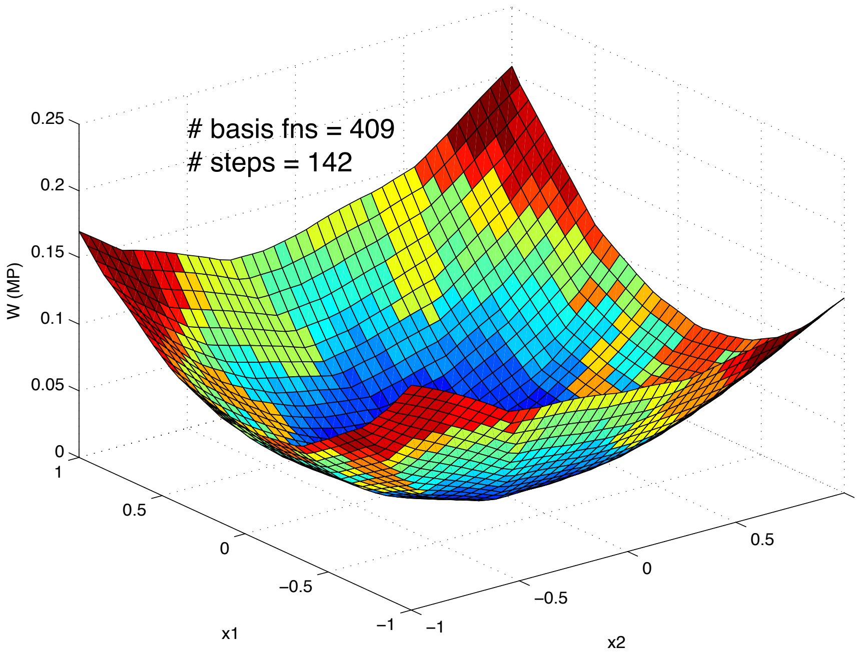
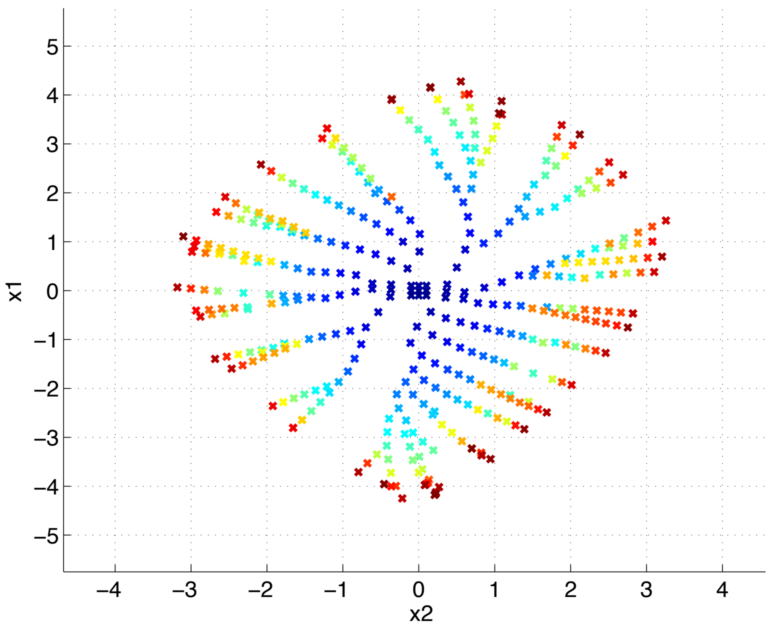
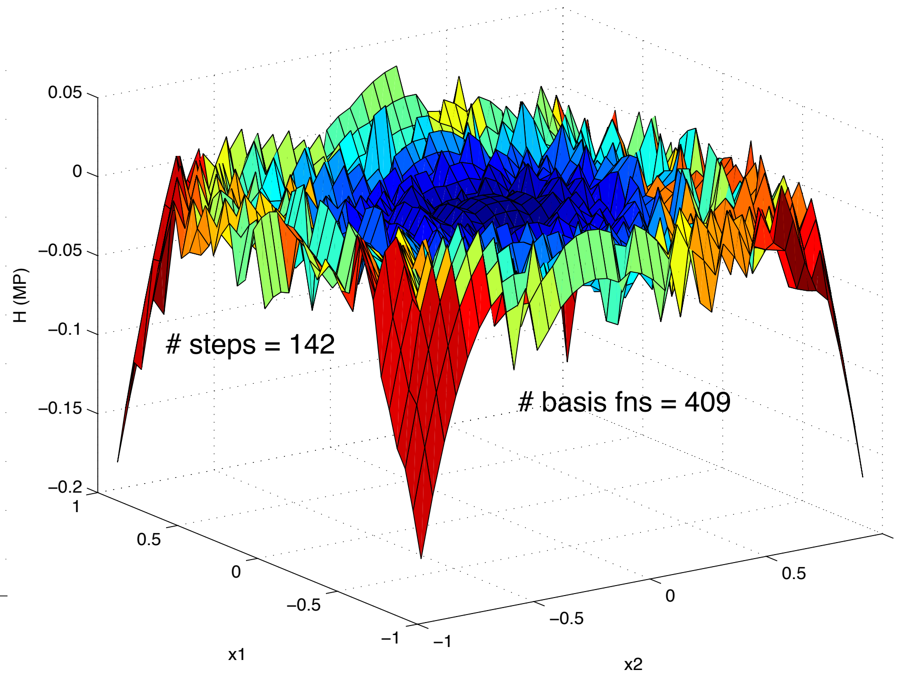
Max-plus fundamental solution semigroups for optimal control







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Outline

"Max-plus fundamental solution semigroups for optimal control"

- Worst case analysis & optimal control.
  - Dynamic programming.
  - Max-plus linearity.
  - Semi-convexity.

  → max-plus fundamental solution semigroups

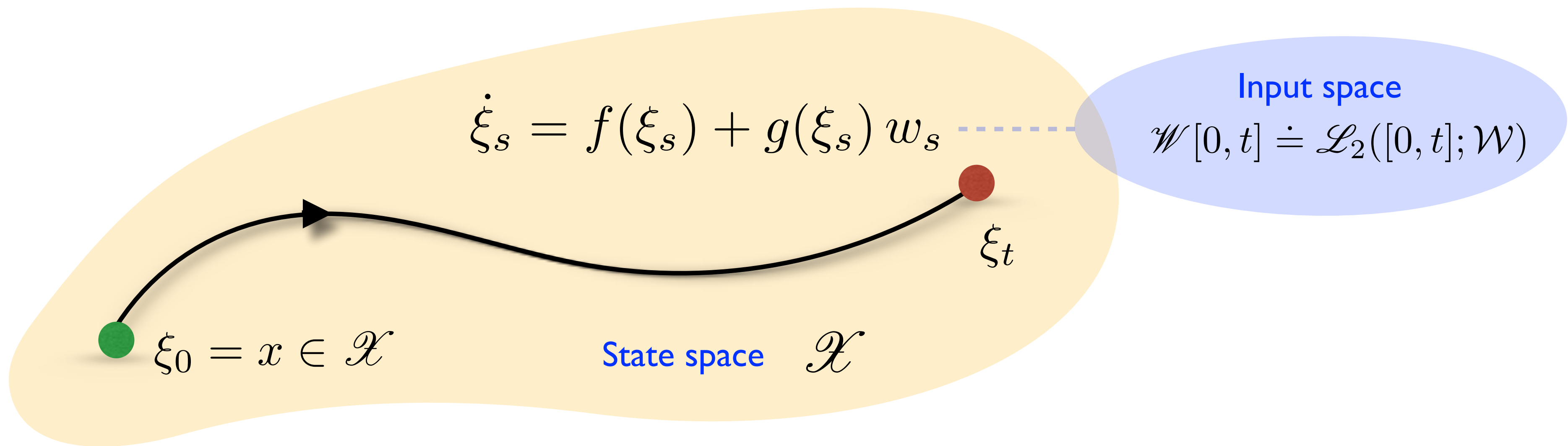
- Value propagation via max-plus fundamental solution semigroups.
  - Kernel approximation.
  - Kernel structure.
  - Problem classes.

  → max-plus methods

Worst case analysis & optimal control

Value $$W_t(x) \doteq [\mathcal{S}_t\, \Psi](x) \doteq \sup_{w \in \mathscr{W}[0,t]} \left\{ \int_0^t \ell(\xi_s) - \tfrac{1}{2}\, |w_s|^2\, ds + \Psi(\xi_t) \right\} \qquad W_t : \mathscr{X} \to \mathbb{R}^-$$

Dynamics



Optimal input $$w^* \doteq \operatorname*{argmax}_{w \in \mathscr{W}[0,t]} J_t[\Psi](x, w) \qquad J_t[\Psi](x,w) \doteq \int_0^t l(\xi_s) - \tfrac{1}{2}\, |w_s|^2\, ds + \Psi(\xi_t)$$

Objective Compute the value function and the optimal control

state feedback characterization

Worst case analysis & optimal control

Value $$W_t(x) \doteq [\mathcal{S}_t\, \Psi](x) \doteq \sup_{w \in \mathscr{W}[0,t]} \left\{ \int_0^t \ell(\xi_s) - \tfrac{1}{2} |w_s|^2 \, ds + \Psi(\xi_t) \right\} \quad W_t : \mathscr{X} \to \mathbb{R}^-$$

dynamic programming

unique viscosity solution

e.g. [M06] Thms 3.20 & 4.9

Hamiltonian

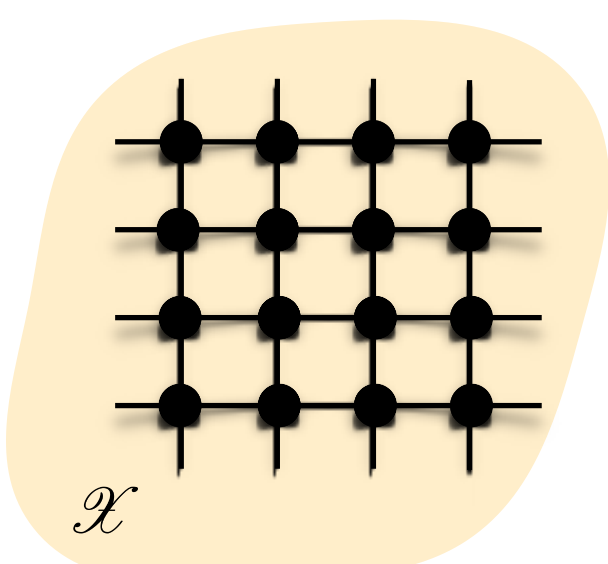
$$H(x,p) \doteq -\sup_{w \in \mathcal{W}} \left\{ \langle p, f(x) + g(x) w \rangle + l(x) - \tfrac{1}{2} |w|^2 \right\}$$

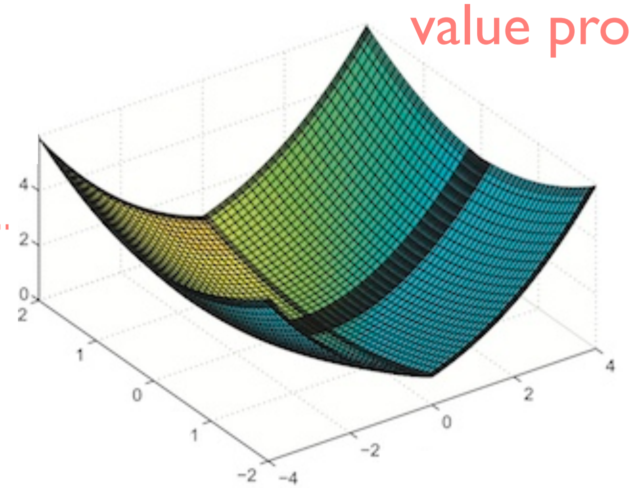
HJB PDE $$0 = \frac{\partial}{\partial t} W_t(x) + H(x, \nabla W_t(x)) \qquad W_0 = \Psi$$

Hamilton-Jacobi
Bellman equation

finite difference
approximation

value propagation





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Worst case analysis & optimal control

Value $$W_t(x) \doteq [\mathcal{S}_t\, \Psi](x) \doteq \sup_{w \in \mathscr{W}[0,t]} \left\{ \int_0^t \ell(\xi_s) - \tfrac{1}{2}\, |w_s|^2\, ds + \Psi(\xi_t) \right\} \quad W_t : \mathscr{X} \to \mathbb{R}^-$$

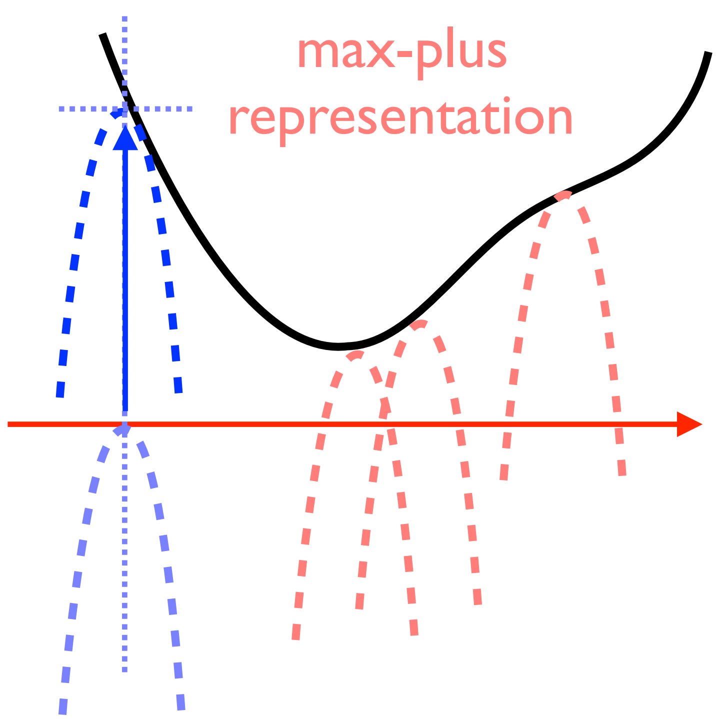
dynamic programming

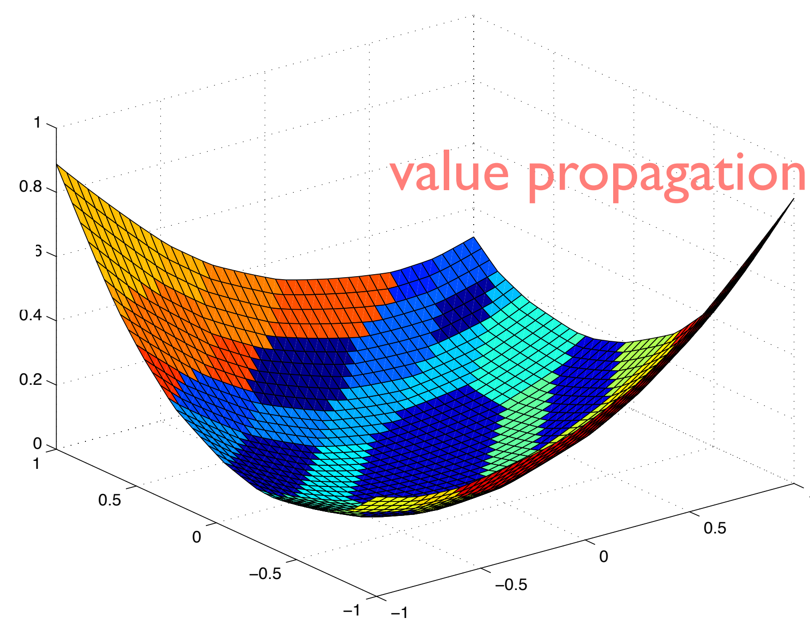
DPP $$W_{t+\tau}\, \Psi = \mathcal{S}_{t+\tau}\, \Psi = \mathcal{S}_\tau\, \mathcal{S}_t\, \Psi = \mathcal{S}_\tau\, W_t \qquad W_0 = \Psi$$

Dynamic programming principle

algebraic structure

- semigroup
- max-plus linearity
- semiconvexity preserving





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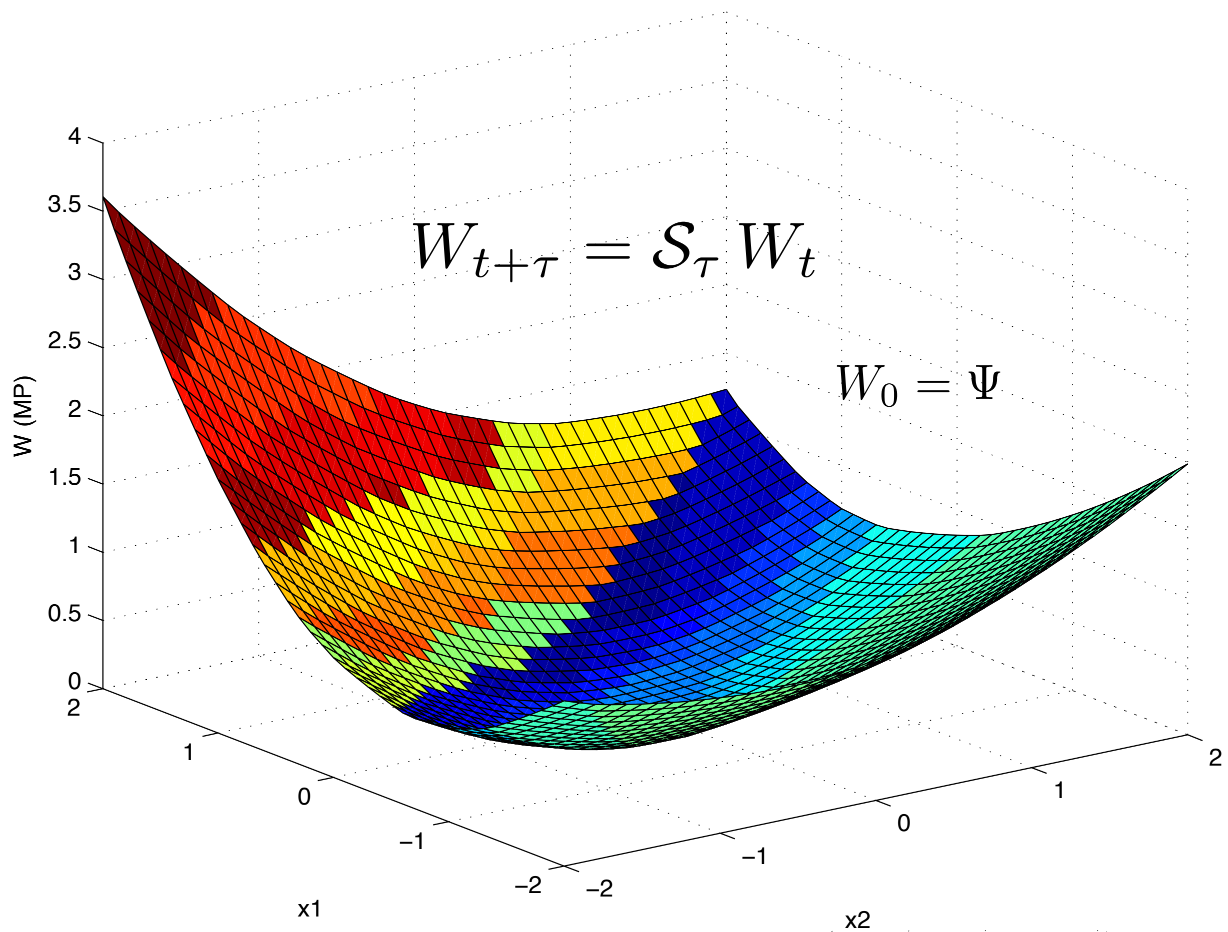
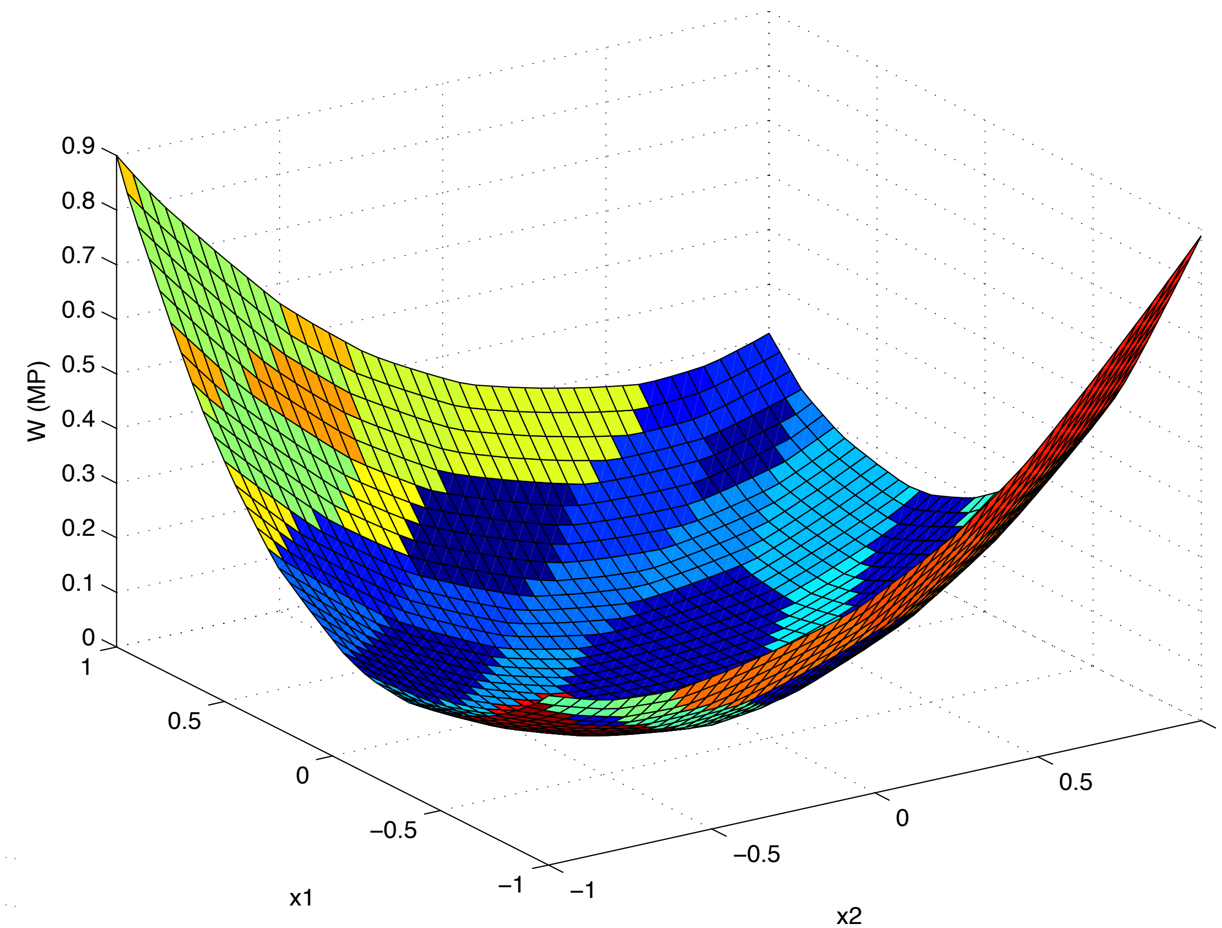
Worst case analysis & optimal control

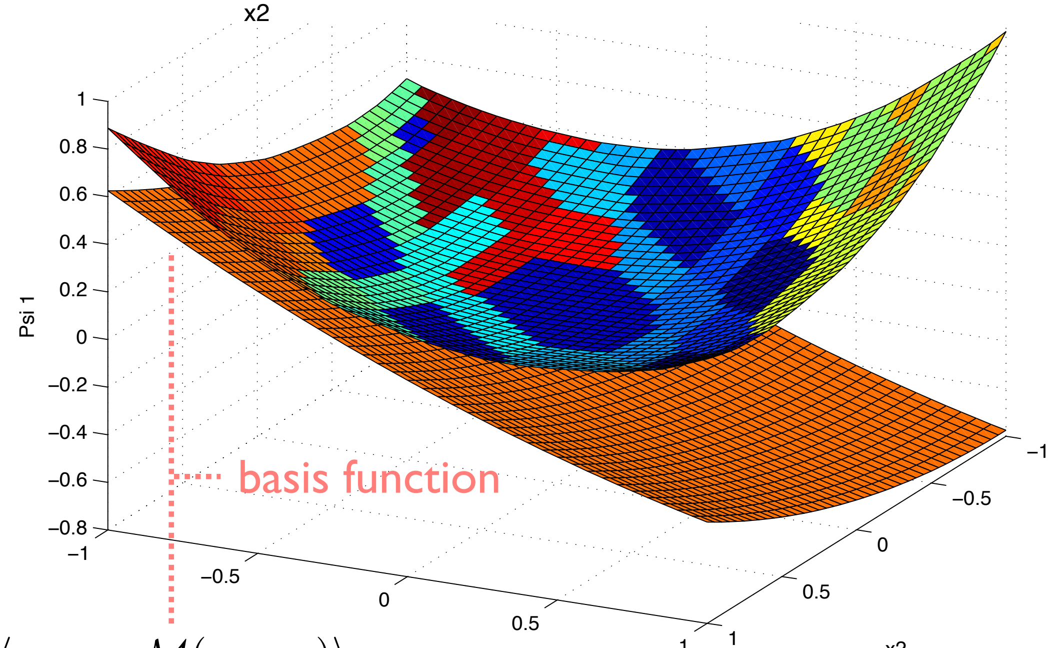


$$W_{t+\tau} = \mathcal{S}_\tau \, W_t$$

$$W_0 = \Psi$$



max-plus representation



basis function

$$\psi_i = \varphi(x, z_i) = \frac{1}{2}\langle x - z_i, \mathcal{M}(x - z_i)\rangle$$

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Fundamental solution semigroups

Value $W_t(x) \doteq [\mathcal{S}_t \, \Psi](x) \doteq \sup_{w \in \mathscr{W}[0,t]} \left\{ \int_0^t \ell(\xi_s) - \tfrac{1}{2} |w_s|^2 \, ds + \Psi(\xi_t) \right\}$ $\quad W_t : \mathscr{X} \to \mathbb{R}^-$

dynamic programming

DPP $W_{t+\tau} \, \Psi = \mathcal{S}_{t+\tau} \, \Psi = \mathcal{S}_\tau \, \mathcal{S}_t \, \Psi = \mathcal{S}_\tau \, W_t \qquad W_0 = \Psi$

Dynamic programming principle

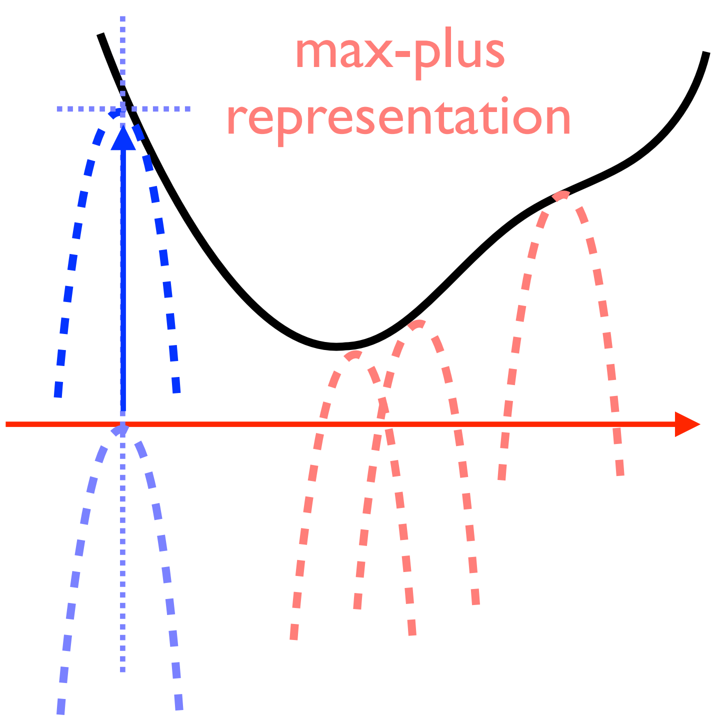
algebraic structure ?

semigroup

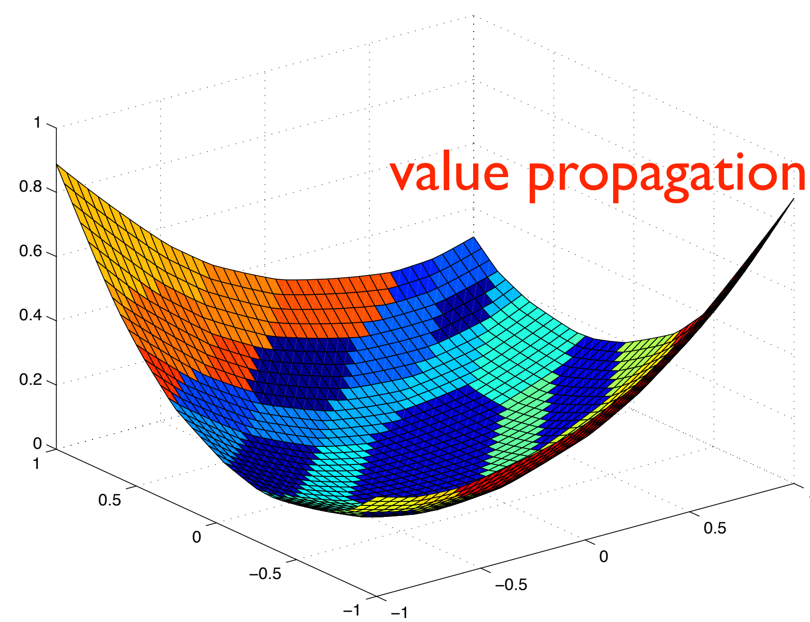
max-plus linearity

semiconvexity preserving

max-plus representation



value propagation



Fundamental solution semigroups

Value $$W_t(x) \doteq [\mathcal{S}_t\, \Psi](x) \doteq \sup_{w \in \mathscr{W}[0,t]} \left\{ \int_0^t \ell(\xi_s) - \tfrac{1}{2}\, |w_s|^2\, ds + \Psi(\xi_t) \right\} \qquad W_t : \mathscr{X} \to \mathbb{R}^-$$

dynamic programming

DPP $$W_{t+\tau}\, \Psi = \mathcal{S}_{t+\tau}\, \Psi = \mathcal{S}_\tau\, \mathcal{S}_t\, \Psi = \mathcal{S}_\tau\, W_t \qquad W_0 = \Psi$$

Dynamic programming principle

algebraic structure ······ semigroup

I — dynamic programming (Lax-Oleinik) semigroup

$$\mathcal{S}_{t+\tau} = \mathcal{S}_\tau\, \mathcal{S}_t \qquad \mathcal{S}_0 = \mathcal{I}$$

Fundamental solution semigroups

Value $$W_t(x) \doteq [\mathcal{S}_t\, \Psi](x) \doteq \sup_{w\in\mathscr{W}[0,t]} \left\{ \int_0^t \ell(\xi_s) - \tfrac{1}{2}\, |w_s|^2\, ds + \Psi(\xi_t) \right\} \qquad W_t : \mathscr{X} \to \mathbb{R}^-$$

dynamic programming

DPP $$W_{t+\tau}\, \Psi = \mathcal{S}_{t+\tau}\, \Psi = \mathcal{S}_\tau\, \mathcal{S}_t\, \Psi = \mathcal{S}_\tau\, W_t \qquad W_0 = \Psi$$

Dynamic programming principle

algebraic structure — semigroup

max-plus linearity

max-plus algebra

$$(\mathbb{R}^-, \oplus, \otimes)$$

$$a \oplus b \doteq \max(a,b) \quad a \otimes b \doteq a + b$$

$$\int_{\mathscr{P}}^{\oplus} f(p)\, dp \doteq \sup_{p\in\mathscr{P}} f(p)$$

2 **DP evolution operator is linear**

$$\mathcal{S}_t(\psi \oplus [c \otimes \phi]) = \mathcal{S}_t\, \psi \oplus [c \otimes \mathcal{S}_t\, \phi]$$

$$[\mathcal{S}_t\, \Psi](x) \equiv \int_{\mathscr{W}[0,t]}^{\oplus} I_t(x,w) \otimes \Psi(\xi_t)\, dw \qquad I_t(x,w) \doteq \int_0^t l(\xi_s) - \tfrac{1}{2}\, |w_s|^2\, ds$$

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Fundamental solution semigroups

Value $$W_t(x) \doteq [\mathcal{S}_t\, \Psi](x) \doteq \sup_{w \in \mathscr{W}[0,t]} \left\{ \int_0^t \ell(\xi_s) - \tfrac{1}{2}\, |w_s|^2\, ds + \Psi(\xi_t) \right\} \qquad W_t : \mathscr{X} \to \mathbb{R}^-$$

dynamic programming

DPP $$W_{t+\tau}\, \Psi = \mathcal{S}_{t+\tau}\, \Psi = \mathcal{S}_\tau\, \mathcal{S}_t\, \Psi = \mathcal{S}_\tau\, W_t \qquad W_0 = \Psi$$

Dynamic programming principle

algebraic structure

- semigroup
- max-plus linearity
- semiconvexity preserving
  - semiconvex functions

$$\mathscr{S}_+^{-\mathcal{M}} \doteq \left\{ f : \mathscr{X} \to \mathbb{R}^- \;\middle|\; \begin{array}{c} f + \frac{1}{2}\langle \cdot, -\mathcal{M}\,\cdot \rangle \\ \text{closed convex} \end{array} \right\}$$

3 semiconvexity preserving

$$\mathcal{S}_t : \mathscr{S}_+^{-\mathcal{M}} \to \mathscr{S}_+^{-\mathcal{M}}$$

Fundamental solution semigroups

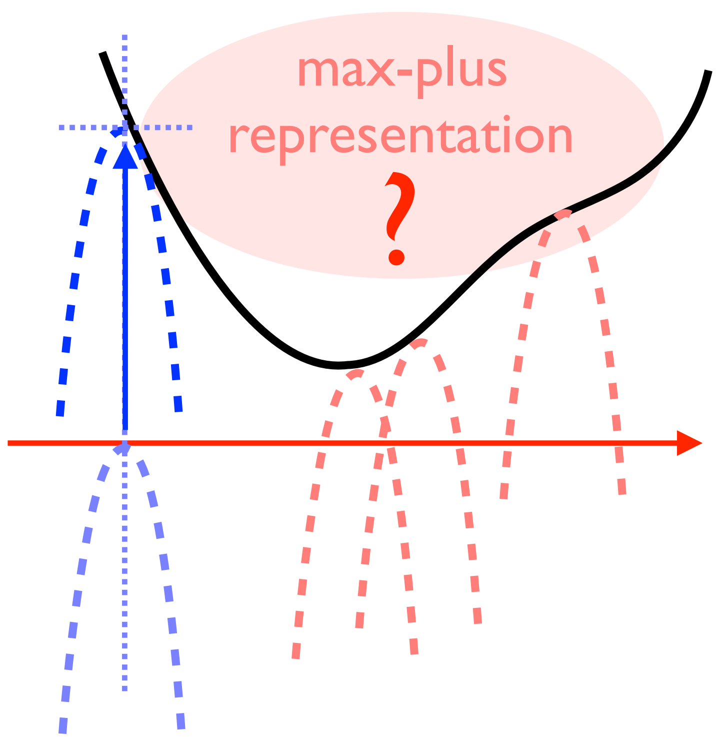
Value $$W_t(x) \doteq [\mathcal{S}_t\,\Psi](x) \doteq \sup_{w\in\mathscr{W}[0,t]} \left\{ \int_0^t \ell(\xi_s) - \tfrac{1}{2}\,|w_s|^2\, ds + \Psi(\xi_t) \right\} \quad W_t : \mathscr{X} \to \mathbb{R}^-$$

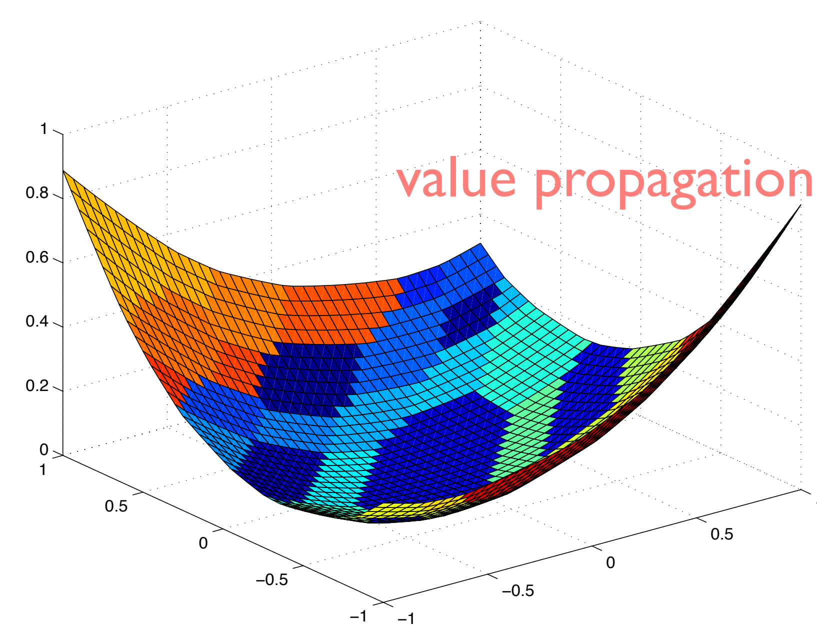
dynamic programming

DPP $$W_{t+\tau}\,\Psi = \mathcal{S}_{t+\tau}\,\Psi = \mathcal{S}_\tau\,\mathcal{S}_t\,\Psi = \mathcal{S}_\tau\,W_t \qquad W_0 = \Psi$$

Dynamic programming principle

algebraic structure
- semigroup
- max-plus linearity
- semiconvexity preserving





Fundamental solution semigroups

3

semiconvexity preserving

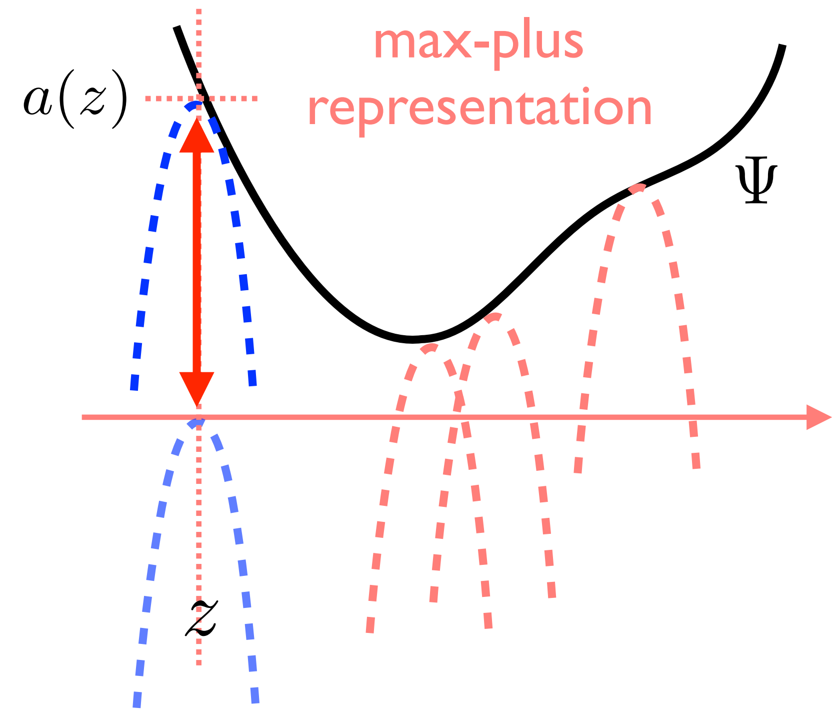
$$\mathcal{S}_t : \mathscr{S}_+^{-\mathcal{M}} \to \mathscr{S}_+^{-\mathcal{M}}$$

$$\Psi \in \mathscr{S}_+^{-\mathcal{M}}$$

$$\mathscr{S}_+^{-\mathcal{M}} \doteq \left\{ f : \mathscr{X} \to \mathbb{R}^- \;\middle|\; \begin{array}{c} f + \frac{1}{2}\langle \cdot, -\mathcal{M}\, \cdot \rangle \\ \text{closed convex} \end{array} \right\}$$

$$\inf_{x \in \mathscr{X}} \{\Psi(x) - \varphi(x,z)\}$$ exists

$$\varphi(\cdot, z) \doteq \tfrac{1}{2}\, \langle \cdot - z, \mathcal{M}(\cdot - z) \rangle$$



Semiconvex duality

$$a(z) = \mathcal{D}_\varphi\, \Psi \doteq -\int_{\mathscr{X}}^{\oplus} \varphi(x,\cdot) \otimes [-\Psi(x)]\, dx \qquad \Psi \in \mathscr{S}_+^{-\mathcal{M}}$$

$$\Psi(x) = \mathcal{D}_\varphi^{-1}\, a(z) \doteq \int_{\mathscr{X}}^{\oplus} \varphi(\cdot, z) \otimes a(z)\, dz \qquad a \in \mathscr{S}_-^{-\mathcal{M}}$$

$$\int_{\mathscr{P}}^{\oplus} f(p)\, dp \doteq \sup_{p \in \mathscr{P}} f(p)$$

max-plus integral operator

semiconcave functions

$$= \sup_{z \in \mathscr{X}} \{\varphi(\cdot, z) + a(z)\}$$

Fundamental solution semigroups

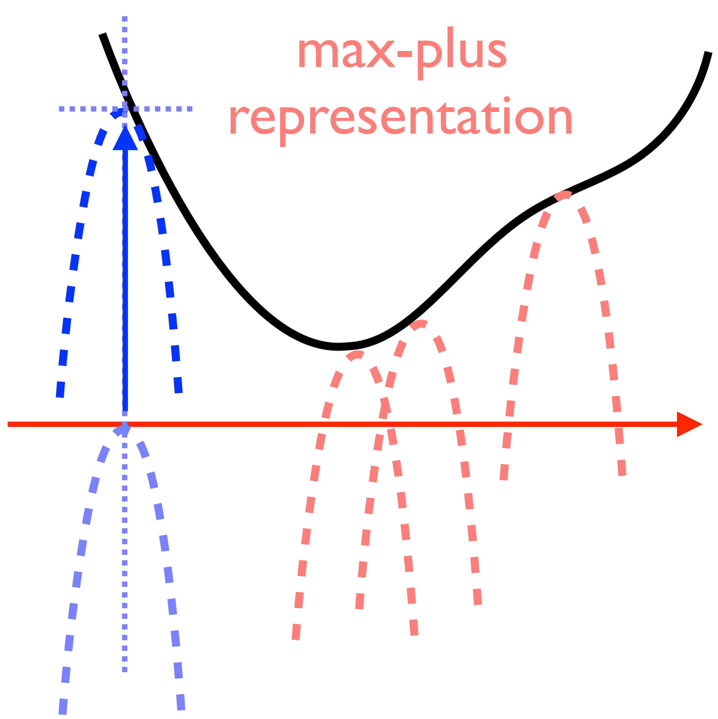
Value $$W_t(x) \doteq [\mathcal{S}_t\, \Psi](x) \doteq \sup_{w \in \mathscr{W}[0,t]} \left\{ \int_0^t \ell(\xi_s) - \tfrac{1}{2}\, |w_s|^2\, ds + \Psi(\xi_t) \right\} \quad W_t : \mathscr{X} \to \mathbb{R}^-$$

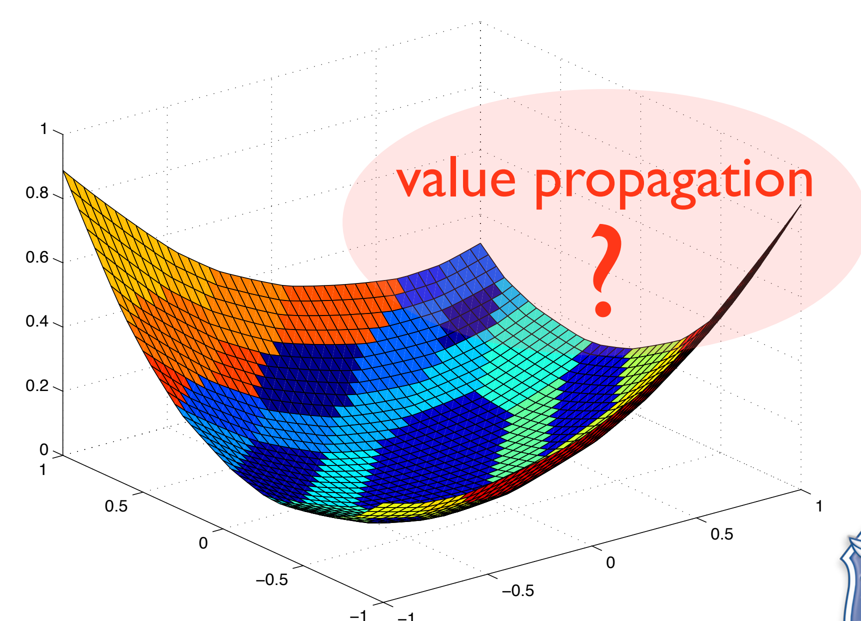
dynamic programming

DPP $$W_{t+\tau}\, \Psi = \mathcal{S}_{t+\tau}\, \Psi = \mathcal{S}_\tau\, \mathcal{S}_t\, \Psi = \mathcal{S}_\tau\, W_t \qquad W_0 = \Psi$$

Dynamic programming principle

algebraic structure
- semigroup
- max-plus linearity
- semiconvexity preserving





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Fundamental solution semigroups

Value propagation via semigroups

$$W_t = \mathcal{S}_t\, \Psi = \mathcal{S}_t\, \mathcal{D}_\varphi^{-1}\, a$$

$\Psi \in \mathscr{S}_+^{-\mathcal{M}}$

$a \doteq \mathcal{D}_\varphi\, \Psi$

composition of a pair of **max-plus integral operators**

$I_t(x,w) \doteq \int_0^t l(\xi_s) - \frac{1}{2}|w_s|^2\, ds$

2 $$[\mathcal{S}_t\, \Psi](x) = \int_{\mathscr{W}[0,t]}^{\oplus} I_t(x,w) \otimes \Psi(\xi_t)\, dw$$

sup over the input space

$$[\mathcal{D}_\varphi^{-1}\, a](x) = \int_{\mathscr{X}}^{\oplus} \varphi(x,z) \otimes a(z)\, dz$$ 3

sup over the state space

$$[\mathcal{S}_t\, \mathcal{D}_\varphi^{-1}\, a](x) = \int_{\mathscr{W}[0,t]}^{\oplus} I_t(x,w) \otimes \int_{\mathscr{X}}^{\oplus} \varphi(\xi_t, z) \otimes a(z)\, dz\, dw$$

swap sup order

$$[\mathcal{S}_t\, \mathcal{D}_\varphi^{-1}\, a](x) = \int_{\mathscr{X}}^{\oplus} [\mathcal{S}_t\, \varphi(\cdot, z)](x) \otimes a(z)\, dz$$

2 3 integral operator over **the state space**

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Fundamental solution semigroups

Value propagation via semigroups

$$\mathcal{S}_t\, \mathcal{D}_\varphi^{-1}$$

$$[\mathcal{S}_t\, \mathcal{D}_\varphi^{-1}\, a](x) = \int_{\mathscr{X}}^{\oplus} [\mathcal{S}_t\, \varphi(\cdot, z)](x) \otimes a(z)\, dz$$

2 3

integral operator over the state space

$$[\mathcal{D}_\varphi^{-1}\, a](x) = \int_{\mathscr{X}}^{\oplus} \varphi(x,z) \otimes a(z)\, dz$$

decompose

$$\mathcal{S}_t\, \mathcal{D}_\varphi^{-1} = \mathcal{G}_t^{\oplus}\, \mathcal{D}_\varphi^{-1}$$

$$\mathcal{S}_t\, \mathcal{D}_\varphi^{-1} = \mathcal{D}_\varphi^{-1}\, \mathcal{B}_t^{\oplus}$$

integral operators over the state space

$$\mathcal{G}_t^{\oplus}\, \mathcal{D}_\varphi^{-1}$$

$$\mathcal{D}_\varphi^{-1}\, \mathcal{B}_t^{\oplus}$$

integral operators over the state space

max-plus primal space fundamental solution

$$\mathcal{G}_t^{\oplus}\, \psi \doteq \int_{\mathscr{X}}^{\oplus} G_t(\cdot, y) \otimes \psi(y)\, dy$$

$$G_t(x,y) = [\mathcal{D}_\varphi\, S_t(x, \cdot)](y)$$

max-plus dual space fundamental solution

$$\mathcal{B}_t^{\oplus}\, a \doteq \int_{\mathscr{X}}^{\oplus} B_t(\cdot, z) \otimes a(z)\, dz$$

$$B_t(y,z) = [\mathcal{D}_\varphi\, S_t(\cdot, z)](y)$$

where $S_t(x,y) = [\mathcal{S}_t\, \varphi(\cdot, y)](x)$

Fundamental solution semigroups

Value
propagation via semigroups

$$\mathcal{S}_t \, \mathcal{D}_\varphi^{-1}$$

2 3

integral operator over the **state space**

$\mathcal{S}_t \, \mathcal{D}_\varphi^{-1} = \mathcal{G}_t^\oplus \, \mathcal{D}_\varphi^{-1}$ — **decompose** — $\mathcal{S}_t \, \mathcal{D}_\varphi^{-1} = \mathcal{D}_\varphi^{-1} \, \mathcal{B}_t^\oplus$

$$\mathcal{S}_t = \mathcal{G}_t^\oplus \qquad\qquad \mathcal{S}_t = \mathcal{D}_\varphi^{-1} \, \mathcal{B}_t^\oplus \, \mathcal{D}_\varphi$$

1

DP

$$\mathcal{S}_{t+\tau} = \mathcal{S}_\tau \, \mathcal{S}_t$$

Max-plus
fundamental solution semigroups

max-plus primal space
fundamental solution semigroup

$$\mathcal{G}_{t+\tau}^\oplus = \mathcal{G}_\tau^\oplus \, \mathcal{G}_t^\oplus$$

max-plus dual space
fundamental solution semigroup

$$\mathcal{B}_{t+\tau}^\oplus = \mathcal{B}_\tau^\oplus \, \mathcal{B}_t^\oplus$$

semigroups of max-plus integral operators over the **state space**

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Idempotent methods

Value $$W_t(x) \doteq [\mathcal{S}_t\, \Psi](x) \doteq \sup_{w \in \mathscr{W}[0,t]} \left\{ \int_0^t \ell(\xi_s) - \tfrac{1}{2}\,|w_s|^2\, ds + \Psi(\xi_t) \right\} \quad W_t : \mathscr{X} \to \mathbb{R}^-$$

max-plus fundamental solution semigroups

e.g. $$\mathcal{B}^{\oplus}_{t+\tau} = \mathcal{B}^{\oplus}_{\tau}\, \mathcal{B}^{\oplus}_{t}$$

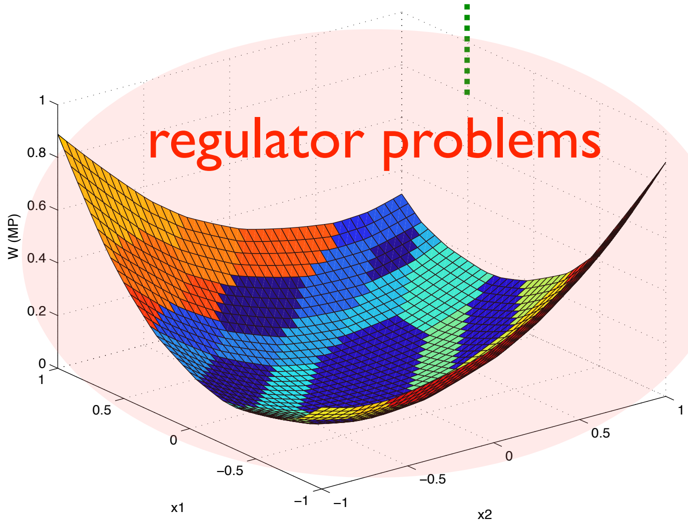
$$\mathcal{B}^{\oplus}_t\, a \doteq \int^{\oplus}_{\mathscr{X}} B_t(\cdot, z) \otimes a(z)\, dz$$

kernel approximation

kernel structure

curse-of-dimensionality free problems

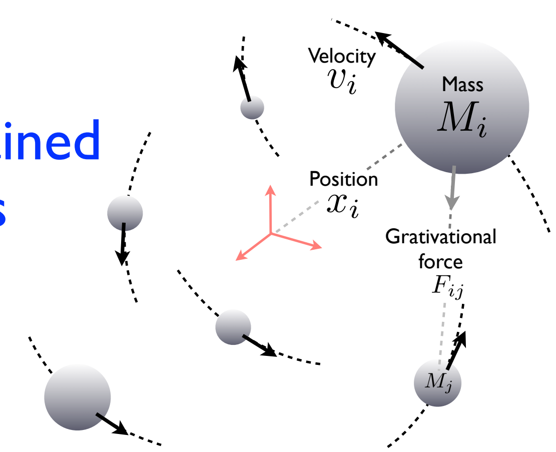
regulator problems



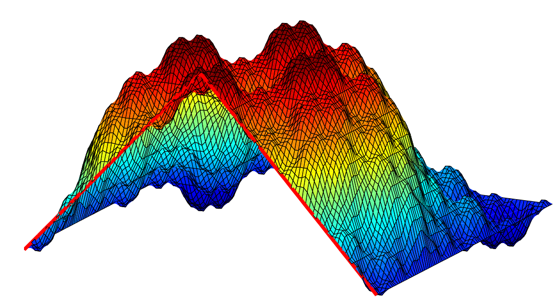
linear non-quadratic regulator problems

state constrained problems

gravitational N-body



wave equation groups



Kernel approximation — regulator problems

Value $W(x) \doteq \lim_{t\to\infty} [\mathcal{S}_t\, \Psi_0](x)$ $\qquad [\mathcal{S}_t\, \Psi](x) \doteq \sup_{w\in\mathscr{W}[0,t]} \left\{ \int_0^t \ell(\xi_s) - \frac{\gamma^2}{2}|w_s|^2\, ds + \Psi(\xi_t) \right\}$ $\qquad \Psi_0(x) \doteq 0$

dual space fundamental solution $\quad \mathcal{S}_t = \mathcal{D}_\varphi^{-1}\, \mathcal{B}_t^\oplus\, \mathcal{D}_\varphi$

[McEneaney, 2006]

$$\mathcal{B}_t^\oplus\, a \doteq \int_{\mathscr{X}}^\oplus B_t(\cdot, z) \otimes a(z)\, dz$$

kernel

$$B_\tau(y,z) \doteq [\mathcal{D}_\varphi S_\tau(\cdot, z)](y)$$

$$S_\tau(x,y) \doteq [\mathcal{S}_\tau\, \varphi(\cdot, y)](x)$$

τ small

evaluated via a Taylor series

e.g. grid $\{z_i\}_{i\in\mathbb{N}}$

basis for semiconvex fns

$$\psi_i(x) \doteq \varphi(x, z_i) = \tfrac{1}{2}\langle x - z_i, \mathcal{M}(x - z_i)\rangle$$

kernel propagation

$$\widehat{B}_{(k+1)\tau} \doteq \widehat{B}_\tau \otimes \widehat{B}_{k\tau} \qquad [\widehat{B}_\tau]_{ij} \doteq B_\tau(z_i, z_j)$$

$k \in \mathbb{N}$

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Kernel approximation — regulator problems

Value $W(x) \doteq \lim_{t\to\infty} [\mathcal{S}_t\, \Psi_0](x) \qquad [\mathcal{S}_t\, \Psi](x) \doteq \sup_{w\in\mathscr{W}[0,t]} \left\{ \int_0^t \ell(\xi_s) - \frac{\gamma^2}{2}|w_s|^2\, ds + \Psi(\xi_t) \right\} \qquad \Psi_0(x) \doteq 0$

kernel propagation

$$\widehat{B}_{(k+1)\tau} \doteq \widehat{B}_\tau \otimes \widehat{B}_{k\tau} \qquad [\widehat{B}_\tau]_{ij} \doteq B_\tau(z_i, z_j)$$

$$\mathcal{S}_{k\tau}\Psi_0 = \mathcal{D}_\varphi^{-1} \mathcal{B}_{k\tau}\, \widehat{a}_0$$

$$\widehat{a}_k \doteq \widehat{B}_{k\tau} \otimes \widehat{a}_0 \qquad [\widehat{a}_0]_i \doteq (\mathcal{D}_\varphi\, \Psi_0)(z_i)$$

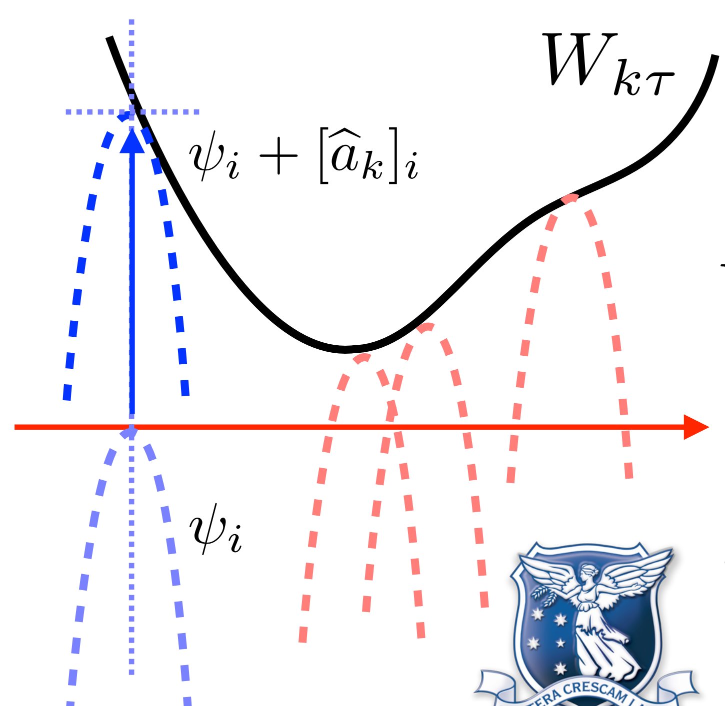
$$\psi_i(x) \doteq \varphi(x, z_i) = \tfrac{1}{2}\langle x - z_i, \mathcal{M}(x - z_i)\rangle$$

max-plus power method

$$\widehat{a}_{k+1} = \widehat{B}_\tau \otimes \widehat{a}_k \quad k \in \mathbb{Z}_{\ge 0}$$

$$W_{k\tau} = \bigoplus_{i\in\mathbb{N}} \psi_i \otimes [\widehat{a}_k]_i$$

$$W_{k\tau} \approx \bigoplus_{i=1}^{\nu} \psi_i \otimes [\widehat{a}_k]_i \qquad \nu \in \mathbb{N}$$



$$W = \lim_{k\to\infty} W_{k\tau}$$

Kernel approximation — regulator problems

max-plus power method

$$\widehat{a}_{k+1} = \widehat{B}_\tau \otimes \widehat{a}_k \quad k \in \mathbb{Z}_{\geq 0}$$

$$W_{k\tau} = \bigoplus_{i \in \mathbb{N}} \psi_i \otimes [\widehat{a}_k]_i$$

$$W_{k\tau} \approx \bigoplus_{i=1}^{\nu} \psi_i \otimes [\widehat{a}_k]_i \qquad \nu \in \mathbb{N}$$

$$[\widehat{B}_\tau]_{ij} \doteq B_\tau(z_i, z_j) \qquad [\widehat{a}_0]_i \doteq (\mathcal{D}_\varphi \, \Psi_0)(z_i)$$

$$B_\tau(y, z) \doteq [\mathcal{D}_\varphi S_\tau(\cdot, z)](y)$$

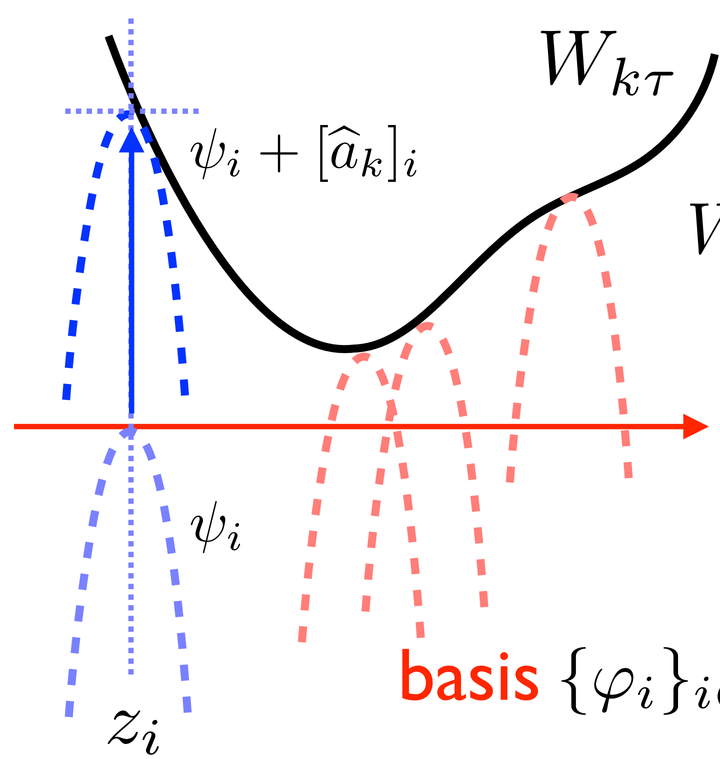
$$S_\tau(x, y) \doteq [\mathcal{S}_\tau \, \varphi(\cdot, y)](x)$$

How to **select basis** for kernel approximation?

$$\psi_i(x) \doteq \varphi(x, z_i) = \tfrac{1}{2}\langle x - z_i, \mathcal{M}(x - z_i)\rangle$$

Hessian?

location? $\{z_i\}_{i \in \mathbb{N}_{\leq \nu}}$



$$W = \lim_{k \to \infty} W_{k\tau}$$

basis $\{\varphi_i\}_{i \in \mathbb{N}_{\leq \nu}}$

$\nu \in \mathbb{N}$

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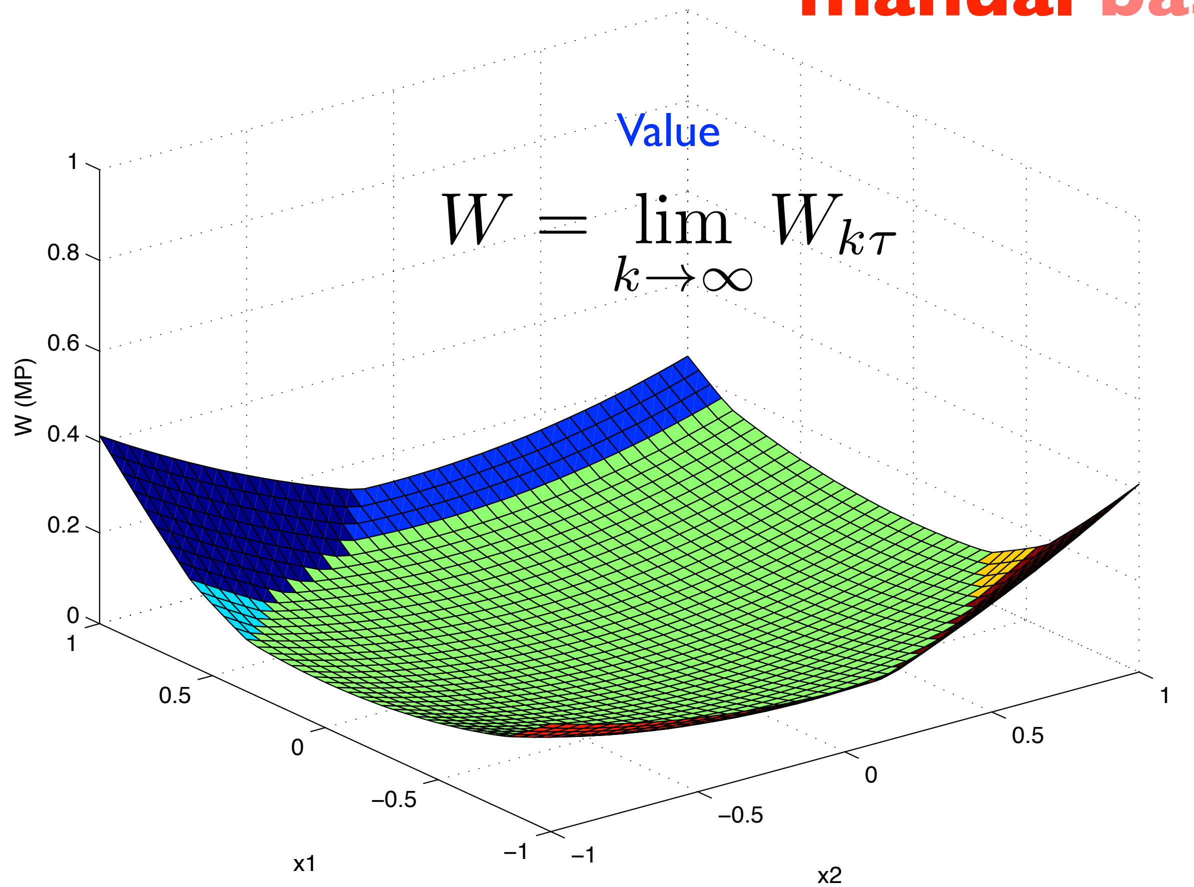
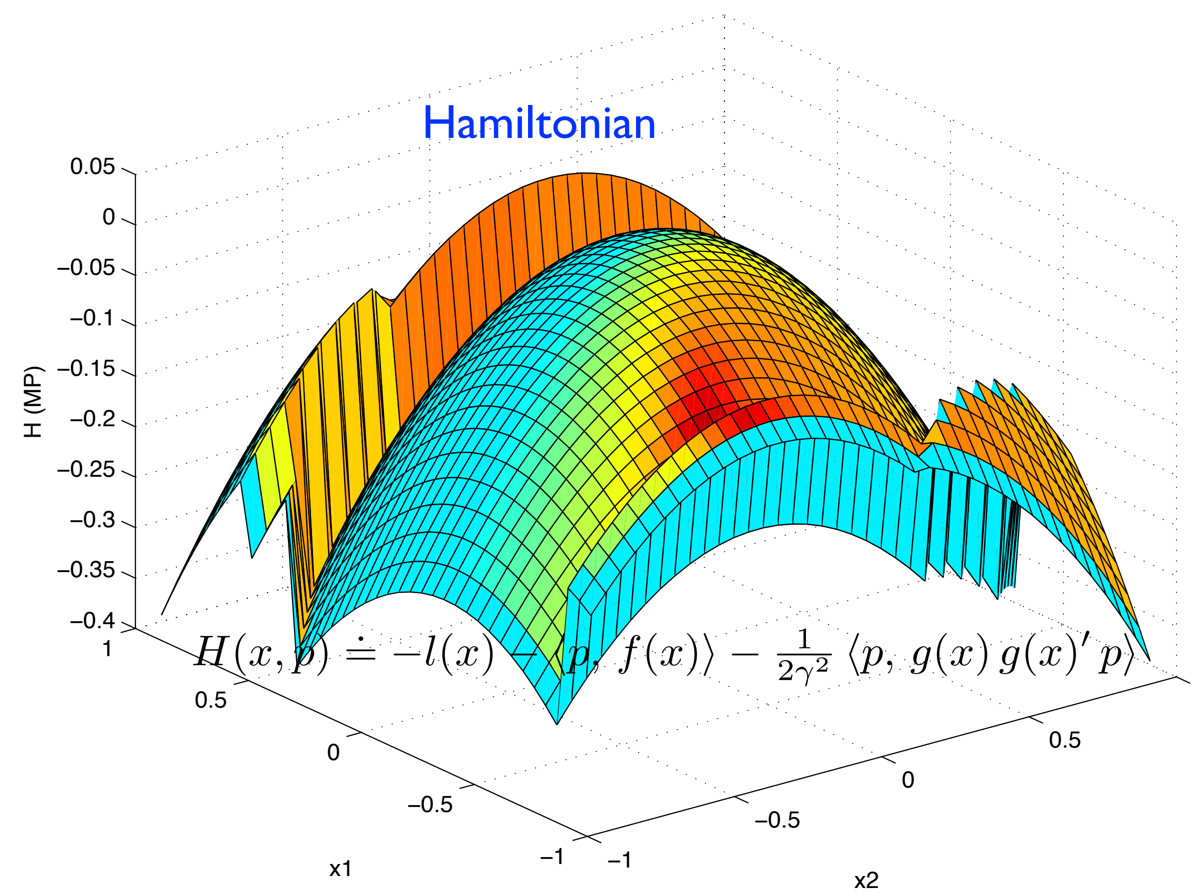
manual basis selection



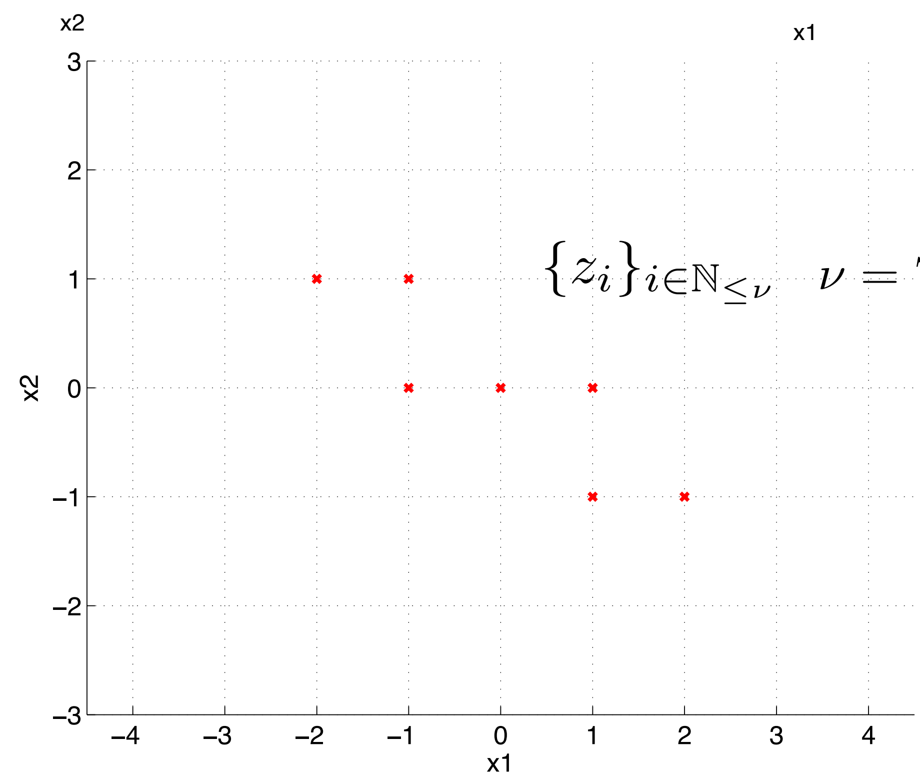


linear dynamics

$$A \doteq \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \quad B \doteq \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$$

$$l(\xi) \doteq \tfrac{1}{2}\,|\xi|^2 \qquad \gamma \doteq 2$$

$$\mathcal{M} \doteq 0.2\,\mathcal{I}$$



7 basis functions

$$\psi_i = \varphi(x, z_i) = \tfrac{1}{2}\langle x - z_i, \mathcal{M}(x - z_i)\rangle$$

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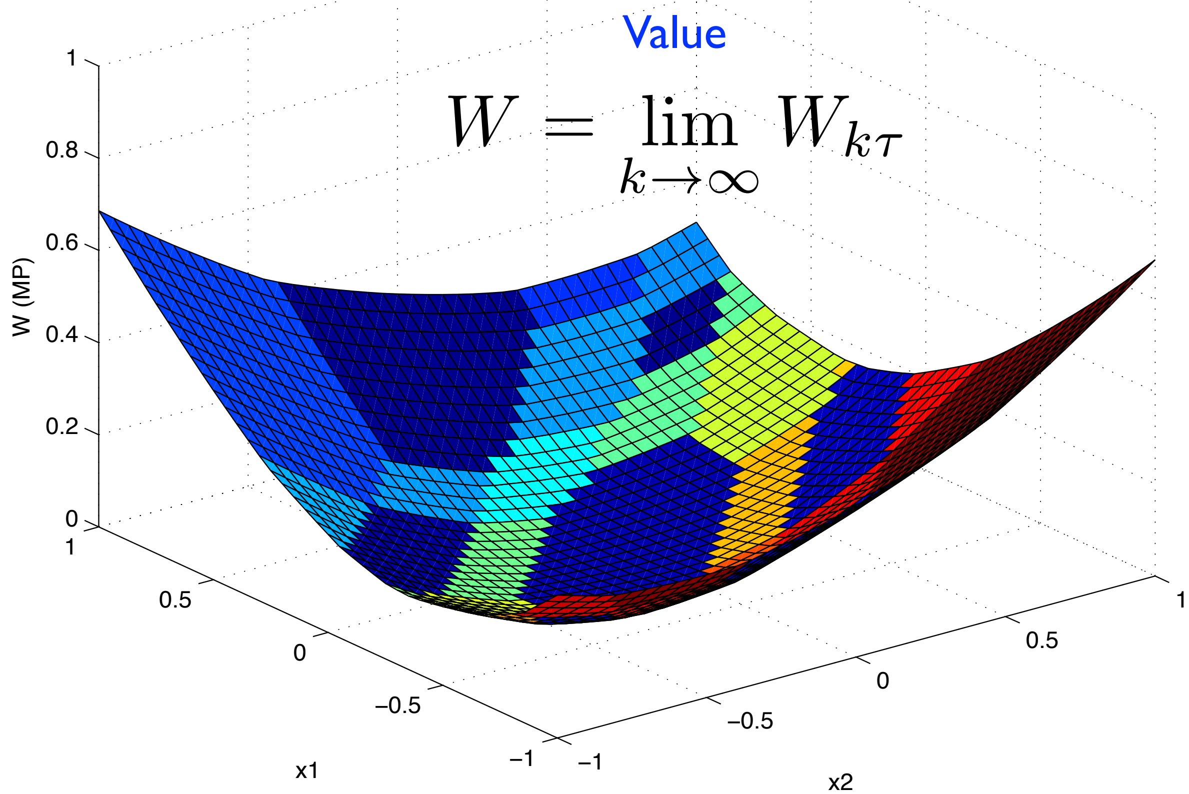
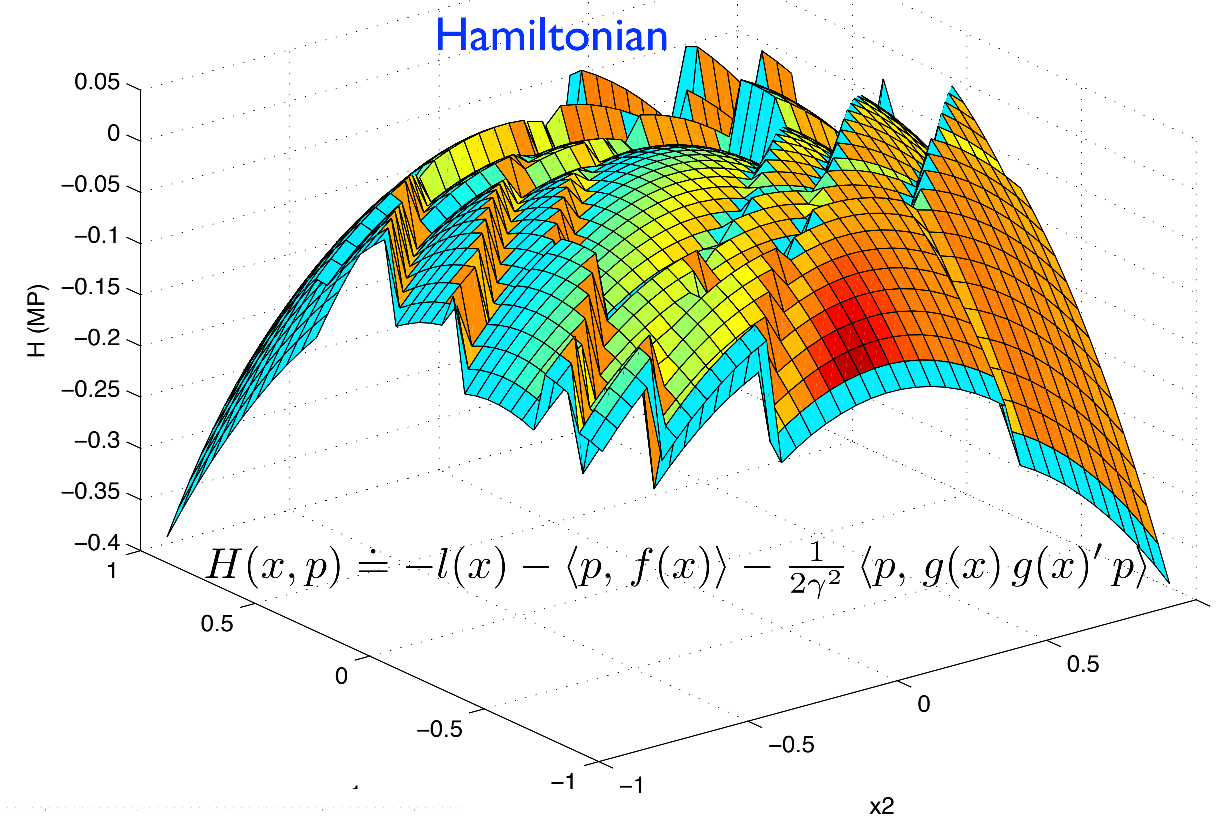
Kernel approximation — regulator problems

manual basis selection





linear dynamics

$$A \doteq \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \quad B \doteq \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$$

$$l(\xi) \doteq \tfrac{1}{2}\,|\xi|^2 \qquad \gamma \doteq 2$$

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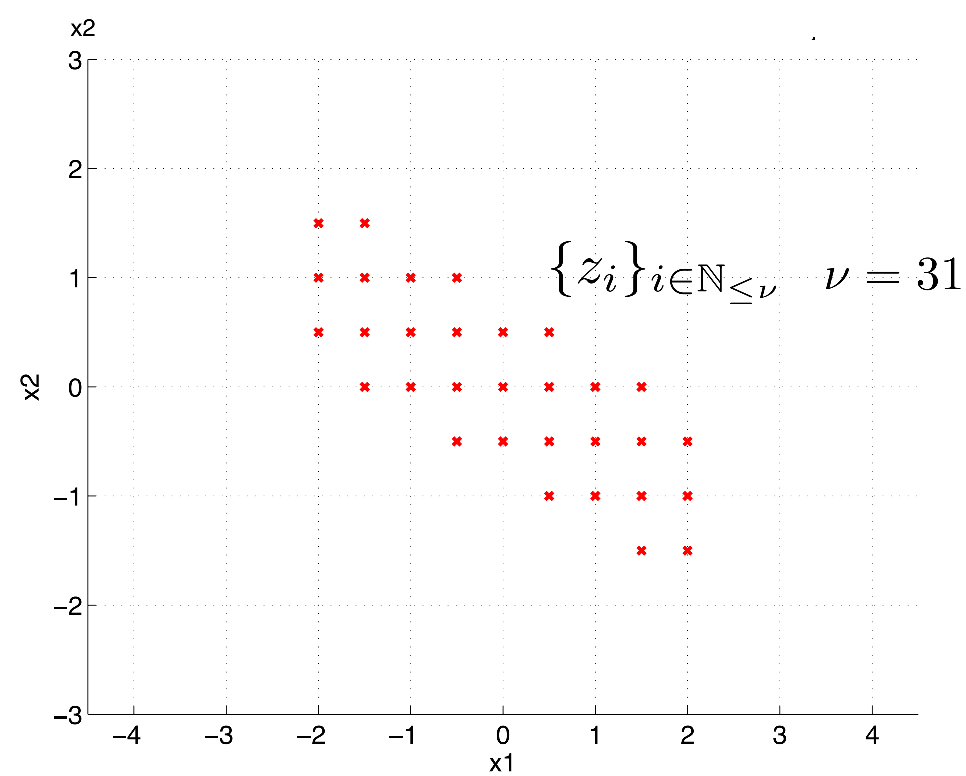

31 basis functions

$$\psi_i = \varphi(x, z_i) = \tfrac{1}{2}\langle x - z_i, \mathcal{M}(x - z_i)\rangle$$

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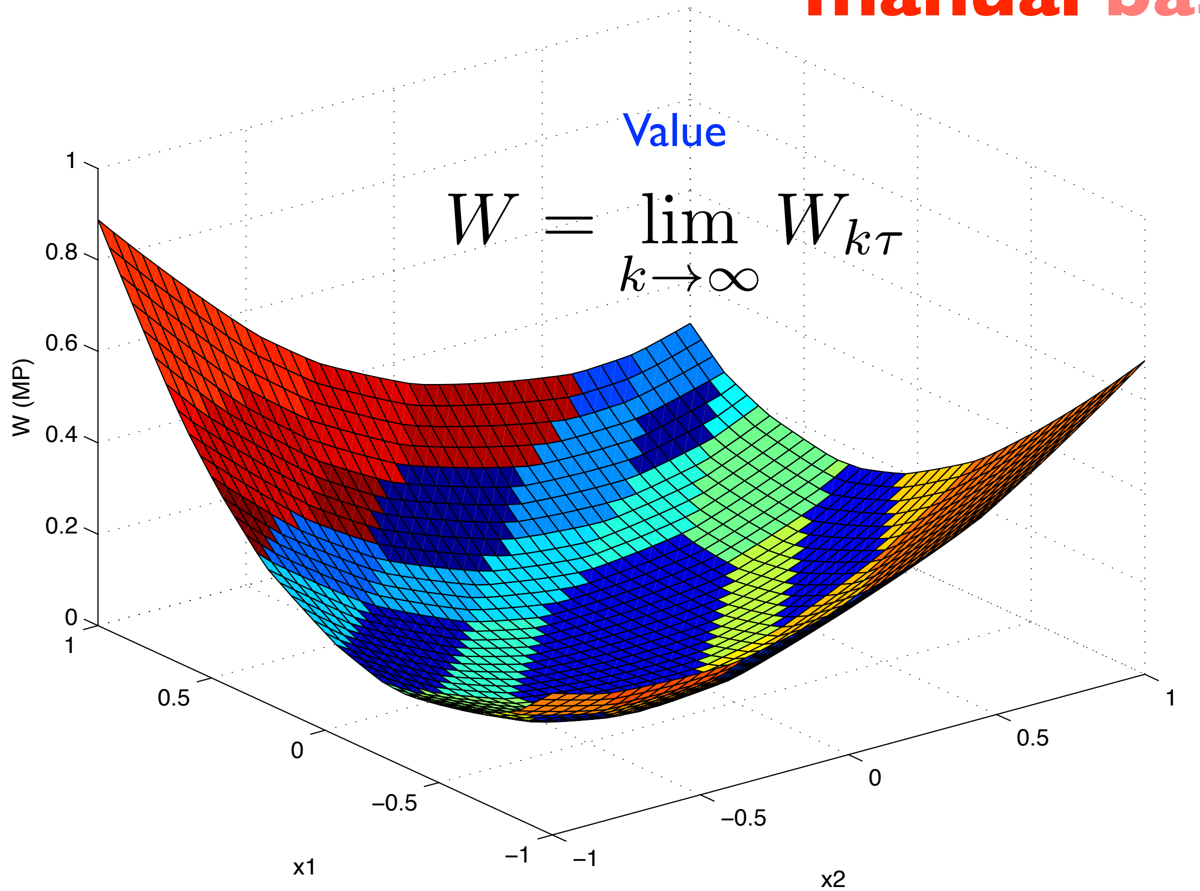
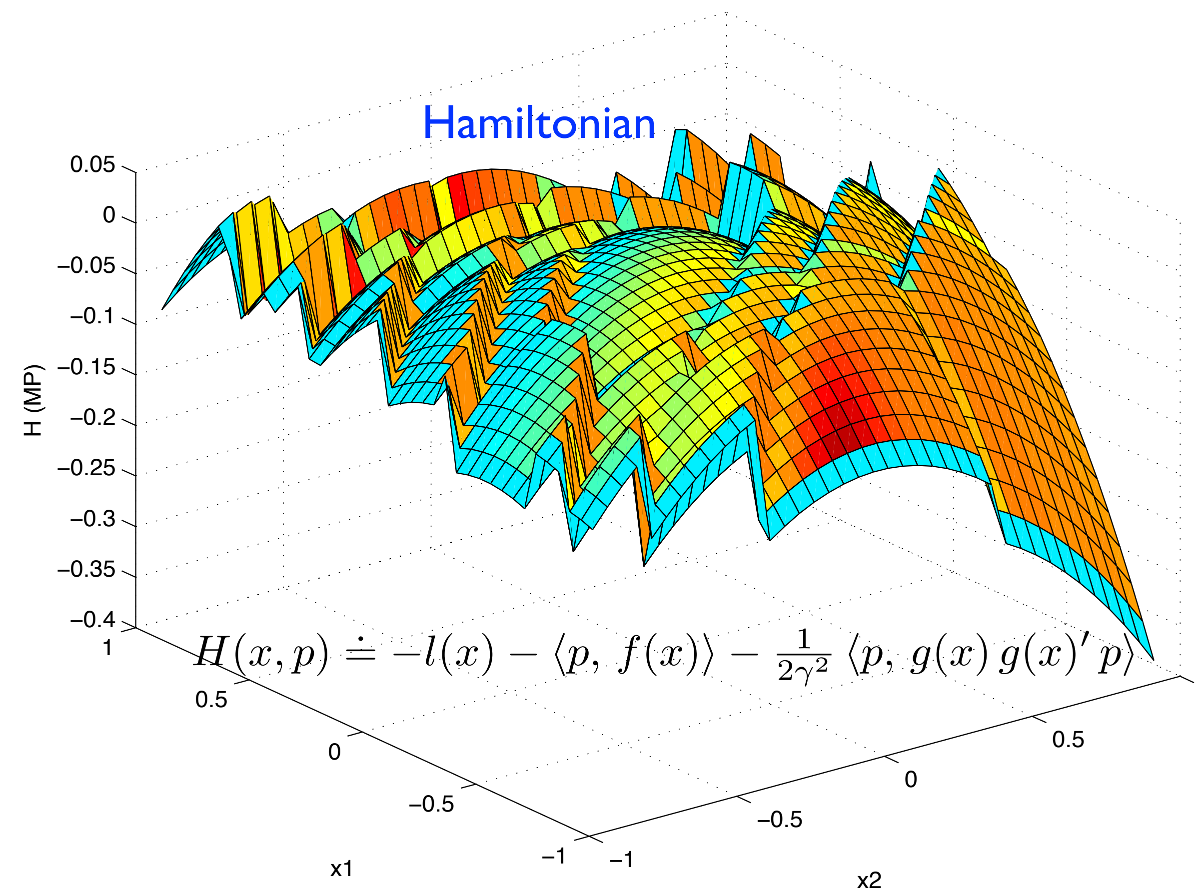
Kernel approximation — regulator problems

manual basis selection



$$W = \lim_{k \to \infty} W_{k\tau}$$



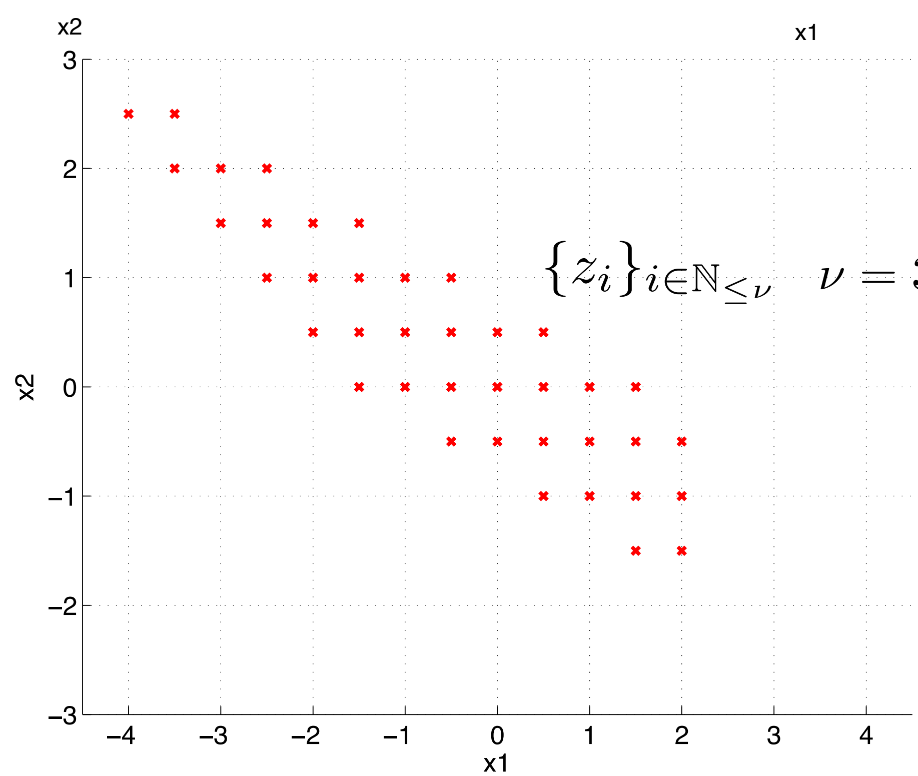
$$H(x,p) \doteq -l(x) - \langle p,\, f(x) \rangle - \frac{1}{2\gamma^2} \langle p,\, g(x)\, g(x)'\, p \rangle$$

linear dynamics

$$A \doteq \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix} \quad B \doteq \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$$

$$l(\xi) \doteq \tfrac{1}{2} |\xi|^2 \qquad \gamma \doteq 2$$

$$\mathcal{M} \doteq 0.2 \mathcal{I}$$



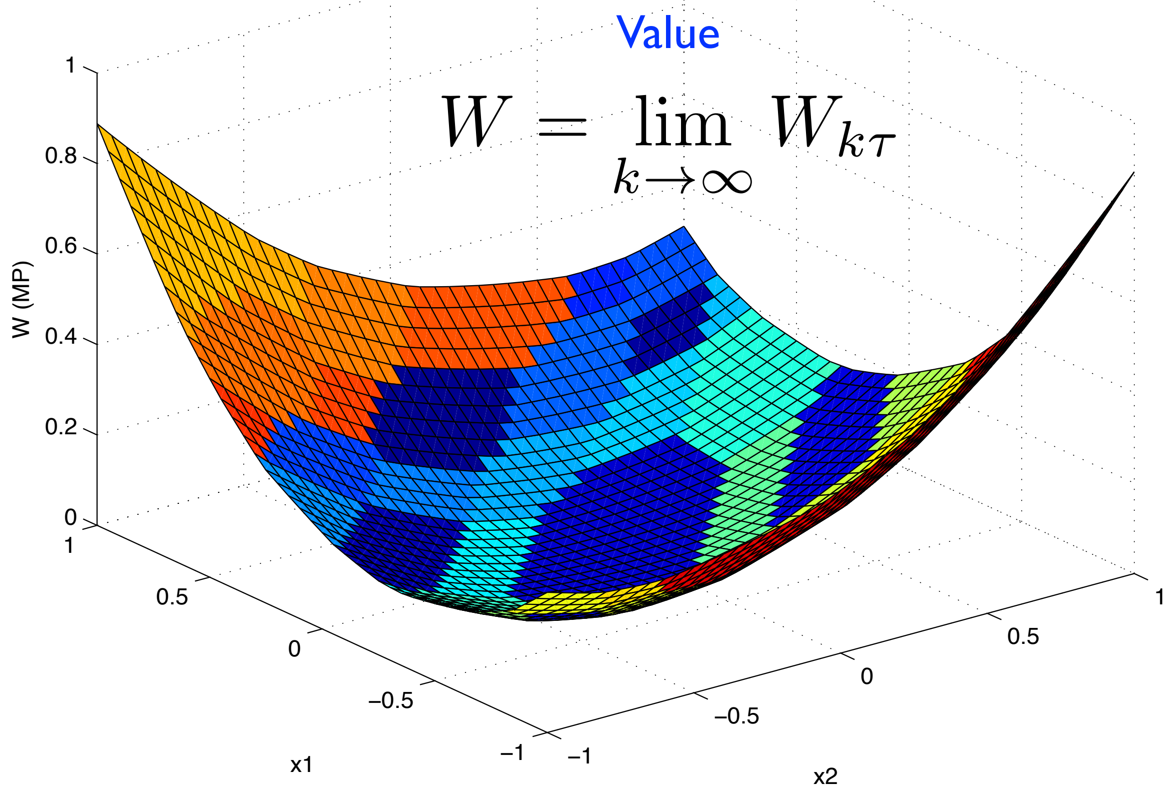
$$\{z_i\}_{i \in \mathbb{N}_{\le \nu}} \quad \nu = 39$$

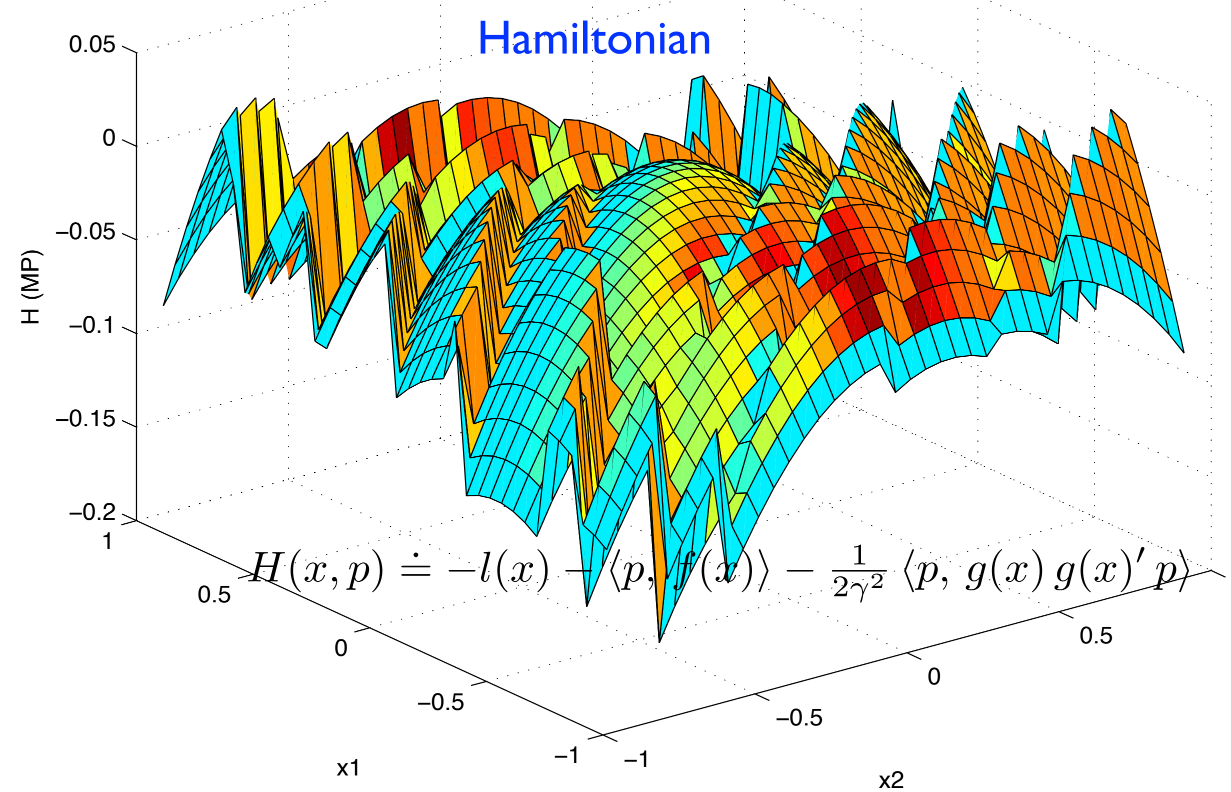
39 basis functions

$$\psi_i = \varphi(x, z_i) = \tfrac{1}{2} \langle x - z_i, \mathcal{M}(x - z_i) \rangle$$

Kernel approximation — regulator problems

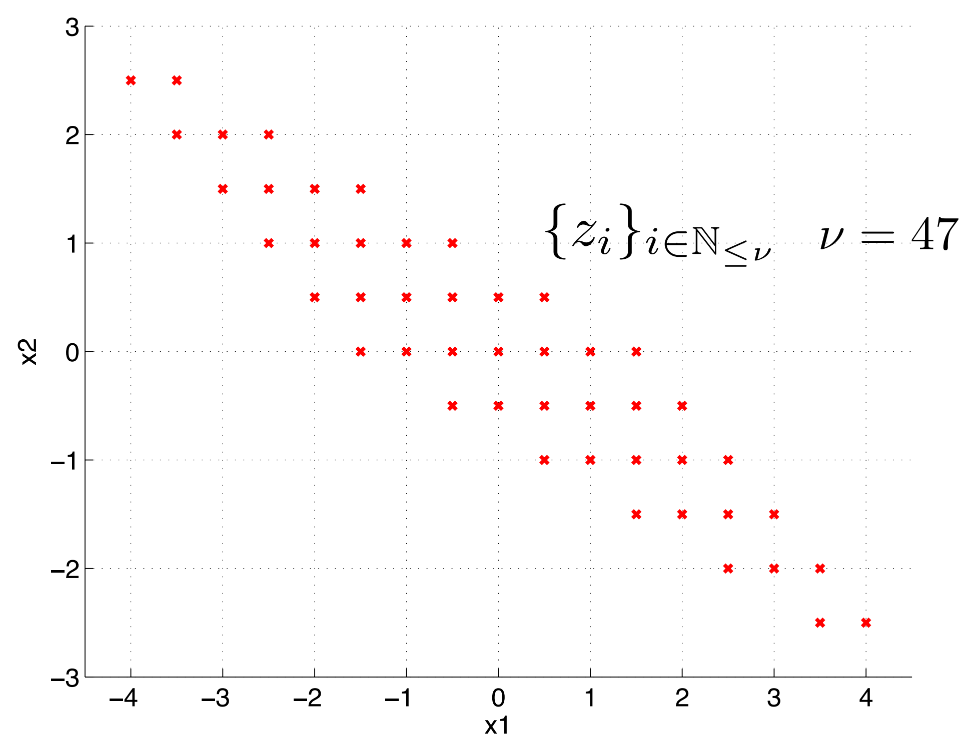
manual basis selection





linear dynamics

automated basis selection?



47 basis functions

$$\psi_i = \varphi(x, z_i) = \tfrac{1}{2}\langle x - z_i, \mathcal{M}(x - z_i)\rangle$$

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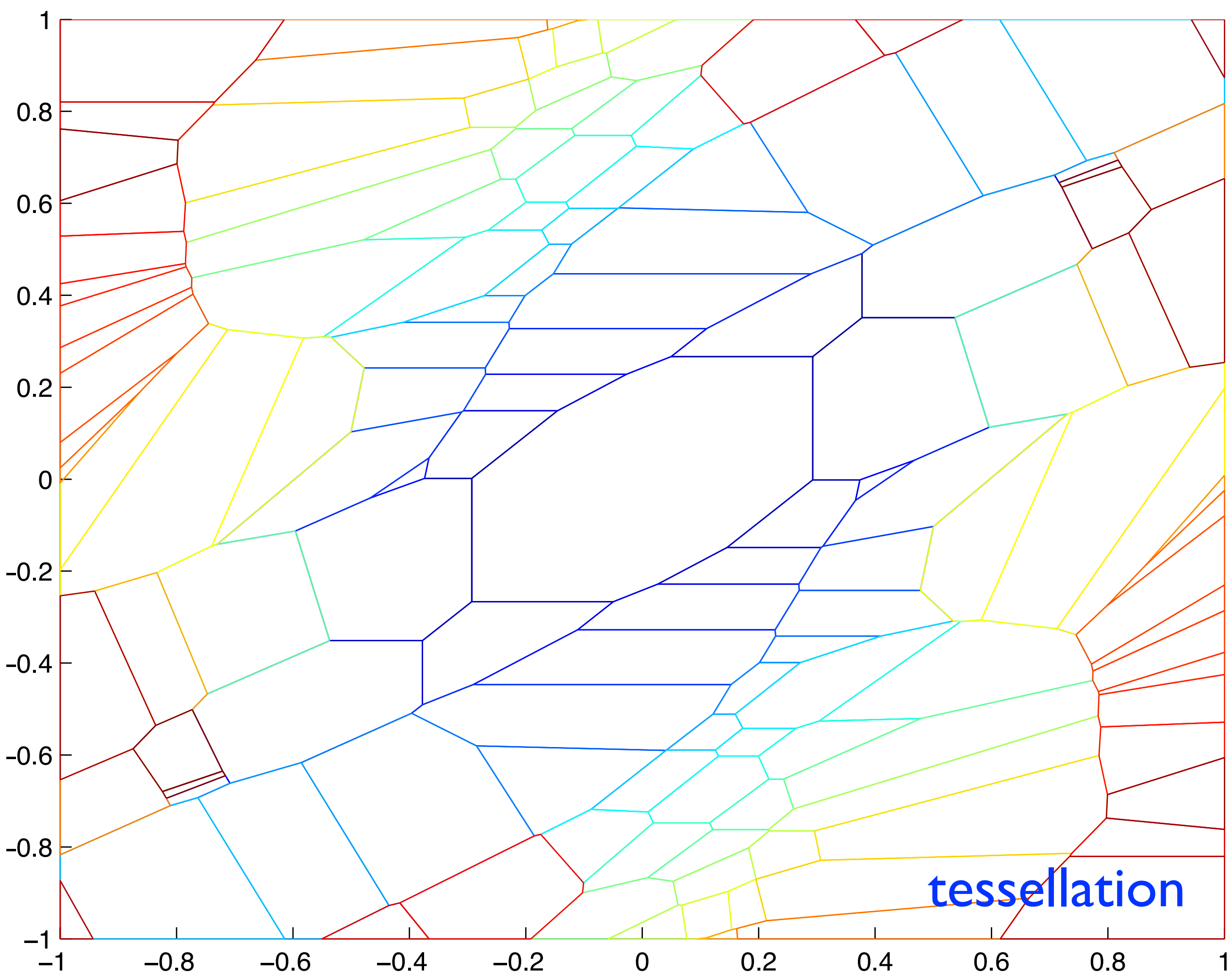
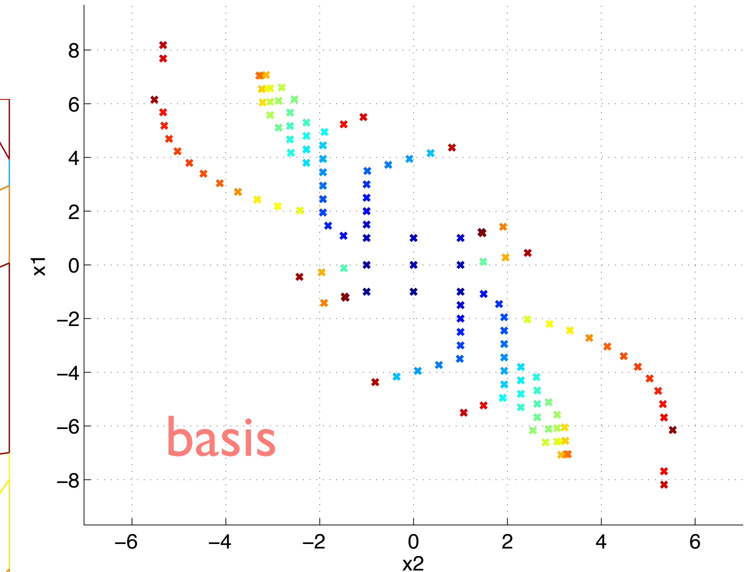
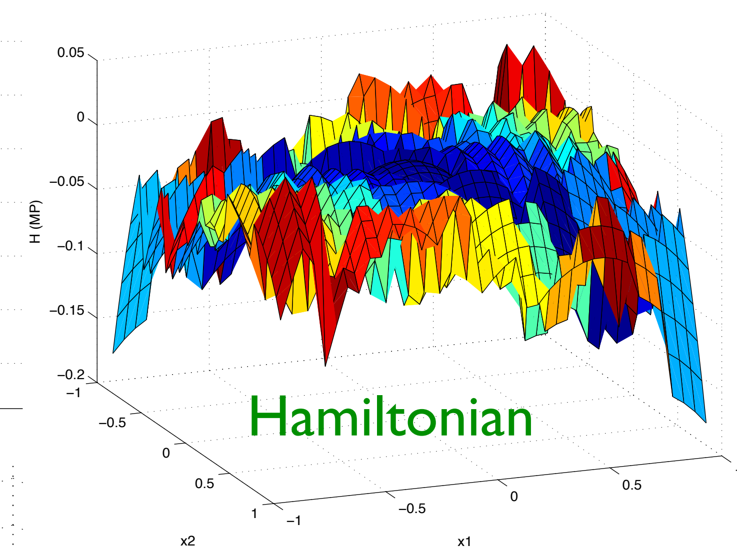
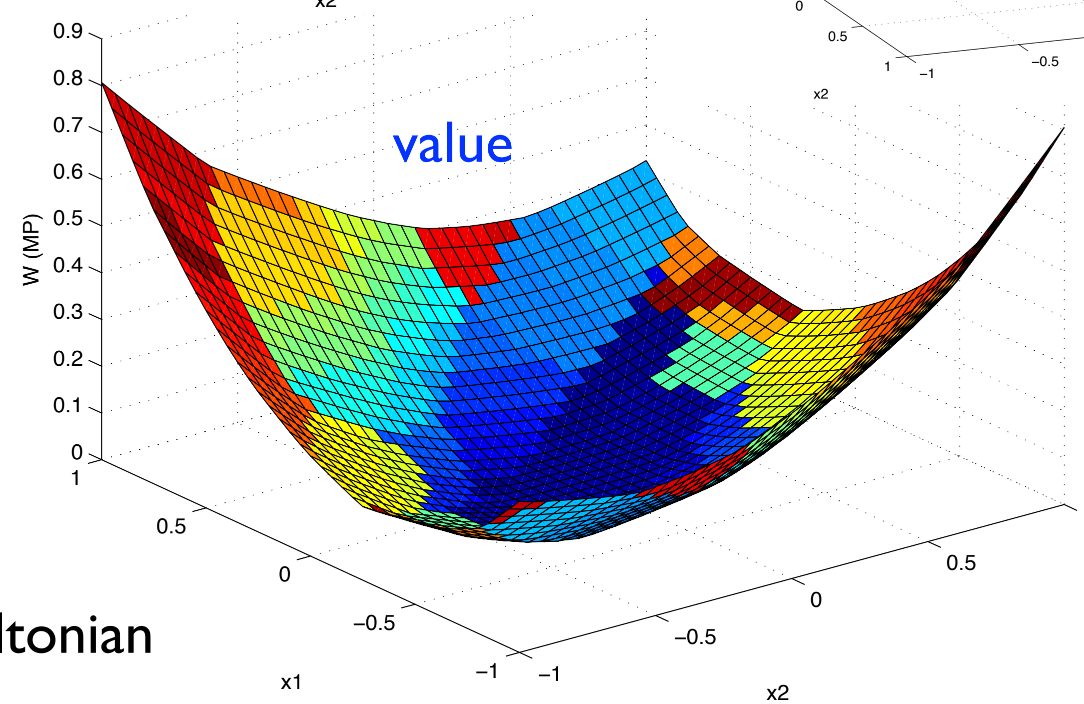
automated basis selection

1 **Tesselate** the state space via $W \approx \bigoplus_{i=1}^{\nu} \psi_i \otimes [\widehat{a}_\infty]_i$ (for fixed basis Hessian $\mathcal{M}$)









2 **Sort** the convex polytopes by worst-case Hamiltonian

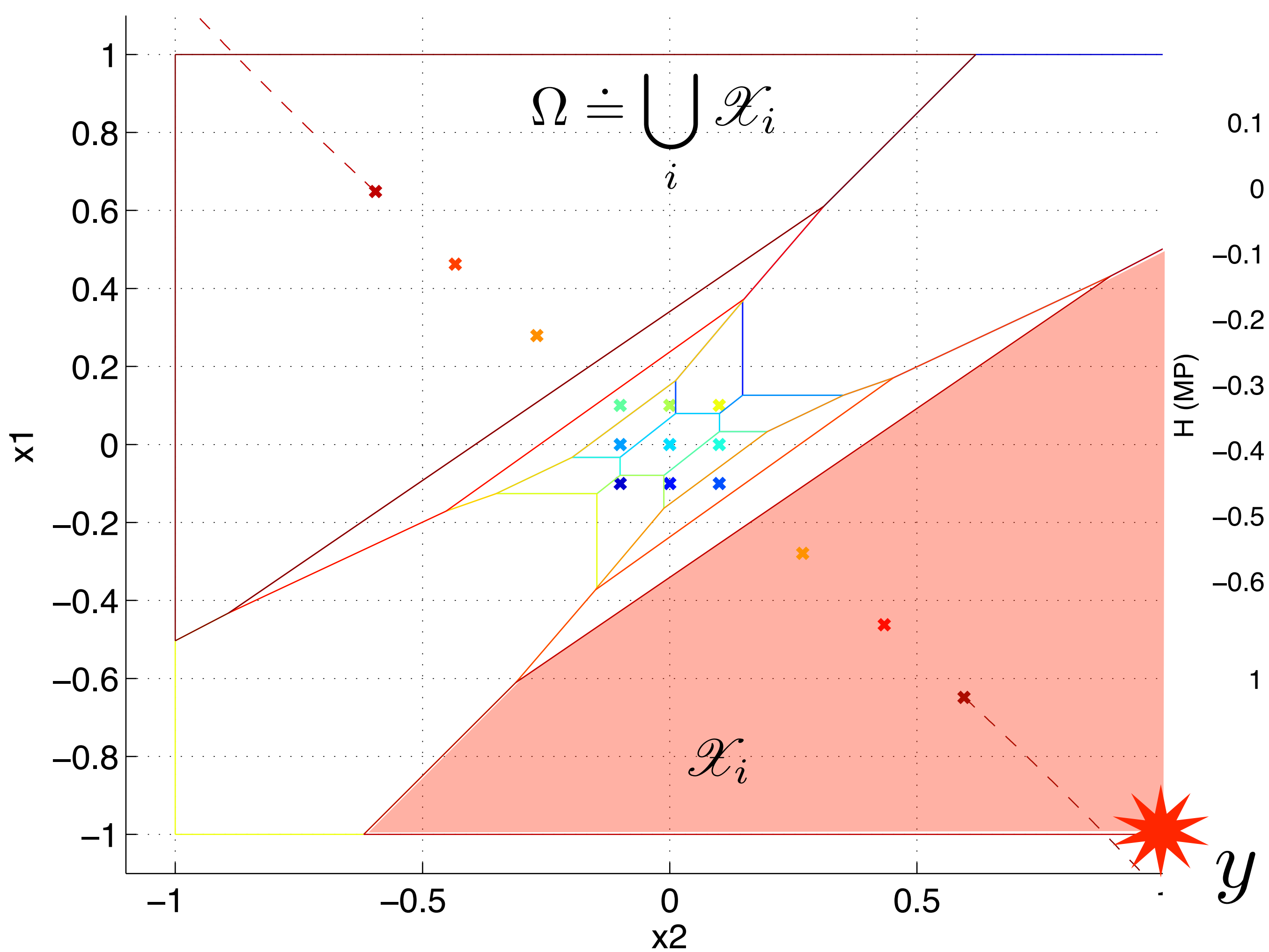
3 **Evolve** existing basis functions to new ones to correct the worst-case

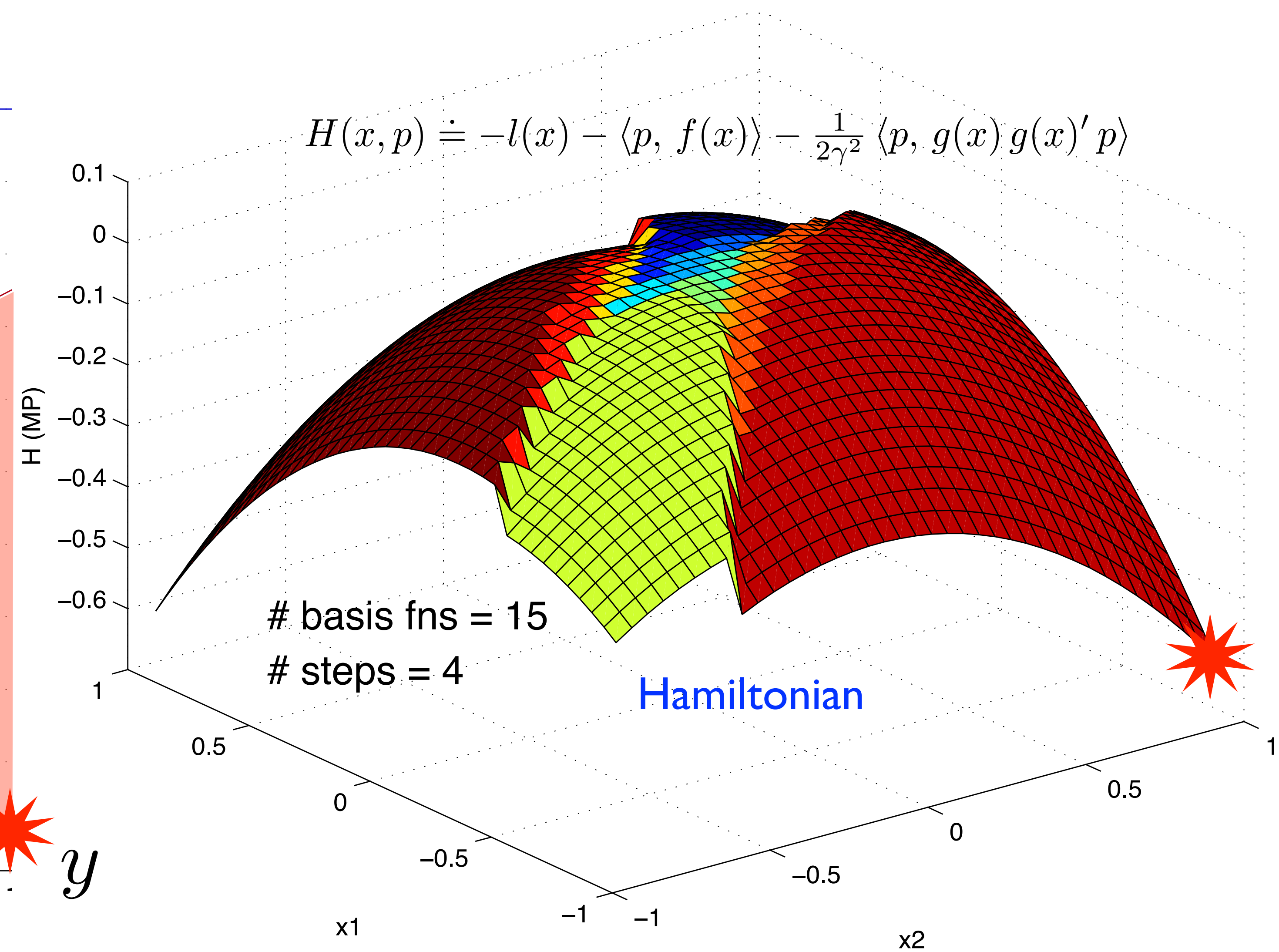
Kernel approximation — regulator problems

automated basis selection

1 Tessellate

2 Sort





sort polytopes by their
"worst" vertex
(by Hamiltonian)

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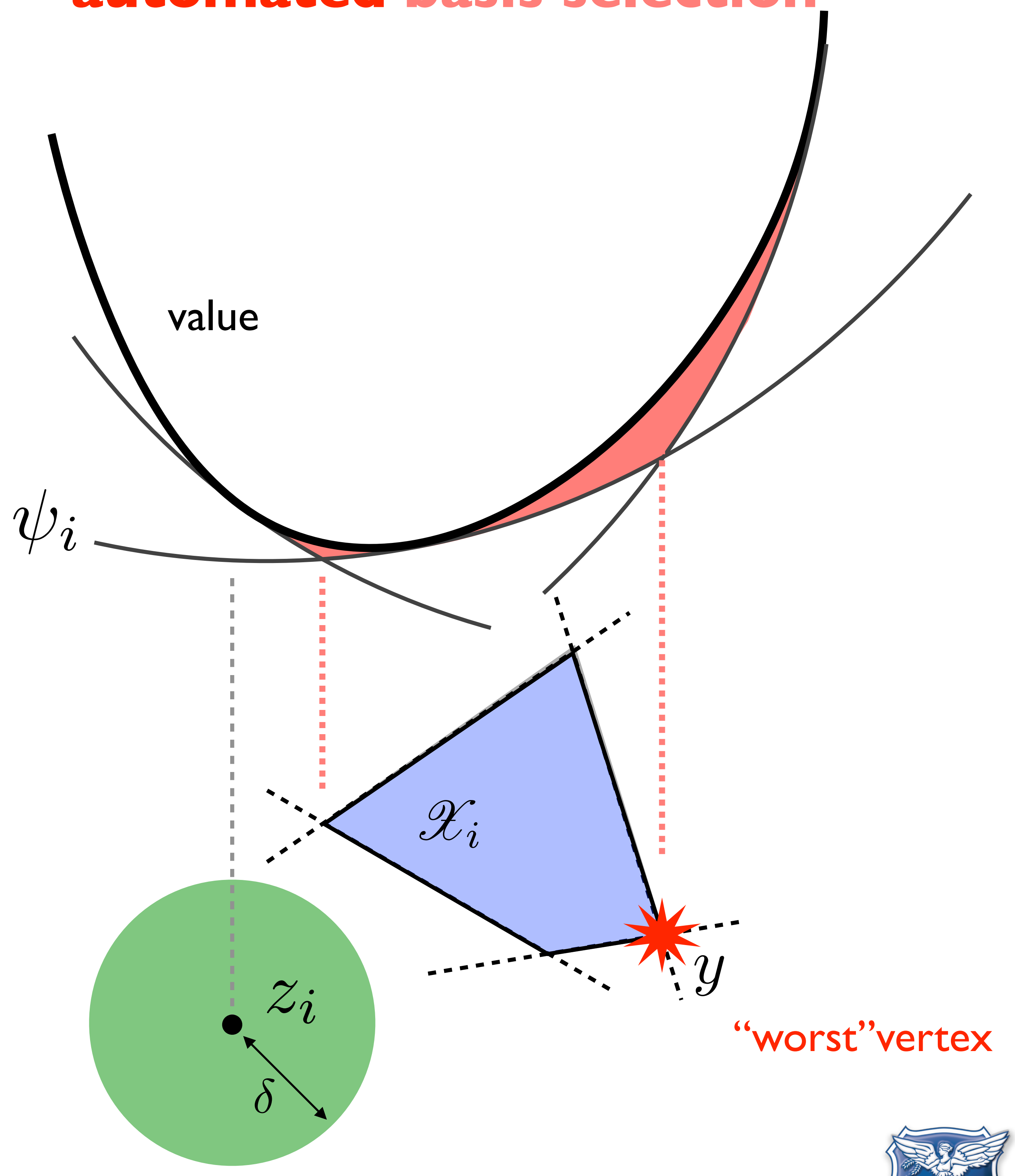
Kernel approximation — regulator problems

automated basis selection

3 Evolve



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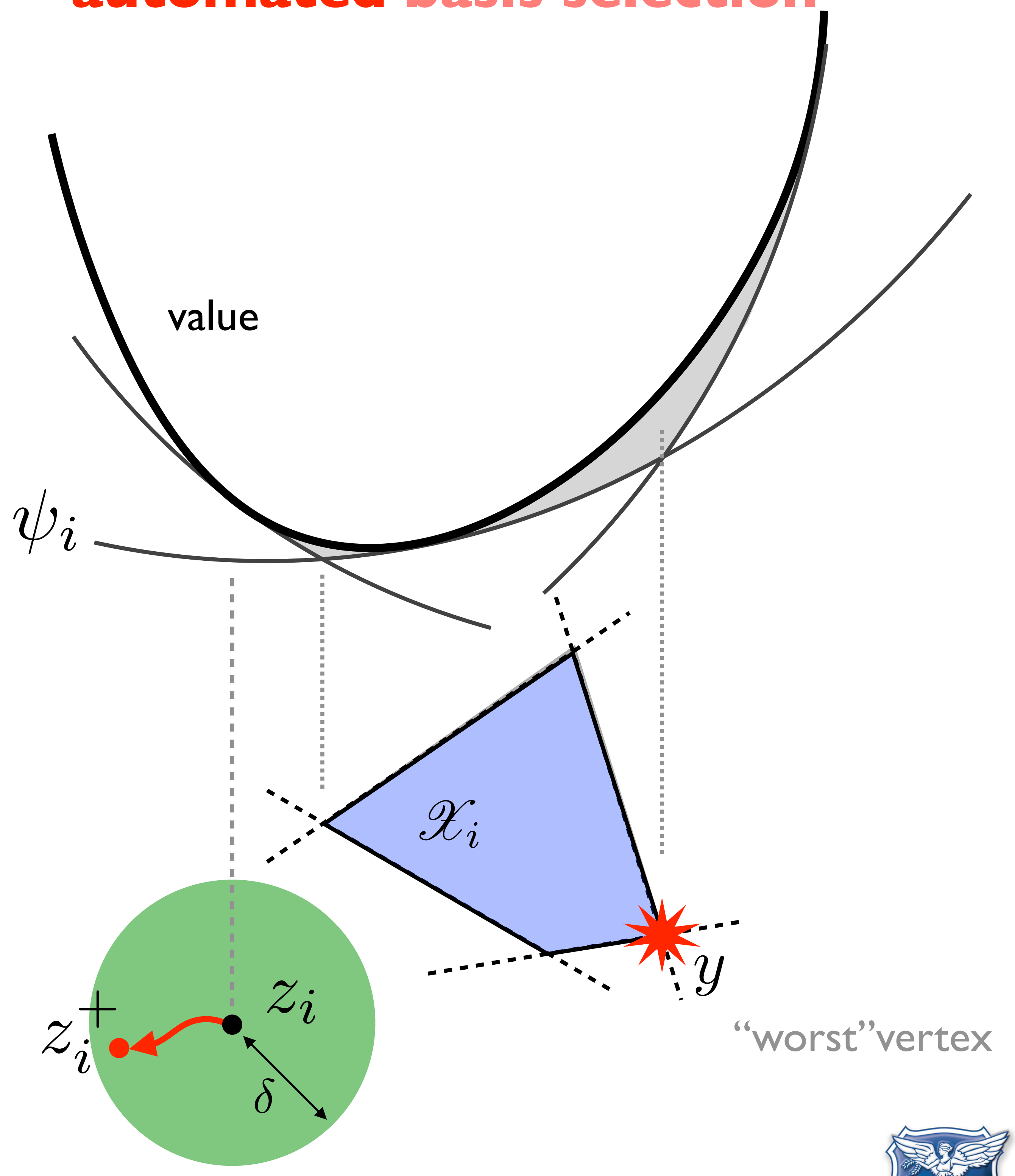
Kernel approximation — regulator problems

automated basis selection

3 Evolve

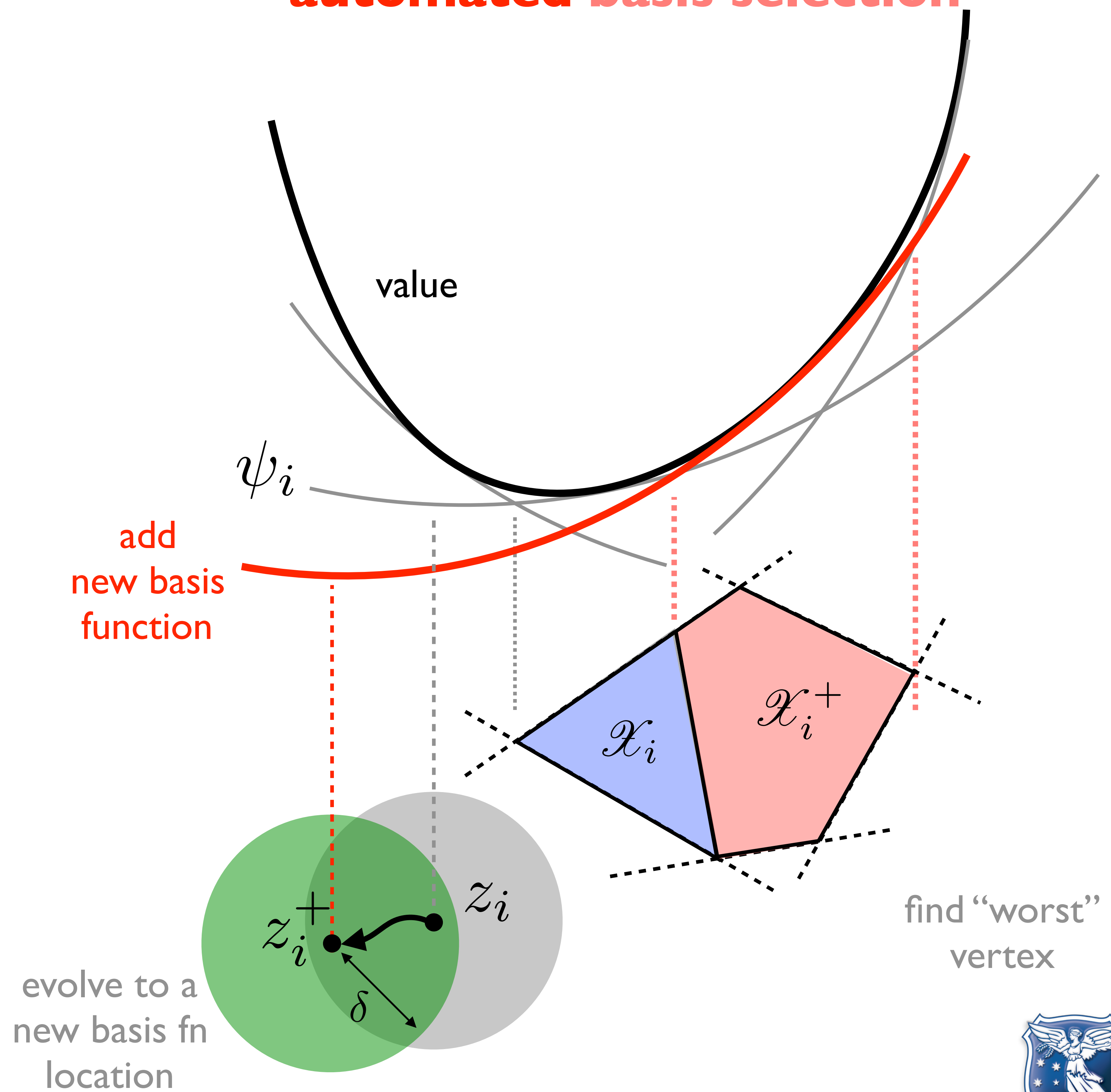


evolve to a new basis fn location

Kernel approximation — regulator problems

automated basis selection

3 Evolve



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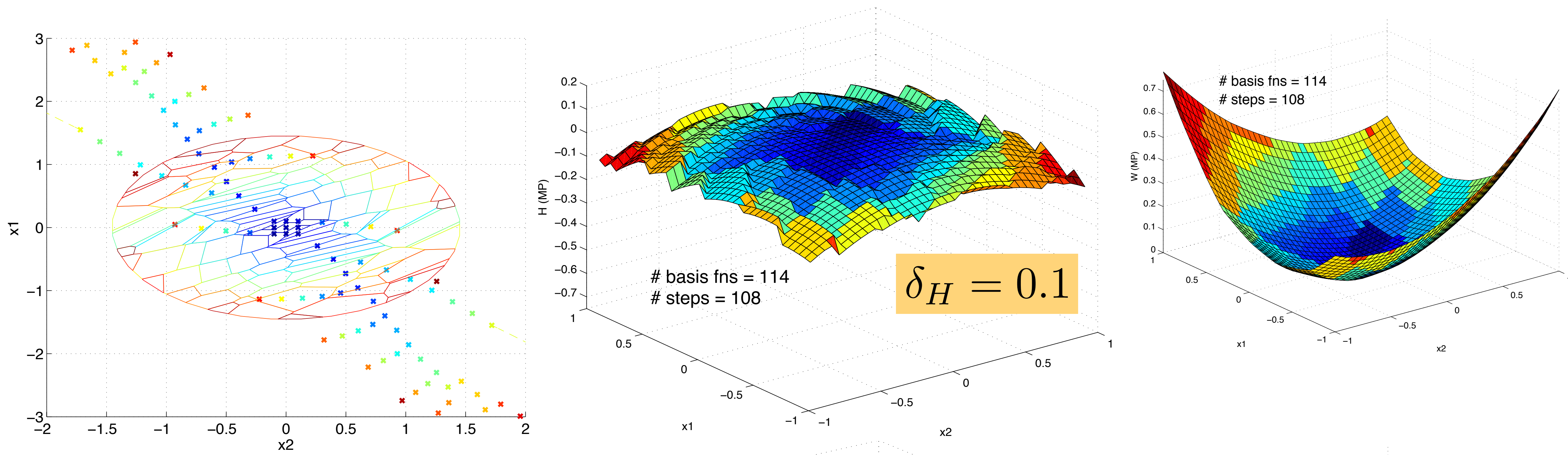
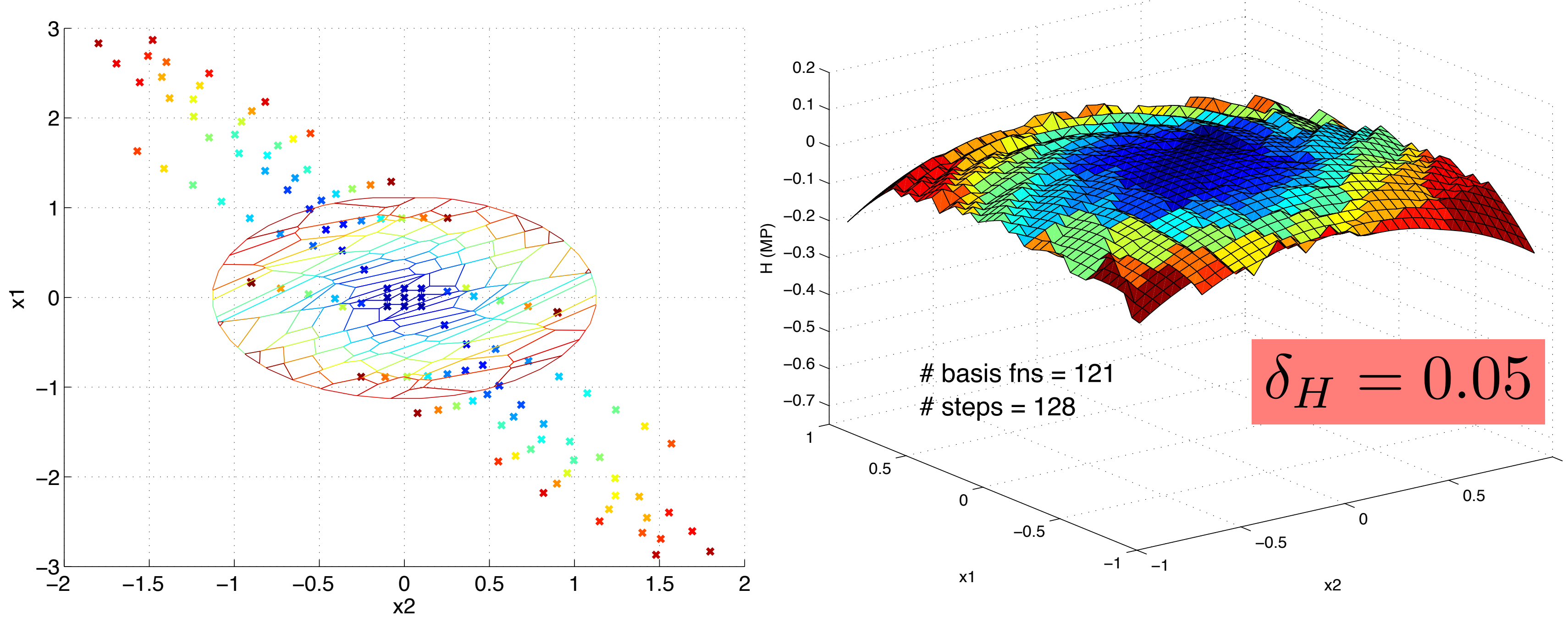
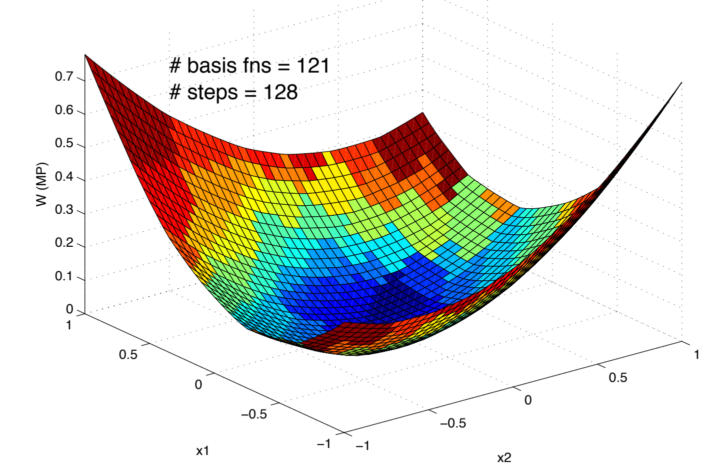
example I

linear dynamics $A \doteq \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}$ $B \doteq \begin{pmatrix} 0.5 \\ 0 \end{pmatrix}$ $l(\xi) \doteq \frac{1}{2}|\xi|^2$ $\gamma \doteq 2$ $\mathcal{M} \doteq 0.2\,\mathcal{I}$







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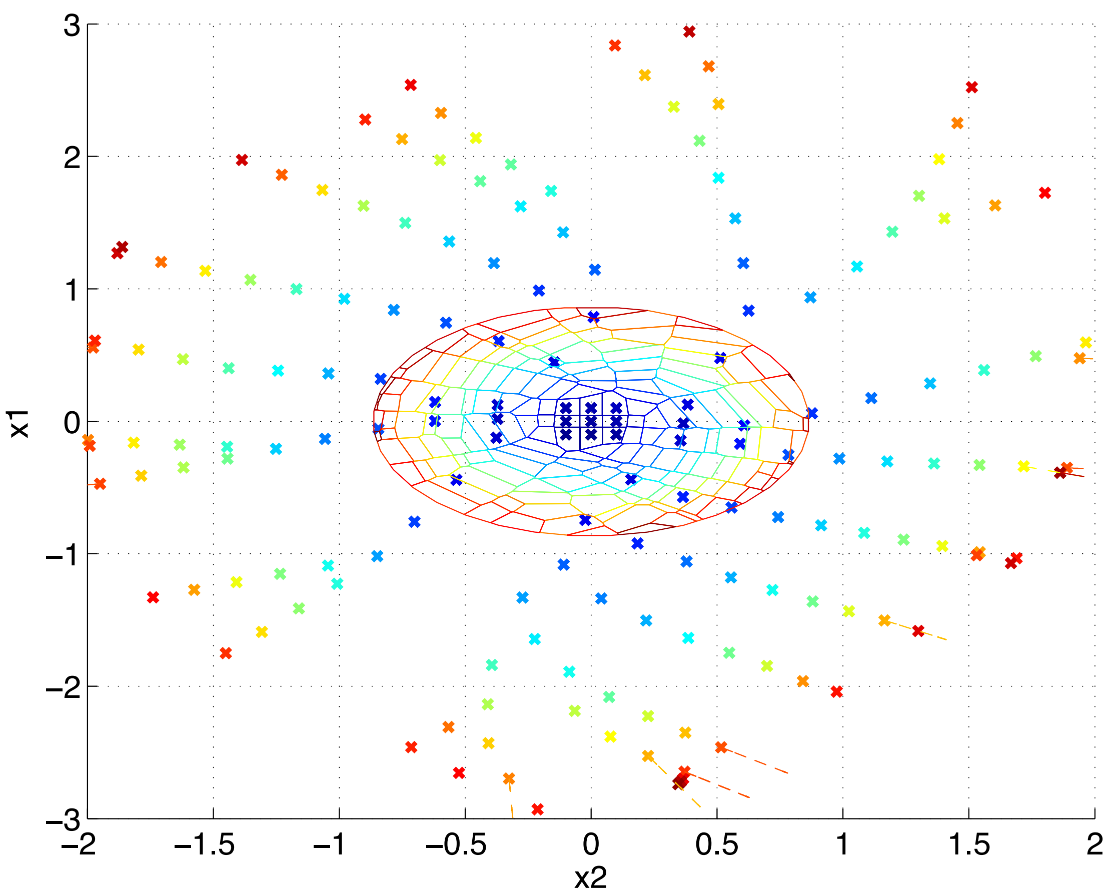
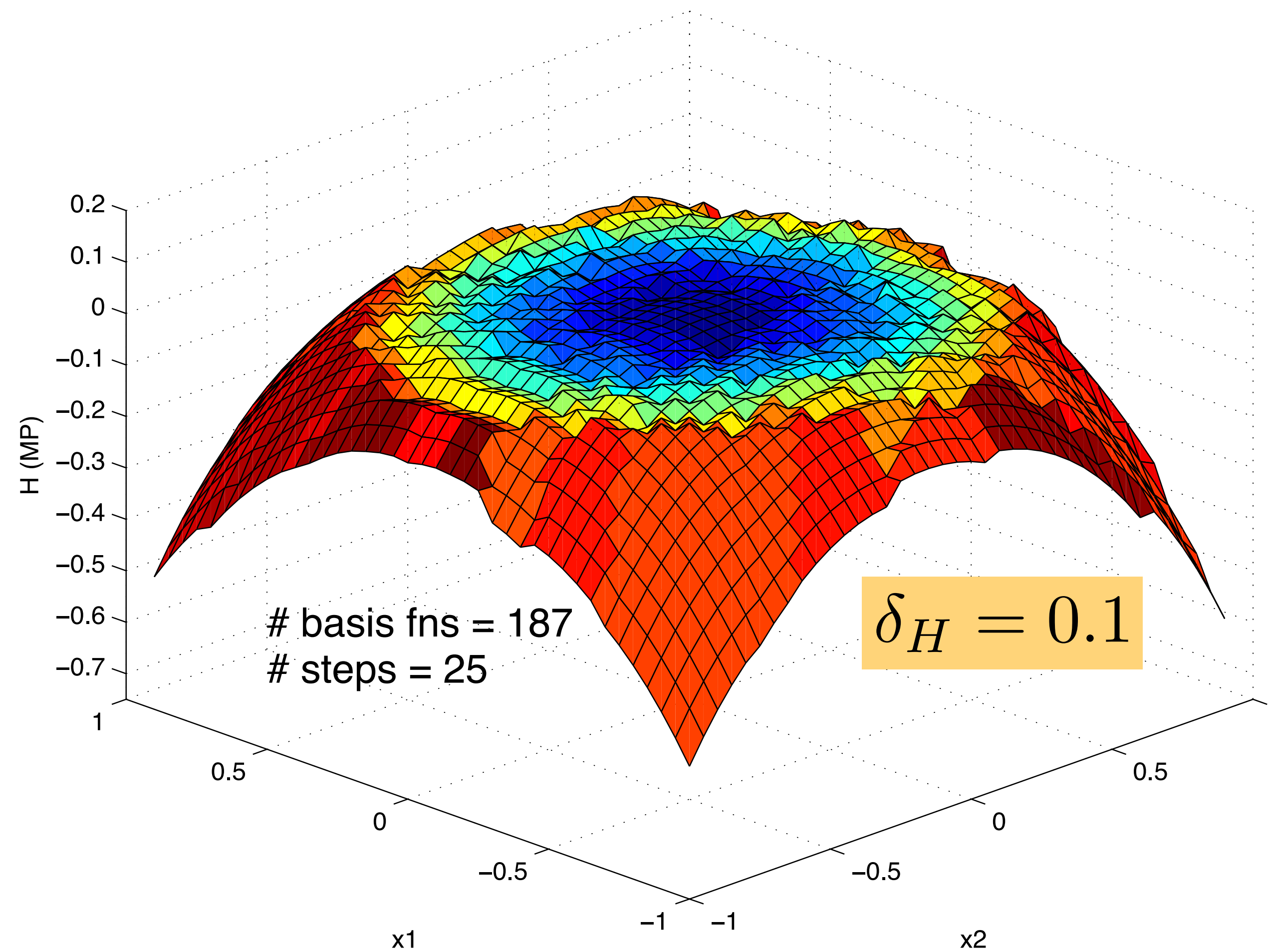
example 2

nonlinear dynamics

$$\dot{\xi}_s = f(\xi_s) + w_s \qquad f(\xi) \doteq \begin{pmatrix} -2\,\xi_1\,[1+\frac{1}{2}\,\tan^{-1}(3\xi_2^2/2)] \\ \frac{1}{2}\,\xi_1 - 3\,\xi_2\,\exp(-\xi_1/3) \end{pmatrix}$$

$$l(\xi) \doteq \frac{1}{2}\,|\xi|^2 \qquad \gamma^2 \doteq 1 \qquad \mathcal{M} \doteq -0.1\,\mathcal{I}$$





"Max-plus fundamental solution semigroups for optimal control"

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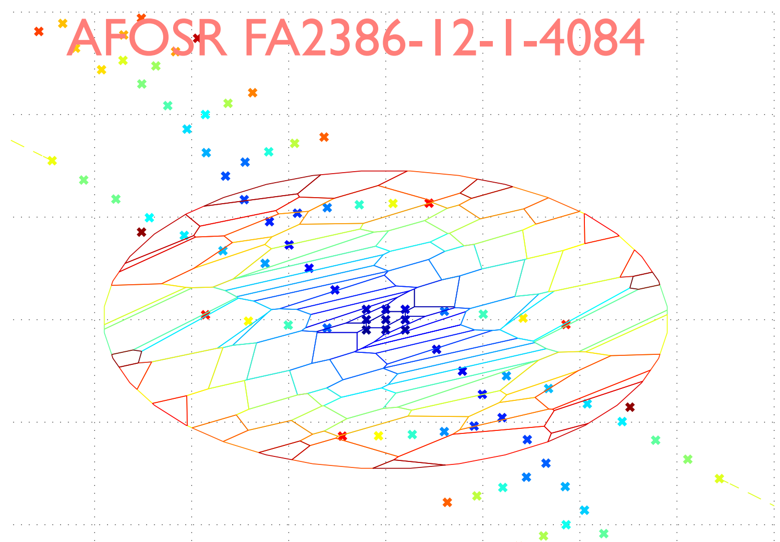
Department of Electrical & Electronic Engineering
University of Melbourne
Australia

AFOSR FA2386-12-1-4084



[McEneaney, 2006] — Max-plus methods for nonlinear control and estimation (Springer)

[Akian, Gaubert, Lakhoua, 2008] — The max-plus finite element method for solving deterministic optimal control problems: Basic properties and convergence analysis (SICON)

[Qu, 2015] — A max-plus based randomized algorithm for solving a class of HJB PDEs (CDC)

[Dower, McEneaney, 2017] — Solving infinite dimensional two point boundary value problems for a wave equation via the principle of stationary action and optimal control (SICON, to appear)

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