A multiscale method for reducing the complexity of (controlled) large multi-agent systems

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Plan

General formulation of the problem

We consider a system composed of a large number of interacting agents for which a microscopic and a MACROscopic description are available.

Microscopic level

$$dX^k = v_m[X]dt + N\, dB_t^k, \qquad k = 1, \dots, N_p,$$

where $X = (X^1, \dots, X^{N_p})$.

Macroscopic level

$$\partial_t \rho(t,x) + \nabla \cdot \Big(\rho(t,x) v_M[\rho(t,\cdot)]\Big) = D \triangle \rho(t,x), \qquad t > 0,\ x \in \mathbb{R}^d$$

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Goal

Coupling the models to improve the single-scale descriptions.

Multiscale abstract approach: advection

The states of the system $\{X^k\}$ and ρ are kept separate, while the coupling acts on the velocity field only.

$$\dot{X}^k = v_m[X], \; k = 1, \dots, N_p, \qquad v_m[X(t)](X^k) = \sum_{X^h \in \mathcal{S}(X^k)} K(X^k; X^h)$$

$$\partial_t \rho + \nabla \cdot (\rho v_M[\rho]) = 0, \qquad v_M[\rho(t,\cdot)](x) = \int_{\mathcal{S}(x)} K(x;y)\rho(t,y)dy$$

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New blended velocity field

$$v_{mM}[X,\rho](t,x) = \theta v_m[X(t)](x) + (1-\theta)\Lambda v_M[\rho(t,\cdot)](x), \quad \theta \in [0,1]$$

Multiscale abstract approach: advection

For abstract measure lovers... There is also a

Measure-based approach

$$\frac{\partial \mu_t}{\partial t} + \nabla \cdot (\mu_t \; v[\mu_t]) = 0$$

$$\mu_t = \theta \sum_{k=1}^{N} \delta_{X^k(t)} + (1-\theta)\Lambda\rho(\cdot,t)\mathcal{L}^d, \quad \theta \in [0,1]$$

Multiscale abstract approach: diffusion

Microscopic model: Brownian motion

$$\begin{cases} dX_t^k = \sqrt{2D}dB_t^k \\ X_0^k = 0 \end{cases} \qquad k = 1, \ldots, N_p$$

Macroscopic model: heat equation

$$\begin{cases} \partial_t u - D\partial_x^2 u = 0, & t > 0,\ x \in \mathbb{R} \\ u(0, x) = \delta_0, & x \in \mathbb{R}, \end{cases}$$

The correspondence

$$\mathbb{P}(X_t^k \in A) = \int_A u(t, x)\, dx, \quad \forall A \subseteq \mathbb{R},\ \forall k.$$

Multiscale abstract approach: diffusion

v_m

The *microscopic velocity field* v_m should be defined as the (formal) derivative of the standard Brownian motion which is a (not rigorously defined) process with zero mean and infinite variance.

v_M

The heat equation can be formally written as

$$\partial_t u + \partial_x (u v_M) = 0$$

with

$$v_M(t, x) := -D \frac{\partial_x u(t, x)}{u(t, x)} = \frac{x}{2t}.$$

Multiscale abstract approach: diffusion

The Ito processes satisfying the SDE can be approximated by using the **strongly-consistent Euler scheme**

$$X_{n+1}^k = X_n^k + \sqrt{2D}\Delta B_n^k, \qquad \Delta B_n^k \sim \mathcal{N}(0, \Delta t), \quad \forall n, k.$$

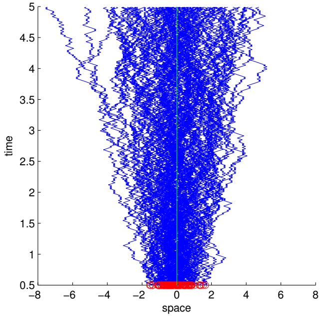
The right scale interpolation in this case is

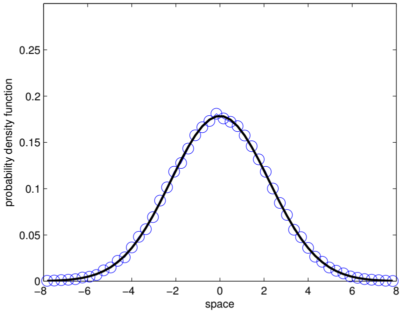
Regularized Brownian motion

$$X_{n+1}^k = \begin{cases} X_n^k + \sqrt{2D}\Delta B_n^k, & \text{with probability } \theta, \\ X_n^k + \frac{X_n^k}{2t_n}\Delta t, & \text{with probability } (1-\theta). \end{cases}$$

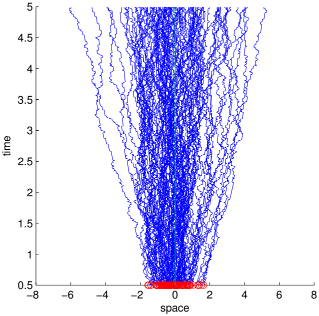
The corresponding probability density function of the particles' positions $\{X_n^k\}_k$ tends to the function $x \to u(t_n, x)$ as $N_p \to \infty$ for any $\theta \in [0, 1]$.

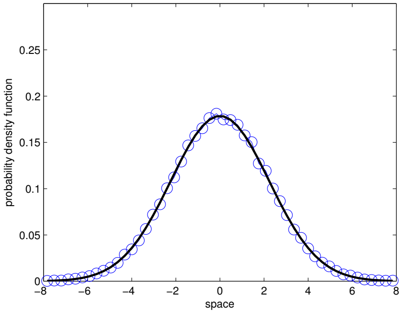
Partial coupling: $\theta = 1$ (pure micro)



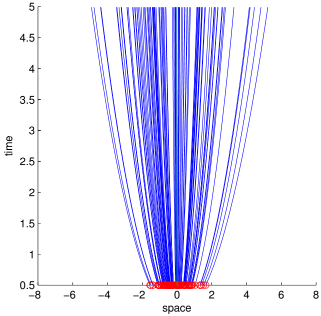


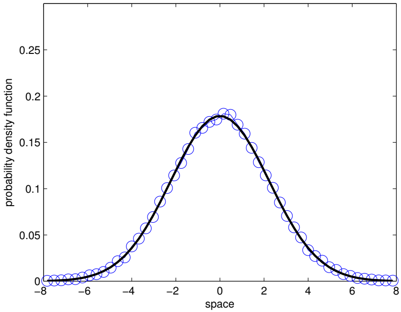
Partial coupling: $\theta = 0.2$





Partial coupling: $\theta = 0$ (pure macro)





Plan

A nonlocal model

Velocity field

$$v(t,x) := v[\mu_t](x) = v_{\text{des}}(x) + v_{\text{int}}[\mu_t](x),$$

Desired velocity

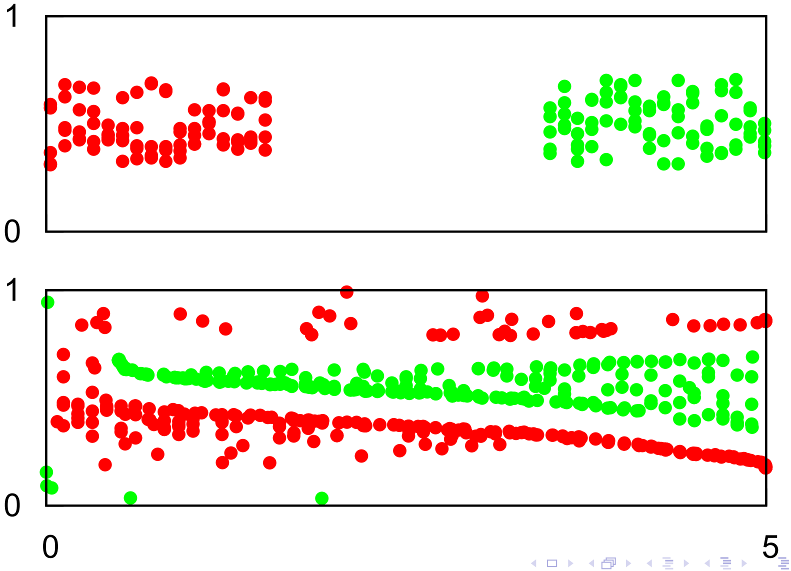
$v_{\text{des}} : \mathbb{R}^2 \to \mathbb{R}^2$ the velocity that pedestrians would set to reach their destination (considering obstacles) if they were alone in the domain.

Interaction velocity

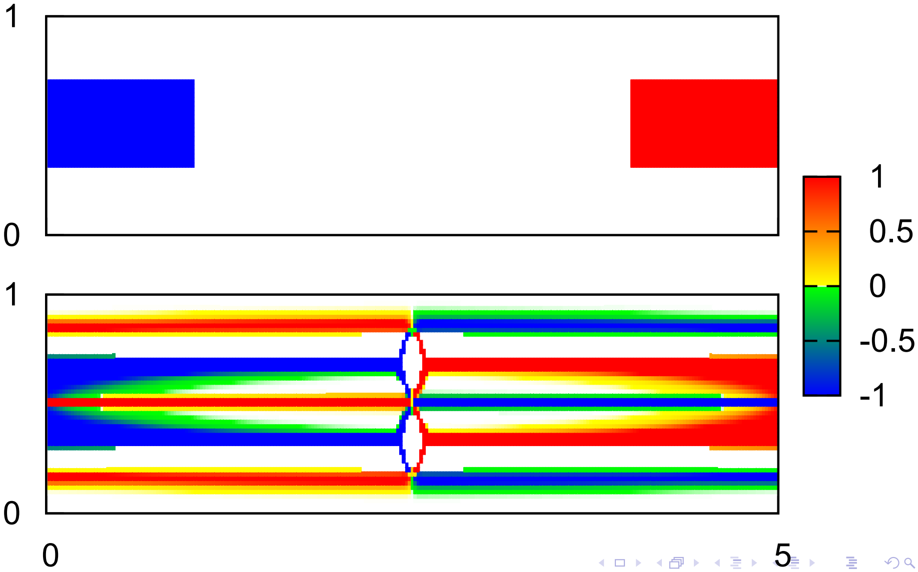
$$v_{\text{int}}[\mu_t](x) = \int_{\mathcal{S}(x)} \mathcal{F}(|y-x|)\frac{y-x}{|y-x|}\, d\mu_t(y),$$

$$\mathcal{F}(s) = -\frac{F_r}{s}\chi_{[0,\,R_r]}(s) \qquad \text{(repulsion effect)}$$

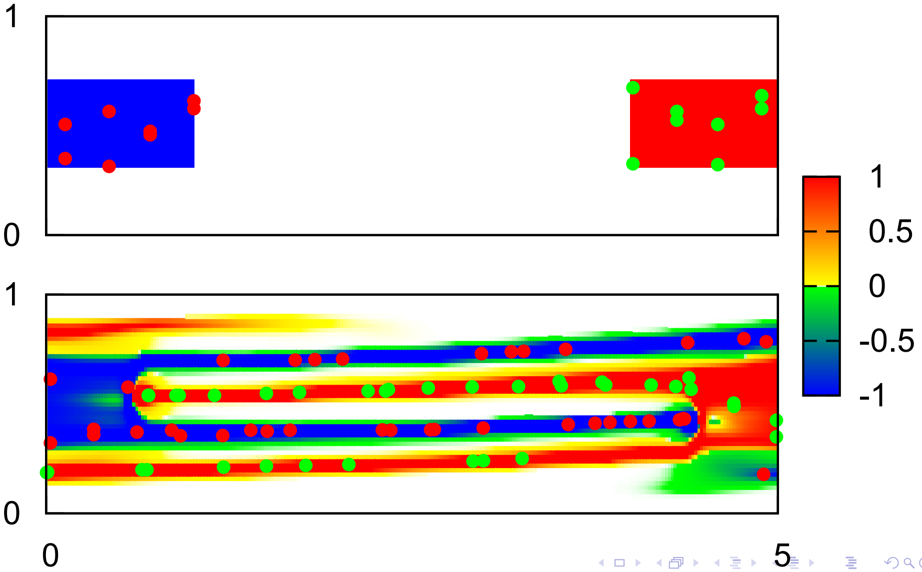
Crossing flows: microscopic model



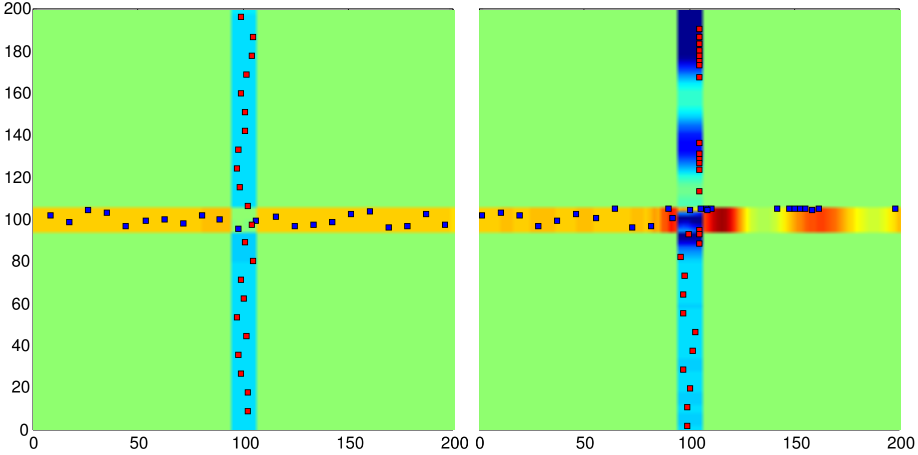
Crossing flows: macroscopic model



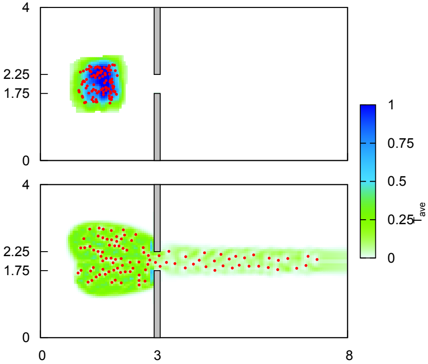
Crossing flows: multiscale model

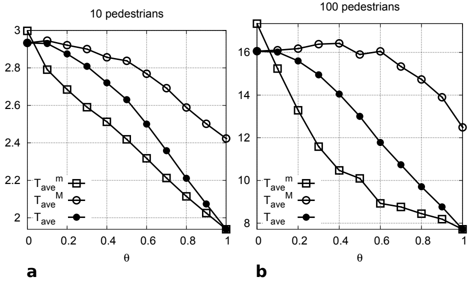


Orthogonal crossing flows

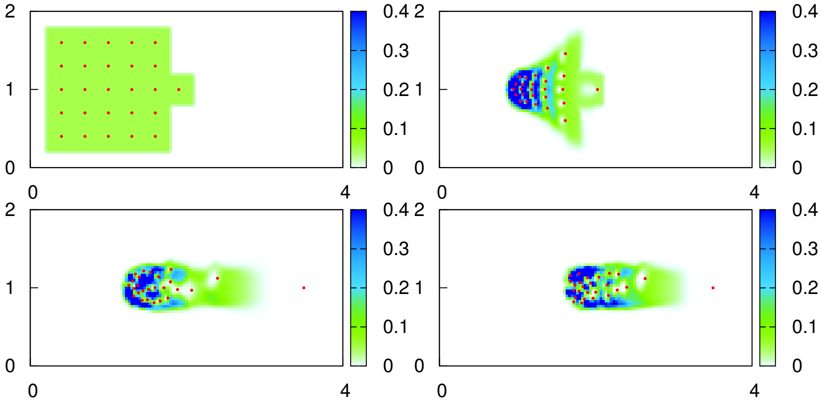


Bottleneck


4
2.25
1.75
0
4
2.25
1.75
0
0
3
8
1
0.75
0.5
0.25
0
T_ave


10 pedestrians
100 pedestrians
T_ave^m
T_ave^M
T_ave
θ
θ
a
b

Tourist guide: multiscale model with control



Plan

A model with opinion polls

An opinion is described by a continuous variable $w(t) \in [-1, 1]$ varying in time due to the interactions with others people.
If we consider political opinions, the sign of the opinion $\mathrm{sgn}(w)$ expresses the voting intent (1 stands for "yes" or "left", -1 stands for "no" or "right"), while the norm $|w|$ gives the degree of conviction.

Microscopic model

$$dw_k = (1 - |w_k|^\delta)\left(\overbrace{C_{\text{int}} \frac{1}{N_p} \sum_{h=1}^{N_p} (w_h - w_k)}^{\text{interactions}} dt + \overbrace{\sqrt{2} C_{\text{noise}} dB_t^k}^{\text{self-thinking}}\right)$$

for $k = 1, .., N_p$, $t \in [0, T]$, where

- $(1 - |w_k|^\delta)$ is the propensity to change opinion;
- B_t^k are independent Brownian motions.

Introducing the break of symmetry by opinion polls

$$P^+ := \frac{\text{card}\{k : w_k(T_{\text{poll}}) > W^0\}}{N_p}, \quad P^- := \frac{\text{card}\{k : w_k(T_{\text{poll}}) < -W^0\}}{N_p}$$

where $W^0 \in [0, 1)$ is a "null threshold" such that if $|w_k| \leq W^0$ then the kth individual is regarded as indecisive.

Result of the poll

$$P := P^+ - P^- \in [-1, 1],$$

The microscopic model with poll

$$dw_k = (1 - |w_k|^{\delta})\Bigg(\overbrace{C_{\text{int}} \frac{1}{N_p} \sum_{h=1}^{N_p} (w_h - w_k)\, dt}^{\text{interactions}} + \overbrace{\sqrt{2} C_{\text{noise}}\, dB_t^k}^{\text{self-thinking}} + \underbrace{C_{\text{poll}}\, b(t, P, w_k)\, dt}_{\text{poll effect}} \Bigg).$$

where

$$b(t, P, w_k) := |P| \, (\text{sgn}\, P - w_k) \chi_{[0,\Delta]}(t - T_{\text{poll}}).$$

The vote

$$V^+ := \frac{\operatorname{card}\{k : \operatorname{sgn} w_k(T_{\text{vote}}) = +1\}}{N_p}$$

$$V^- := \frac{\operatorname{card}\{k : \operatorname{sgn} w_k(T_{\text{vote}}) = -1\}}{N_p}$$

Result of the vote

$$V := V^+ - V^- \in [-1, 1].$$

The macroscopic model

Define the probability density function $\rho = \rho(t, w)$ in such a way that $\rho(t, w)dw$ is the probability that at time t a generic individual has an opinion in $[w, w + dw]$. We get that ρ is the solution to

The Fokker-Planck equation

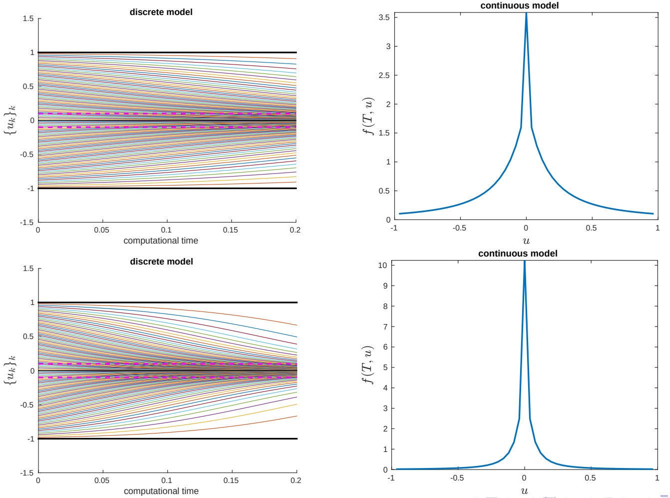
$$\partial_t \rho + \partial_w (K_{\text{adv}}[\rho]\rho) = \partial_w^2 (C_{\text{noise}}^2 (1 - |w|^\delta)^2 \rho),$$

where

$$K_{\text{adv}}[\rho(t,\cdot)](w) = (1-|w|^\delta)\left(C_{\text{int}} \int_{-1}^{1} (w'-w)\rho(t,w')dw' + \underbrace{C_{\text{poll}}\, b(t,P,w)}_{\text{from micro model}} \right)$$

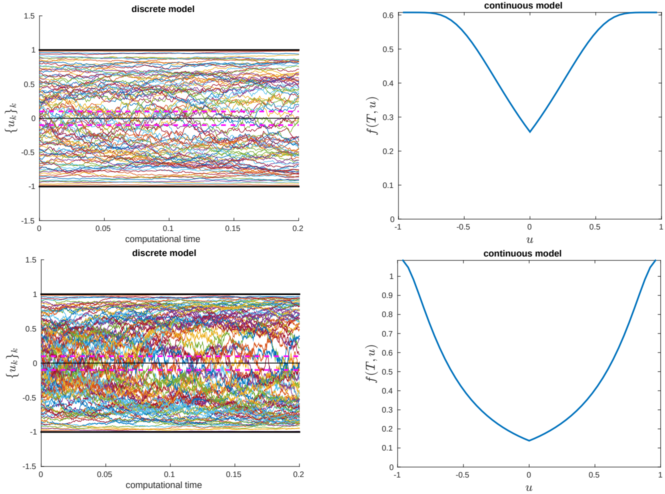
Single-scale model, small N_p

Effect of interactions only (reference and double strength)



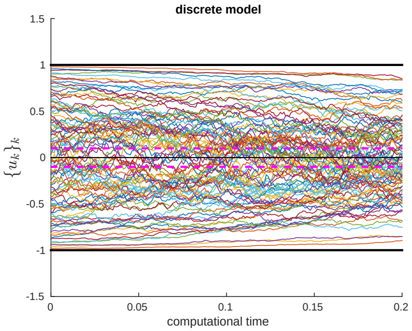
Single-scale model, small N_p

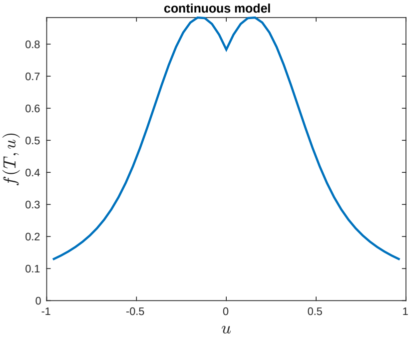
Effect of noise only (reference and double strength)



Single-scale model, small N_p

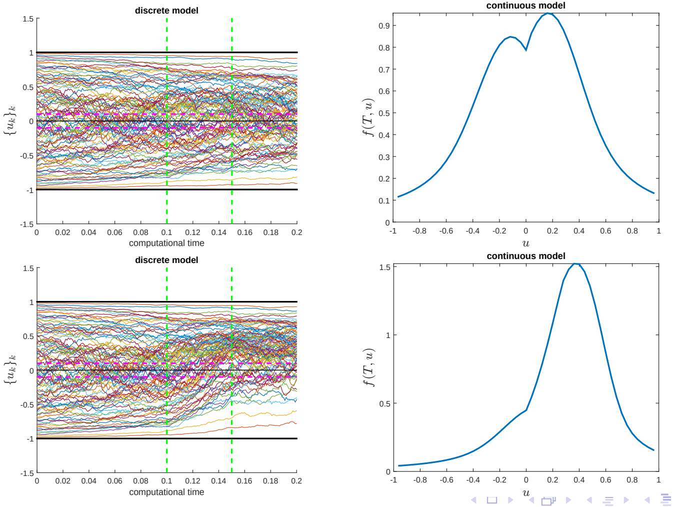
Effect of interaction and noise only





Single-scale model, small N_p

Complete model (reference and 10x poll strength)



The multiscale model

The multiscale model (from the microscopic side) is

$$dw_k = \theta\left[(1-|w_k|^\delta)\left(C_{\text{int}}\frac{1}{N_k}\sum_{h=1}^{N}(w_h-w_k)\Gamma_{hk}+C_{\text{poll}}b(t,P,w_k)\right)dt\right]$$

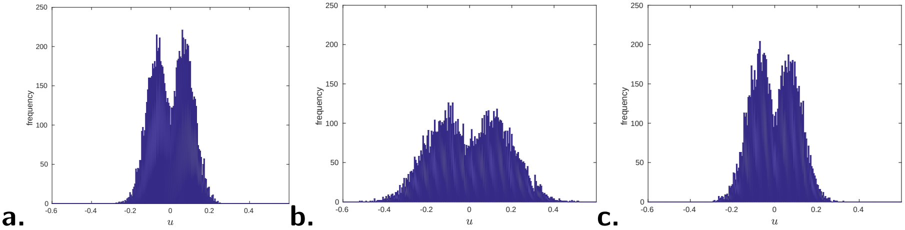
$$+(1-\theta)K_{\text{adv}}[\rho(t,\cdot)](w_k)dt+\begin{cases} C_{\text{noise}}(1-|w_k|^\delta)\sqrt{2}dB_t^k, & \text{with prob. } \theta,\\ K_{\text{diff}}[\rho(t,\cdot)](w_k)dt, & \text{with prob. } (1-\theta)\end{cases}$$

with

$$K_{\text{diff}}[\rho](w)=-C_{\text{noise}}^2\left(2(w-\text{sgn}(w))+(1-|w|^\delta)^2\frac{\partial_w\rho}{\rho}\right).$$

Multi-scale model, one poll

Histograms of the results of the final vote V after 10000 runs
(uniformly-distributed random choice of initial opinions in the space $[-1, 1]$)

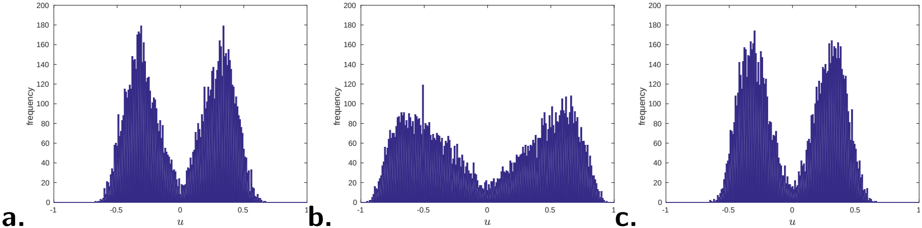


a. Full-size model $N_p = 2000$ ($\theta = 1$).
b. Pure reduced model $N_p^* = 400$ ($\theta = 1$).
c. Optimally-hybridized reduced model $N_p^* = 400$ ($\theta = 0.1$).

Wasserstein distance: from 0.06 to 0.008.

Multi-scale model, two polls

Histograms of the results of the final vote V after 10000 runs
(uniformly-distributed choice of initial opinions in the space $[-1, 1]$)



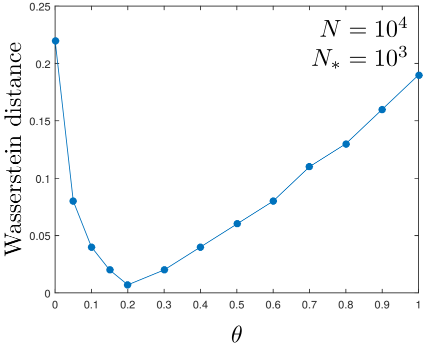
a. Full-size model $N_p = 10000$ ($\theta = 1$).

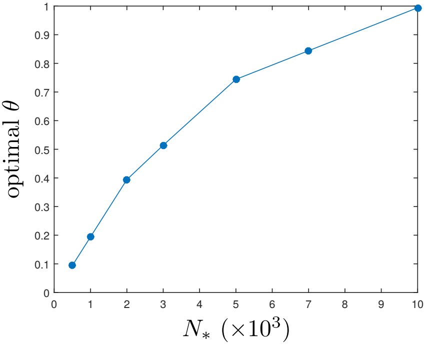
b. Pure reduced model $N_p^* = 1000$ ($\theta = 1$).

c. Optimally-hybridized reduced model $N_p^* = 1000$ ($\theta = 0.2$).

Wasserstein distance: from 0.19 to 0.007.

Multiscale model, optimal blending parameter





Plan

Control of multiscale dynamics

Control and optimization of multiscale dynamics is hard due to the computational effort, but could be advantageous in some situations.

Ideas

- Control micro dynamics and get the effect on MACRO dynamics
- Control MACRO dynamics and get the effect on micro dynamics
- Control both micro dynamics and MACRO dynamics
- Control both a reduced micro dynamics and MACRO dynamics
- Bilevel control?
- ...

The optimization process depends on the coupling itself!

A very preliminary numerical investigation

Microscopic model (Lagrangian control)

$$\dot{X}^k = C_{\text{int}} \sum_{h=1}^{N_p} (X^h - X^k) + u_k, \qquad k = 1, \ldots, N_p$$

$$J^m[X] = \sum_k \int_0^T \Big(|X^k - x^{\mathcal{T}}| + C|u_k(t)| \Big) dt$$

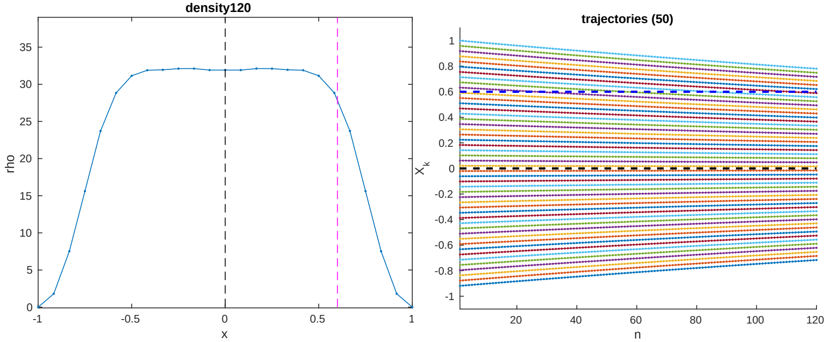
Macroscopic model (Eulerian control)

$$\partial_t \rho + \partial_x(\rho v[\rho]) = 0, \quad v[\rho](x,t) = \int_{\mathbb{R}} C_{\text{int}}(y-x)\rho(y,t)dy + u(x,t)$$

$$J^M[\rho] = \int_{\mathbb{R}} \int_0^T \Big(|x - x^{\mathcal{T}}|\rho(x,t) + C|u(x,t)| \Big) dxdt$$

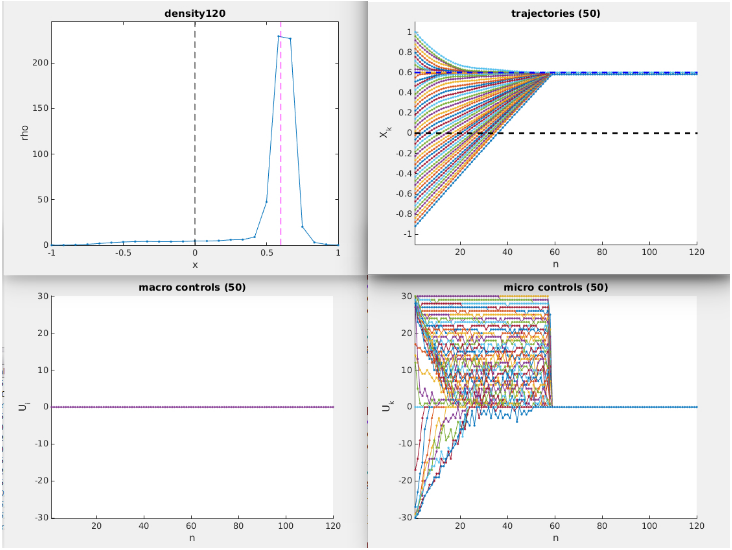
Uncontrolled dynamics

Initial condition: uniform distribution of the agents in $[-1, 1]$

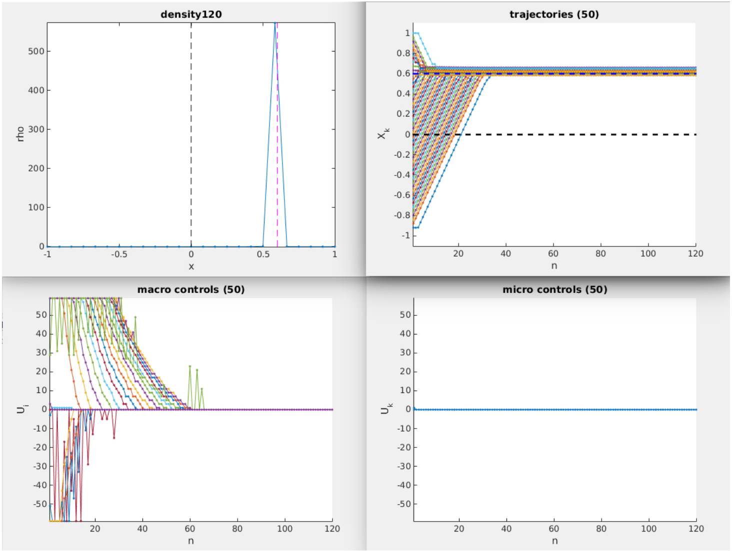


Same results are obtained for any $\theta \in [0, 1]$

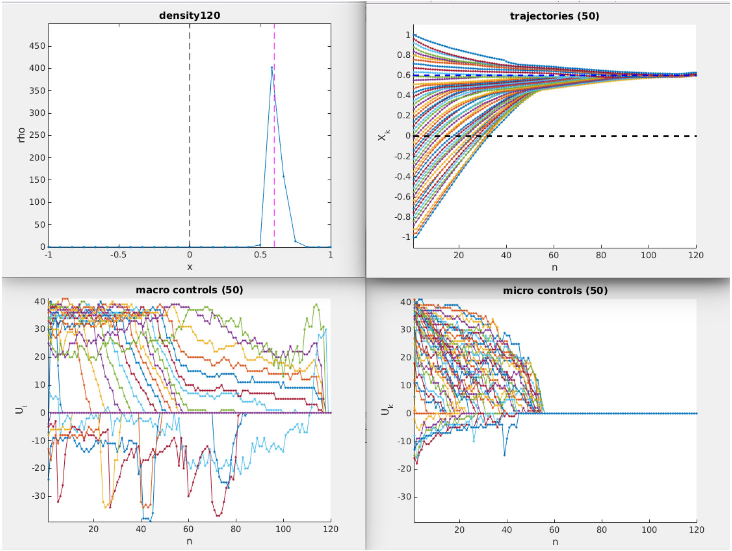
Micro-driven optimal multiscale dynamics



Macro-driven optimal multiscale dynamics



Hybrid optimal multiscale dynamics



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[1] E. Cristiani, A. Tosin, *Reducing complexity of multiagent systems with symmetry breaking: an application to opinion dynamics with polls*, preprint arXiv:1706.03115.

[2] E. Cristiani, B. Piccoli, A. Tosin, *Multiscale modeling of granular flows with application to crowd dynamics*, Multiscale Model. Simul., 9 (2011), 155–182.

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[4] E. Cristiani, *Blending Brownian motion and heat equation*, J. Coupled Syst. Multiscale Dyn., 3 (2015), 351–356.