cdot 10^{-2}$	$1.2683 \cdot 10^{-3}$	1.9998	$1.5543 \cdot 10^{-15}$	$6.2524 \cdot 10^{-15}$
$6.25 \cdot 10^{-3}$	$3.1708 \cdot 10^{-4}$	1.9999	$1.2212 \cdot 10^{-15}$	$7.1576 \cdot 10^{-15}$

We can observe an order of convergence close to 2 for λ_1 and therefore equivalent to one obtained by discretization of the Rayleigh quotient via finite elements.

Some examples

Example: A 2-dimensional non divergence type example

Consider the eigenvalue problem for the Ornstein-Uhlenbeck operator

$$\Delta\phi - x \cdot D\phi + \lambda\phi = 0, \qquad x \in (0,1)^2$$

with homogeneous boundary conditions. The principal eigenvalue and the corresponding eigenfunction are given by

$$\lambda_1 = 4, \qquad \phi_1(x_1, x_2) = (1 - x_1^2)(1 - x_2^2)$$

The Laplacian is discretized by a five-point formula.

h	$Err(\lambda_1)$	$Order(\lambda_1)$
$4.00 \cdot 10^{-1}$	0.1524	
$2.00 \cdot 10^{-1}$	0.0392	1.9592
$1.00 \cdot 10^{-1}$	0.0103	1.9250
$5.00 \cdot 10^{-2}$	0.0027	1.9580

Example 3: the Fucik spectrum The Fucik spectrum of Δ is the set of pairs $(\mu, \alpha\mu) \in \mathbb{R}^2$ for which the equation

$$-\Delta u = \mu u^+ - \alpha\mu u^-$$

has a non-zero solution, where $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \max\{-u(x), 0\}$.

For fixed $0 < \alpha < 1$ the Fucik principal eigenvalue of Δ is λ_1

$$\max\{\Delta u, \frac{1}{\alpha}\Delta u\} + \lambda_1 u = 0$$

The operator $F[u] = \max\{\Delta u, \frac{1}{\alpha}\Delta u\}$ is a nonlinear **convex** operator.
In the 1-d case with $\Omega = [0, \pi]$ and $\alpha = 1/2$ one can compute the exact value of $\lambda_1 = 1$.

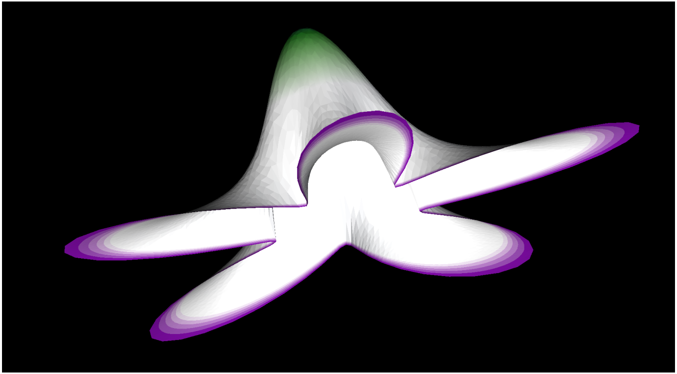
h	$Err(\lambda_1)$	$Order(\lambda_1)$
$1.00 \cdot 10^{-1}$	0.0809	
$5.00 \cdot 10^{-2}$	0.0203	1.9964
$2.50 \cdot 10^{-2}$	0.0051	1.9991
$1.25 \cdot 10^{-2}$	0.0013	1.9998
$6.25 \cdot 10^{-3}$	0.0003	2.0000

Projects and work in progress

- develop a similar theory in **unbounded domains** starting from:

  H. Berestycki and L. Rossi,
  Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains. Comm. Pure Appl. Math. 68 (2015)
  and
  I. Capuzzo Dolcetta, F. Leoni and A. Vitolo (2005),
  The Alexandrov-Bakelman-Pucci weak Maximum Principle for fully nonlinear equations in unbounded domains. Comm.PDE's Vol. 30
  and subsequent works
- try to develop an analogous approach for H-J-B equations in the case of **degenerate diffusions**
- analyse the rate of convergence
- test the various computational issues on more complex models

The principal Dirichlet eigenflower for the Laplacian

copia.png 

Courtesy of S. Cacace. Based on numerical schemes in
I. Birindelli, F. Camilli, ICD 2016.