Receding-horizon optimal control with economic objectives – practical and asymptotic convergence

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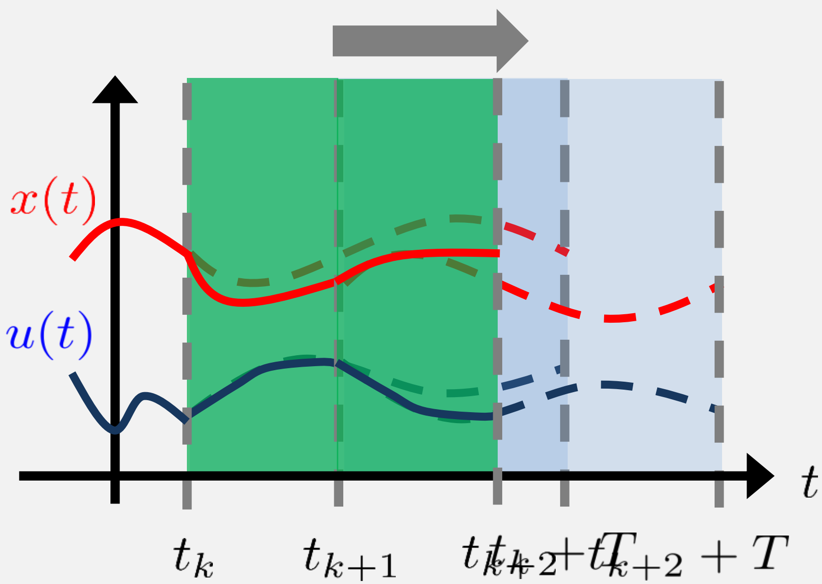
Nonlinear Model Predictive Control (NMPC)

NMPC= repeated optimal control



1. State estimate $\hat{x}(t_k)$ at t_k

2. Solve OCP

$$\min_{u(\cdot)} \int_{t_k}^{t_k+T} F(x(t), u(t)) d\tau + E(x(t_k + T))$$

$$\frac{dx(\tau)}{d\tau} = f(x(\tau), u(\tau)), \; x(t_k) = \hat{x}(t_k)$$

$$x(\tau) \in \mathcal{X}, \quad u(\tau) \in \mathcal{U}$$

$$x(t_k + T) \in \mathcal{E}$$

3. Apply $u^\star(\tau)$ for $\tau \in [t_k, t_{k+1})$

NMPC design

- Stabilization of x_s: $F(x,u) = (x - x_s)^T Q (x - x_s) + (u - u_s)^T R (u - u_s) \geq \|x - x_s\|$
  + suitable terminal constraints / penalties or reachability conditions
- Tracking of $r(t)$: $F(t, x, u) = (h(x) - r(t))^T Q (h(x) - r(t)) + \quad \ldots \quad \geq \|h(x) - r(t)\|$

Observation: control task at hand influences design of OCP.

What is Economic NMPC?

How to improve performance of a continuous process? $\rightarrow$ Solve OCP with long/infinite horizon T_∞.

$$\min_{u(\cdot)} \int_0^{T_\infty} F(x(\tau), u(\tau)) d\tau$$

subject to

$$\frac{dx(\tau)}{d\tau} = f(x(\tau), u(\tau)), \quad x(0) = x_0 \in \mathcal{X}_0$$
$$u(\tau) \in \mathcal{U} \subset \mathbb{R}^{n_u}, x(\tau) \in \mathcal{X} \subset \mathbb{R}^{n_x}$$

Challenges:

- Structure of optimal solutions?
- Numerics?
- ...

Solution: receding horizon approximation with shorter horizon T.

$$\min_{u(\cdot)} \int_{t_k}^{t_k+T} F(x(\tau), u(\tau)) d\tau$$

subject to

$$\frac{dx(\tau)}{d\tau} = f(x(\tau), u(\tau)), \quad x(t_k) = \hat{x}(t_k)$$
$$u(\tau) \in \mathcal{U} \subset \mathbb{R}^{n_u}, x(\tau) \in \mathcal{X} \subset \mathbb{R}^{n_x}$$

- $F \approx$ economic criteria: yield, profit, ...
- F can be non-quadratic: $F = a^T x + b^T u$, ...

Economic NMPC

- NMPC with generalized (economic) objectives.
- Approximation of an infinite-horizon OCP by receding-horizon solutions.

[Rawlings & Amrit `09; Würth et al. `11; Angeli et al. `12; Grüne `13; Ellis et al.`14; ...]

T. Faulwasser | Receding-horizon Optimal Control with Economic Objectives

Outline

Motivation

- Economic MPC

Turnpike properties and dissipativity

- Turnpike conditions and converse results

Asymptotic and practical convergence in EMPC

- Exact and approximate turnpikes

Recovering asymptotic convergence in EMPC

- Terminal constraints and penalties

Summary and outlook

How to describe turnpike behavior in OCPs?

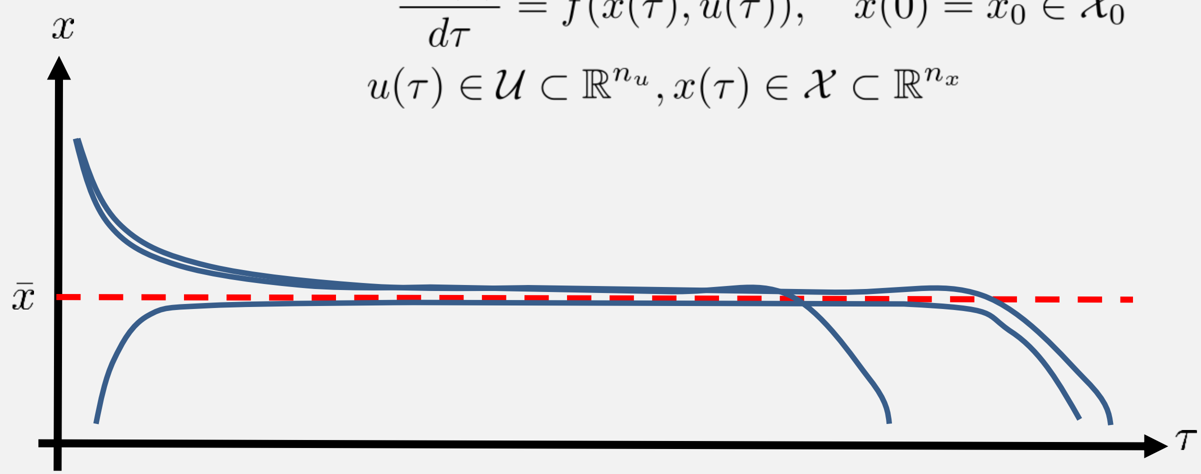
Problem setup

$$\min_{u(\cdot)} \quad \int_0^T F(x(\tau), u(\tau)) d\tau$$

subject to $\qquad (\text{OCP}_T(x_0))$

$$\frac{dx(\tau)}{d\tau} = f(x(\tau), u(\tau)), \quad x(0) = x_0 \in \mathcal{X}_0$$

$$u(\tau) \in \mathcal{U} \subset \mathbb{R}^{n_u}, x(\tau) \in \mathcal{X} \subset \mathbb{R}^{n_x}$$



Conceptual idea of turnpike properties

- Property of OCPs with and without terminal constraints.
- Optimal solutions approach neighborhood of a specifc steady state.
- Time spend at turnpike grows with increasing horizon length T.
- If turnpike at $\bar{x}$, then for $T = \infty$, we have that $\lim_{t \to \infty} x^\star(t) \approx \bar{x}$.
- Different notions for turnpikes: dichotomy in OCPs, hyper-sensitive OCPs, …

[Dorfman, Samuelson & Solow `58; McKenzie `76; Carlson et al. `91; Damm et al. `14; Trelat & Zuazua `14; …]

Parametric Optimal Control Problems

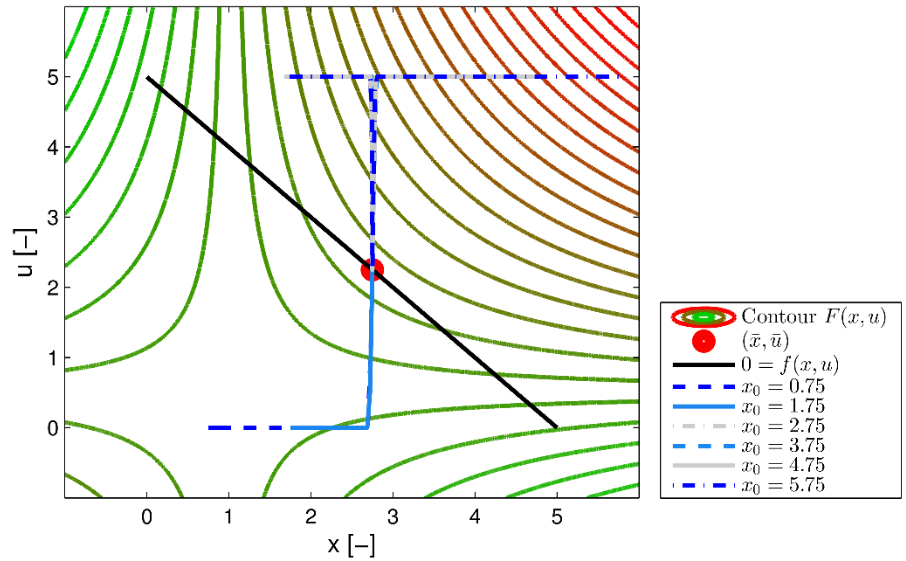
Optimal fish harvest

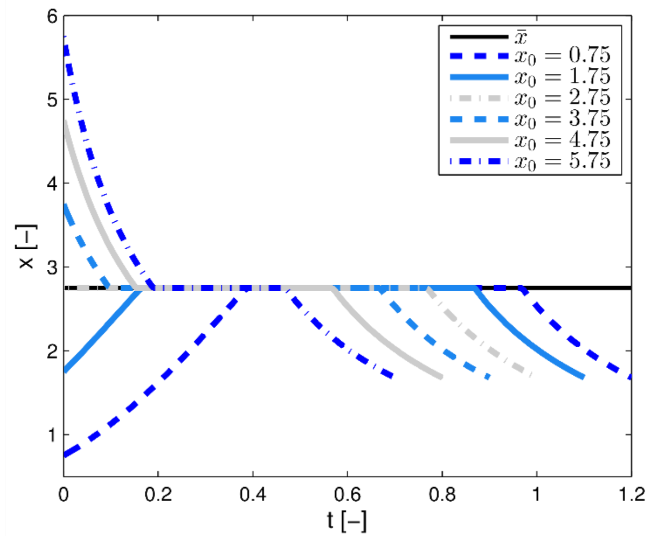
$$\min_{u(\cdot)} \quad \int_0^{T} ax(t) + bu(t) - cx(t)u(t)dt$$

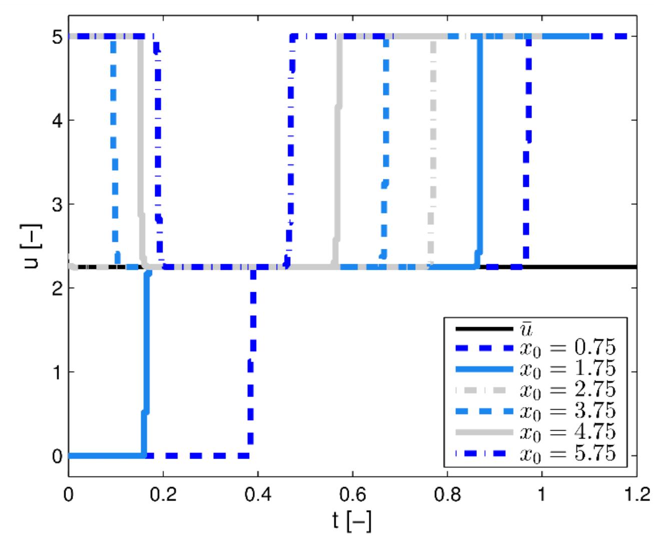
subject to

$$\dot{x} = x(x_S - x - u), \qquad x(0) = x_0$$

$$u(t) \in [0, u_{max}], x(t) \in (0, \infty)$$







→ Similar behavior for different initial conditions and horizon lengths.

→ Similarity properties of solutions of parametric OCPs.

Parametric Optimal Control Problems

Optimal fish harvest (quadratic objective)

$$\min_{u(\cdot)} \quad \int_0^T \tfrac{1}{2}q(x(t)-x_C)^2 + \tfrac{1}{2}r(u(t)-u_C)^2 dt$$

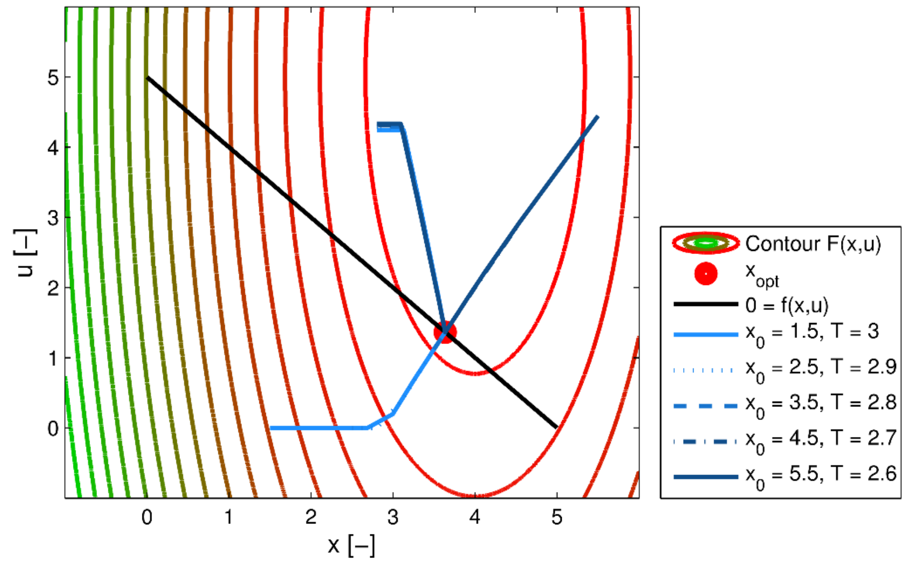
subject to

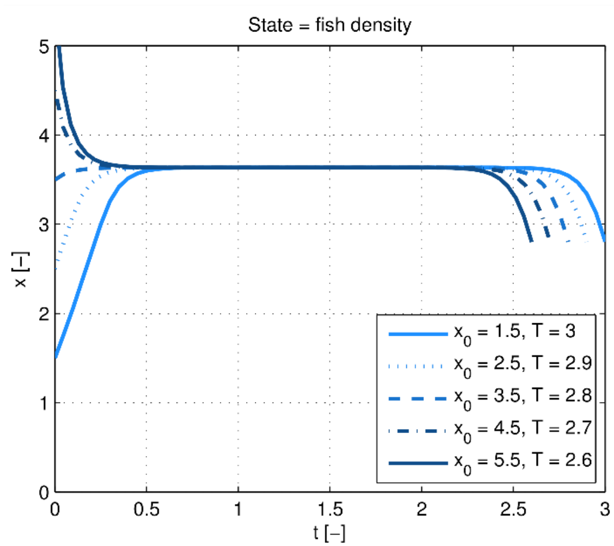
$$\dot{x} = x(x_S - x - u), \quad x(0) = x_0$$

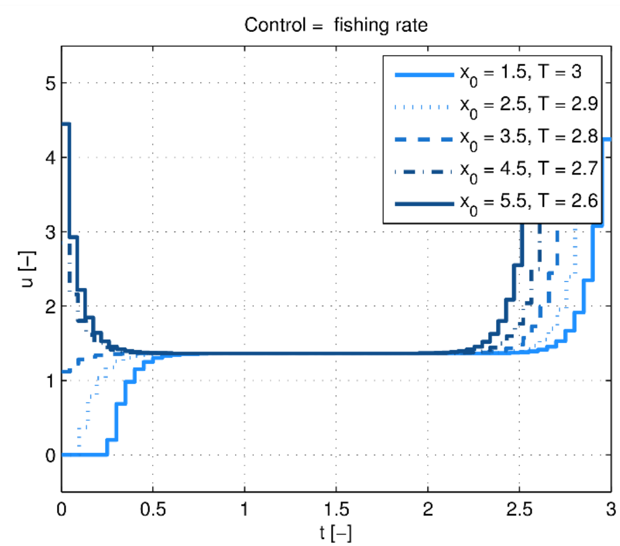
$$u(t) \in [0, u_{max}], x(t) \in (0, \infty)$$

$$u_{max} = 5, x_S = 5$$

$$q = 10, r = 1, x_C = 4, u_C = 5$$

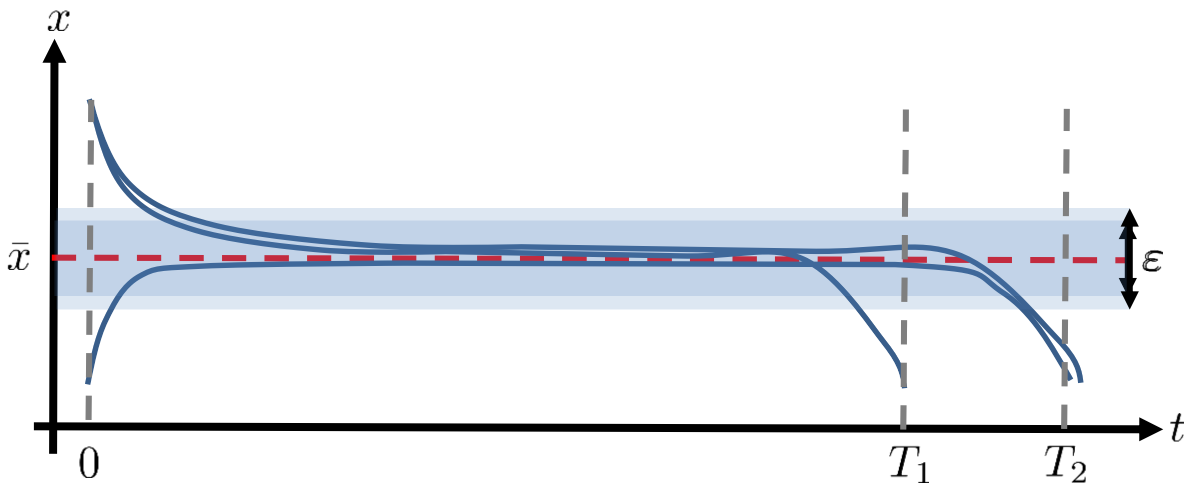






→ Similar behavior for different initial conditions and horizon lengths.
→ Similarity properties of solutions of parametric OCPs.

Turnpike Property of OCPs



Definition (Turnpike property).
Consider the optimal pairs $z^\star(\cdot, x_0)$ and

$$\Theta_{\varepsilon,T} := \{t \in [0,T] : \|z^\star(\cdot, x_0) - \bar{z}\| > \varepsilon\}.$$

The optimal pairs $z^\star(\cdot, x_0)$ of $\mathrm{OCP}_T(x_0)$ have an ***input-state turnpike property*** with respect to $\bar{z}$ if there exists $\nu : [0,\infty) \to [0,\infty)$ s. t.

$$\forall x_0 \in \mathcal{X}_0, \forall T \geq 0, \forall \varepsilon > 0 : \quad \mu[\Theta_{\varepsilon,T}] < \nu(\varepsilon),$$

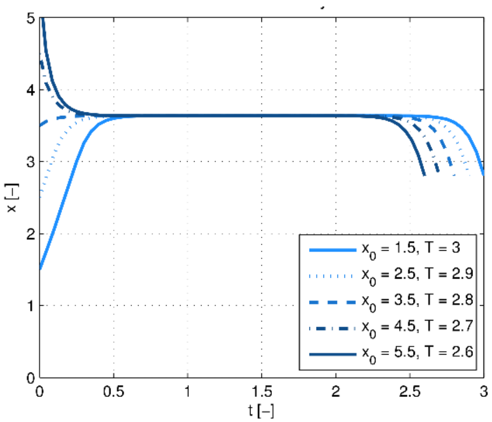
where $\mu[\cdot]$ is the Lebesgue measure on the real line.
The solution pairs $z^\star(\cdot, x_0)$ of $\mathrm{OCP}_T(x_0)$ are said to have an **exact input-state turnpike property** if additionally

$$\mu[\Theta_{\mathbf{0},T}] < \nu(0) < \infty.$$

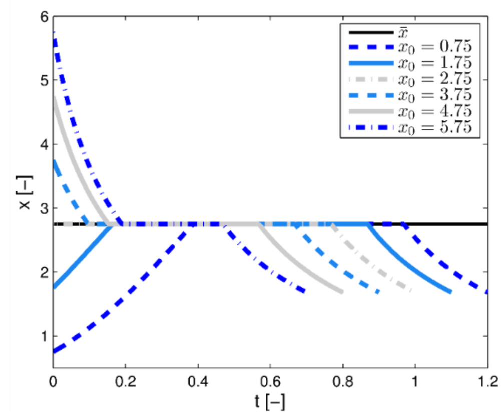
[Carlson et al. \`91, Faulwasser et al. \`14, \`17]

Turnpike Properties of OCPs

Turnpikes are either **approximate** or **exact**.



→ approximate



→ exact

When do turnpikes occur in OCPs?

Definition (Strict dissipativity w.r.t. $(\bar{x},\bar{u})$). $\Sigma : \dot{x} = f(x,u)$ is said to be *strictly dissipative with respect to the steady state pair* $(\bar{x},\bar{u})$ if there exists a bounded non-negative storage function $S : \mathcal{X} \to \mathbb{R}_0^+$ and $\alpha \in \mathcal{K}$ such that for all admissible pairs $z(\cdot, x_0)$, all $x_0 \in \mathcal{X}$, and all horizons $T > 0$

$$S(x(T,x_0)) - S(x_0) \leq \int_0^T -\alpha(\|(x(\tau),u(\tau)) - (\bar{x},\bar{u}))\|) + F(x(\tau),u(\tau)) - F(\bar{x},\bar{u})\mathrm{d}\tau.$$

[Diehl et al. `11; Angeli et al. `12; ...]

Theorem (Dissipativity $\Rightarrow$ turnpike).
Suppose that

- from all $x_0 \in \mathcal{X}_0$ the optimal steady state $\bar{x}$ is exponentially reachable,
- Σ is strictly dissipative w.r.t. to $(\bar{x},\bar{u})$.

Then the optimal pairs $z^\star(\cdot, x_0)$ of $\mathrm{OCP}_T(x_0)$ have a turnpike property with respect to the steady state pair $(\bar{x},\bar{u})$.

[Grüne `13; Faulwasser at al. `14, `17; Damm et al. `14]

Convergence of NMPC based on Exact Turnpikes

NMPC scheme without terminal constraints and without terminal penalty

$$\min_{u(\cdot)} \int_{t_k}^{t_k+T} F(x(\tau), u(\tau)) d\tau$$

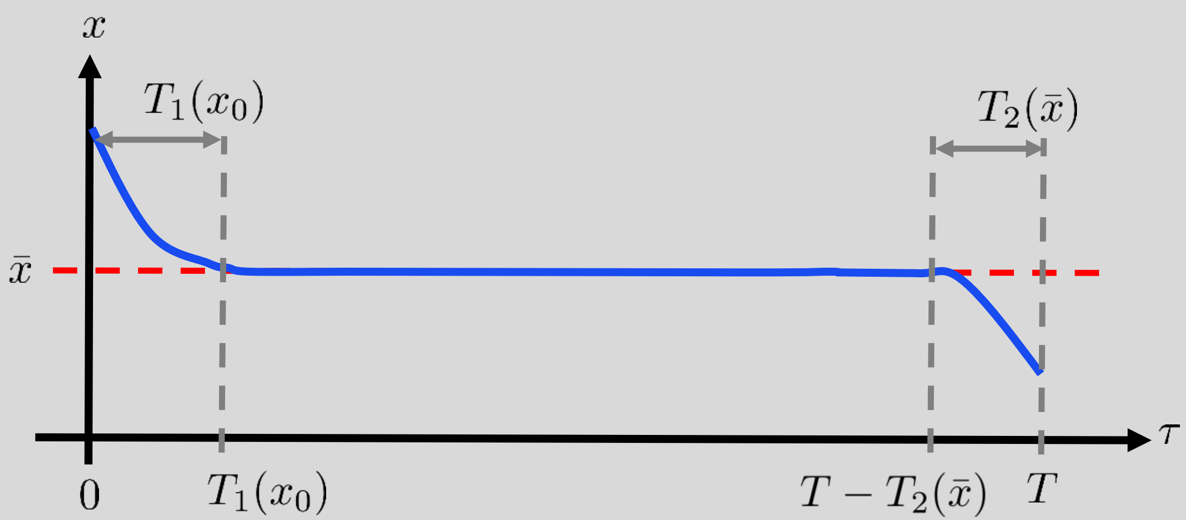
$$\text{subject to} \qquad (\mathrm{OCP}_T(\hat{x}(t_k)))$$

$$\forall \tau \in [t_k, t_k + T]: \ \frac{d}{d\tau} x(\tau) = f(x(\tau), u(\tau)), \quad x(t_k) = \hat{x}(t_k)$$

$$u(\tau) \in \mathcal{U}, \ x(\tau) \in \mathcal{X}$$

Main assumptions

- No terminal constraints, no end penalty.
- No structural assumptions on F $\rightarrow$ economic NMPC.
- **Exact turnpike property** in $\mathrm{OCP}_T(x_0)$:



Convergence of NMPC based on Exact Turnpikes

Theorem (Convergence of NMPC based on exact turnpike).
Suppose that

- Σ is controlled via $\mathrm{OCP}_T(\hat{x}(t_k))$,
- for all $\hat{x}(t_k) \in \mathcal{X}_0$, $\mathrm{OCP}_T(\hat{x}(t_k))$ has an exact turnpike property at $\bar{z}$,
- $\hat{x}(t_0) \in \mathcal{X}_0$.

Then

- $\mathrm{OCP}_T(\hat{x}(t_k))$ is recursively feasible, and
- there exist a horizon length $T \in (0, \infty)$, a sampling time $\delta > 0$ and a time $\bar{t} \geq t_0$ such that

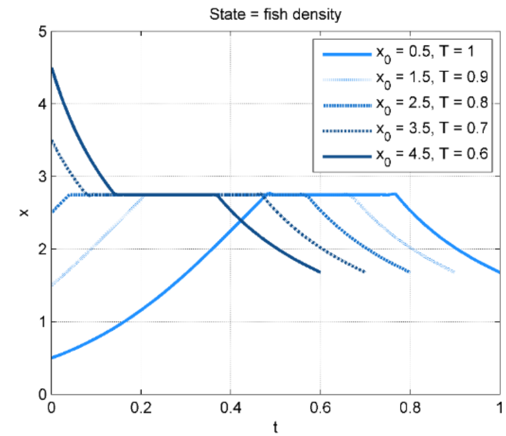
$$\forall t \geq \bar{t}: \quad \hat{x}(t, x_0, u^{mpc}(\cdot)) = \bar{x}.$$

[Faulwasser & Bonvin `15, `17]

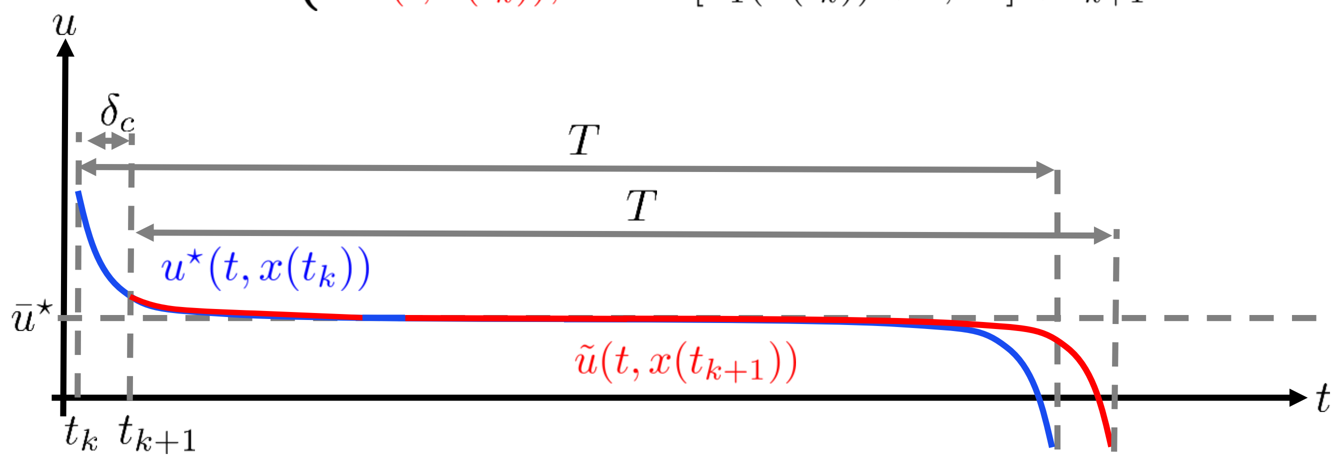
Stability of NMPC based on Exact Turnpikes

Main steps of the proof:

- If, for all $x_0 \in \mathcal{X}_0$, we have an exact turnpike, then $\mathcal{X}_0$ is rendered positively invariant by NMPC scheme.
- End pieces of exact turnpike solutions are identical.
- Construction of admissible (optimal) input trajectory



$$\tilde{u}_{k+1}(t, x(t_{k+1})) = \begin{cases} u^\star(t, x(t_k)), & \forall t \in [0,\ T_1(x(t_k))) + t_{k+1} \\ \bar{u}^\star, & \forall t \in [T_1(x(t_k)),\ T_1(x(t_k))) + \delta) + t_{k+1} \\ u^\star(t, x(t_k)), & \forall t \in [T_1(x(t_k)) + \delta,\ T] + t_{k+1} \end{cases} .$$



Performance? Approximate turnpikes?

Example: Optimal Fish Harvest

$$\min_{u(\cdot)} \int_{t_k}^{t_k+T} ax(\tau) + bu(\tau) - cx(\tau)u(\tau)d\tau$$

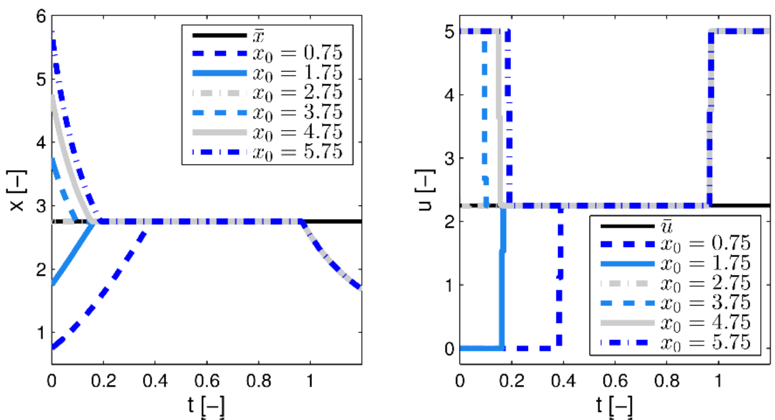
subject to

$$\frac{dx}{d\tau} = x(x_S - x - u), \quad x(t_k) = \hat{x}(t_k)$$

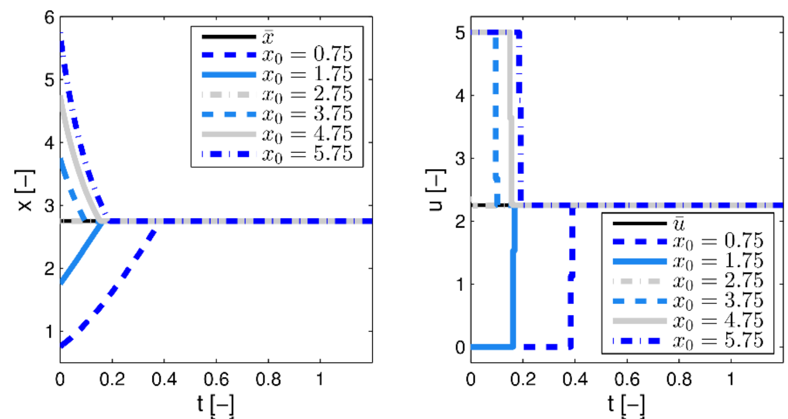
$$u(t) \in [0, u_{max}], x(t) \in (0, \infty)$$

- x fish density
- u fishing rate
- $x_s = 5$ highest sustainable fish density
- $a = 1, b = c = 2, u_{max} = 5$
- $T = 1.2,\ \delta = 0.1$ [Cliff & Vincent `73]

Open-loop turnpike solutions



Closed-loop NMPC solutions



Questions

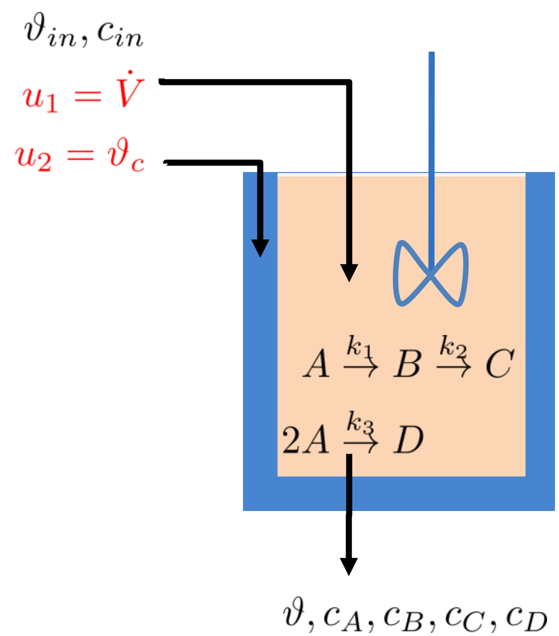
- How to verify turnpikes in OCPs? When are turnpikes exact?
- What if turnpikes are only approximate?

Example – Chemical Reactor

Van de Vusse Reactor $A \overset{k_1}{\rightarrow} B \overset{k_2}{\rightarrow} C, \qquad 2A \overset{k_3}{\rightarrow} D$

Dynamics (partial model)

$$\dot{c}_A = r_A(c_A, \vartheta) + (c_{in} - c_A)u_1$$
$$\dot{c}_B = r_B(c_A, c_B, \vartheta) - c_B u_1$$
$$\dot{\vartheta} = h(c_A, c_B, \vartheta) + \alpha(u_2 - \vartheta) + (\vartheta_{in} - \vartheta)u_1,$$
$$r_A(c_A, \vartheta) = -k_1(\vartheta)c_A - 2k_3(\vartheta)c_A^2$$
$$r_B(c_A, c_B, \vartheta) = k_1(\vartheta)c_A - k_2(\vartheta)c_B$$
$$h(c_A, c_B, \vartheta) = -\delta\Big(k_1(\vartheta)c_A \Delta H_{AB} + k_2(\vartheta)c_B \Delta H_{BC} + 2k_3(\vartheta)c_A^2 \Delta H_{AD}\Big)$$
$$k_i(\vartheta) = k_{i0} \exp \frac{-E_i}{\vartheta + \vartheta_0}, \quad i = 1, 2, 3.$$



Constraints

$$c_A \in [0, 6]\frac{mol}{l} \quad c_B \in [0, 4]\frac{mol}{l} \quad \vartheta \in [70, 150]°C$$
$$u_1 \in [3, 35]\frac{1}{h} \quad u_2 \in [0, 200]°C.$$

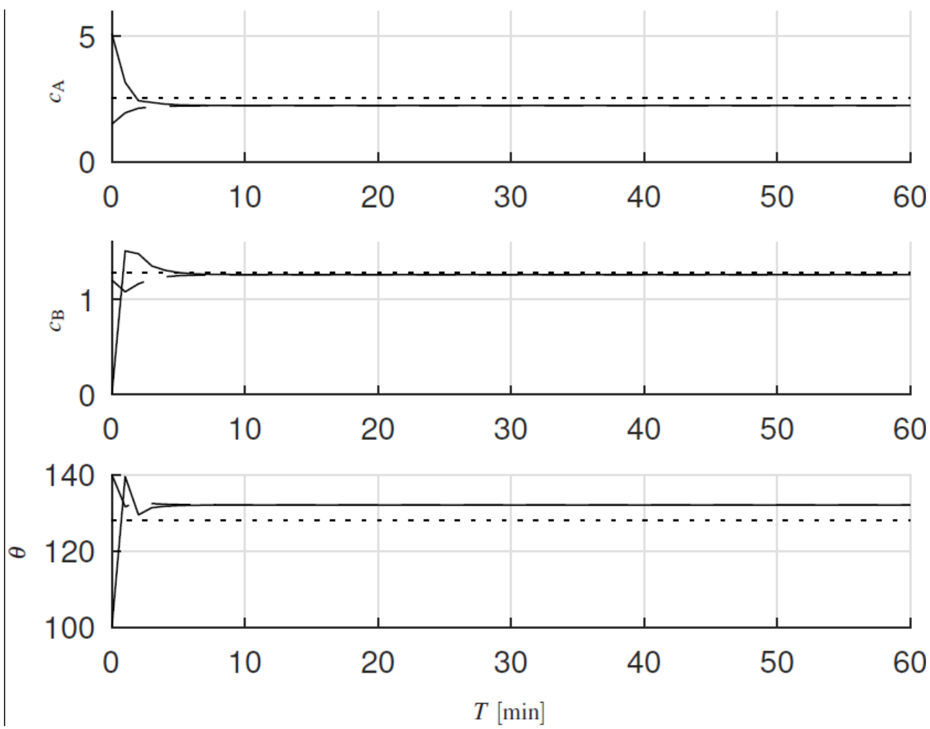
Objective = maximize produced amount of B

$$J_T(x_0, u(\cdot)) = \int_0^T -\beta c_B(t) u_1(t) dt, \qquad \beta > 0$$

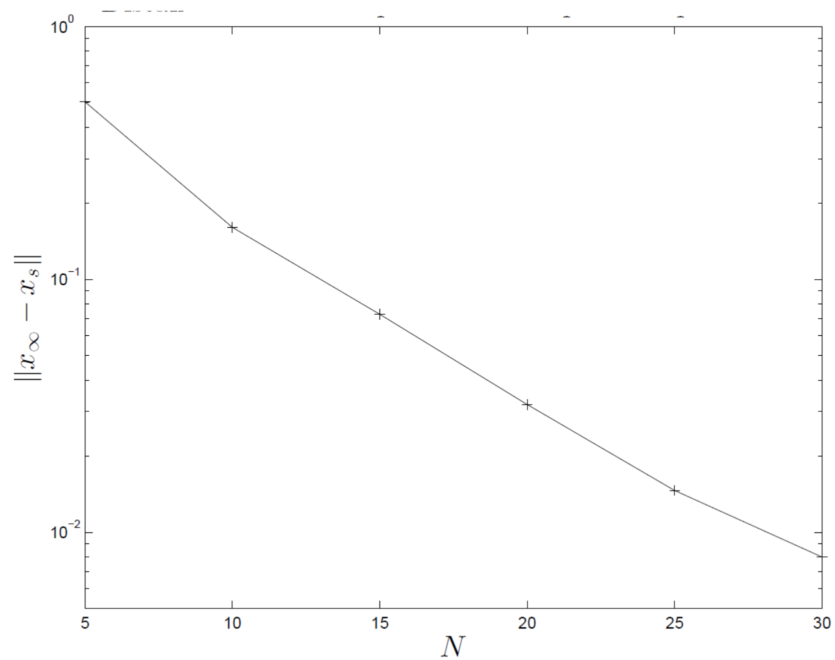
[Chen et al. `95; Rothfuß, Rudolph, Zeitz `96]

Example – Chemical Reactor

$T = 0.01667h$



Distance to equilibrium



Overview – EMPC without Terminal Constraints

Property	Approximate Turnpike	Exact Turnpike
recursive feasibility	for long horizons if controllability at turnpike	for long horizons
performance	approximation of infinite-horizon performance for $T \to \infty$	infinite-horizon performance for $T < \infty$
stability of closed loop	practical stability, i.e. convergence to neighborhood of turnpike	finite-time convergence to turnpike

[Grüne `13; Faulwasser & Bonvin `15,`17; …]

Singular OCPs and Exact Turnpikes

OCP with input box constraints and input affine data

$$\min_{u(\cdot)} \int_0^T F_0(x(\tau)) + \sum_{i=1}^{n_u} F_1^i(x(\tau))u_i(\tau)d\tau$$

subject to (OCP-SING)

$$\Sigma: \quad \frac{dx(\tau)}{d\tau} = f_0(x(\tau)) + \sum_{i=1}^{n_u} f_1^i(x(\tau))u_i(\tau), \quad x(0) = x_0 \in \mathcal{X}_0$$

$$u(\tau) \in [u_1^{min}, u_1^{max}] \times \cdots \times [u_{n_u}^{min}, u_{n_u}^{max}]$$

Necessary conditions of optimality for OCP-SING

$$H(\lambda_0, \lambda, x, u) = \lambda_0 \left(F_0(x) + \sum_{i=1}^{n_u} F_1^i(x)u_i \right) + \lambda^\top \left(f_0(x) + \sum_{i=1}^{n_u} f_1^i(x)u_i \right)$$

$$\frac{dx^\star(\tau)}{d\tau} = \quad H_\lambda(\lambda_0^\star, \lambda^\star(\tau), x^\star(\tau), u^\star(\tau)), \quad x^\star(0) = x_0$$

$$\frac{d\lambda^\star(\tau)}{d\tau} = -H_x(\lambda_0^\star, \lambda^\star(\tau), x^\star(\tau), u^\star(\tau)), \quad \lambda^\star(T) = 0 \qquad \text{(NCO)}$$

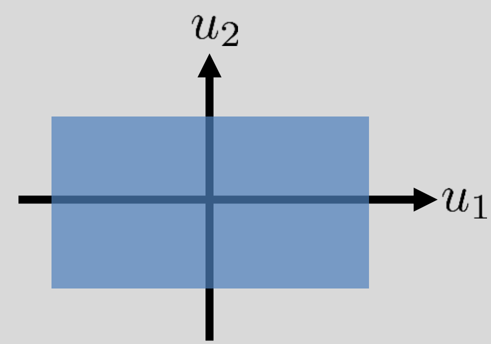
$$\forall \tau \in [0, T] \text{ and } \forall u \in \mathcal{U}$$

$$H(\lambda_0^\star, \lambda^\star(\tau), x^\star(\tau), u^\star(\tau)) \le H(\lambda_0^\star, \lambda^\star(\tau), x^\star(\tau), u),$$

Singular OCPs and Exact Turnpikes

Necessary conditions of optimality imply

$$u_i^\star(\tau) \in \{u_{i,min}, u_{i,max}\} \quad \text{if } s_i(x^\star(\tau), \lambda^\star(\tau)) \neq 0$$
$$u_i^\star(\tau) \in [u_{i,min}, u_{i,max}] \quad \text{if } s_i(x^\star(\tau), \lambda^\star(\tau)) = 0$$
$$s_i(x, \lambda) = \lambda_0 F_1^i(x) + \lambda^\top f_1^i(x), \quad i = 1, \ldots, n_u.$$



$\Rightarrow$ Either optimal inputs are on the boundary of $\mathcal{U}$, or a singular arc with $s_i(\lambda, x) = 0$.

Definition (Steady-state singular OCP).
OCP-SING is said to be **steady-state singular** if, for any non-vanishing interval $[\tau_0, \tau_1] \subset [0, T]$, the condition

$$s_i(x^\star(\tau), \lambda^\star(\tau)) = 0, \forall \tau \in [\tau_0, \tau_1], \forall i = 1, \ldots, n_u$$

implies that

$$(x^\star(\tau), u^\star(\tau), \lambda^\star(\tau)) = (\bar{x}, \bar{u}, \bar{\lambda})$$

where $(\bar{x}, \bar{u}, \bar{\lambda})$ specifies a unique steady state of (NCO).

→ Only singular arc is a steady state! ←

Singular OCPs and Exact Turnpikes

Theorem (Exactness of turnpikes).
Suppose that OCP-SING

(i) is steady-state singular with respect to $(\bar{\lambda}, \bar{x}, \bar{u})$ such that, $\forall i \in \{1, \ldots, n_u\}, \; \bar{u}_i \notin \{u_{i,min}, u_{i,max}\}$, and

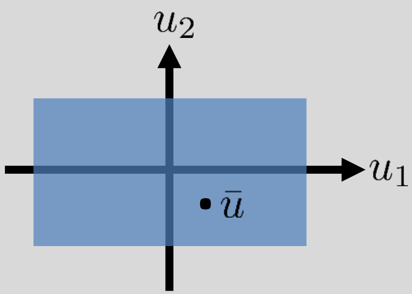
(ii) the optimal solutions to OCP-SING have a turnpike at $\bar{z} = (\bar{x}, \bar{u})$.

Then, the turnpike at $\bar{z}$ is exact. [Faulwasser & Bonvin `17]

Remarks

- Approximate turnpikes can be verified via dissipativity condition.
- Proof uses singular nature of OCP-SING, i.e.,

$$
\begin{aligned}
u_i^\star(\tau) &\in \{u_{i,min}, u_{i,max}\} && \text{if } s_i(x^\star(\tau), \lambda^\star(\tau)) \neq 0 \\
u_i^\star(\tau) &= \bar{u}_i && \text{if } s_i(x^\star(\tau), \lambda^\star(\tau)) = 0
\end{aligned}
$$



- Optimal solutions cannot stay arbitarily close to turnpike, without $u^\star(\tau) = \bar{u} \implies x^\star(\tau) = \bar{x}$.
- Verifying steady-state singularity?

Steady-State Singularity of Linear-Quadratic OCPs

Special case of OCP-SING:

- Linear dynamics $\dot{x} = Ax + Bu, \quad x(0) = x_0 \in \mathcal{X}_0.$
- Quadratic objective with $F(x,u) = \frac{1}{2}x^\top Qx + x^\top Su + q^\top x + r^\top u$ and convex input constraints $\mathcal{U} \subset \mathbb{R}^{n_u}$.

Necessary conditions of optimality on singular arc $\rightarrow$ linear DAE

$$\underbrace{\begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}}_{\tilde{E}} \begin{pmatrix} \dot{x} \\ \dot{\lambda} \\ \dot{u} \end{pmatrix} = \underbrace{\begin{pmatrix} A & 0 & B \\ -Q & -A^T & -S \\ S^T & B^T & 0 \end{pmatrix}}_{\tilde{A}} \begin{pmatrix} x \\ \lambda \\ u \end{pmatrix} + \begin{pmatrix} 0 \\ -q \\ r \end{pmatrix}.$$

Lemma (Steady-state singularity of linear quadratic OCPs).
If for all $s \in \mathbb{C}$

$$\det(s\tilde{E} - \tilde{A}) = p(s) = constant \neq 0,$$

then OCP-SING is steady-state singular.

→ Use of properties of nilpotent DAEs
→ Extension to nonlinear dynamics?

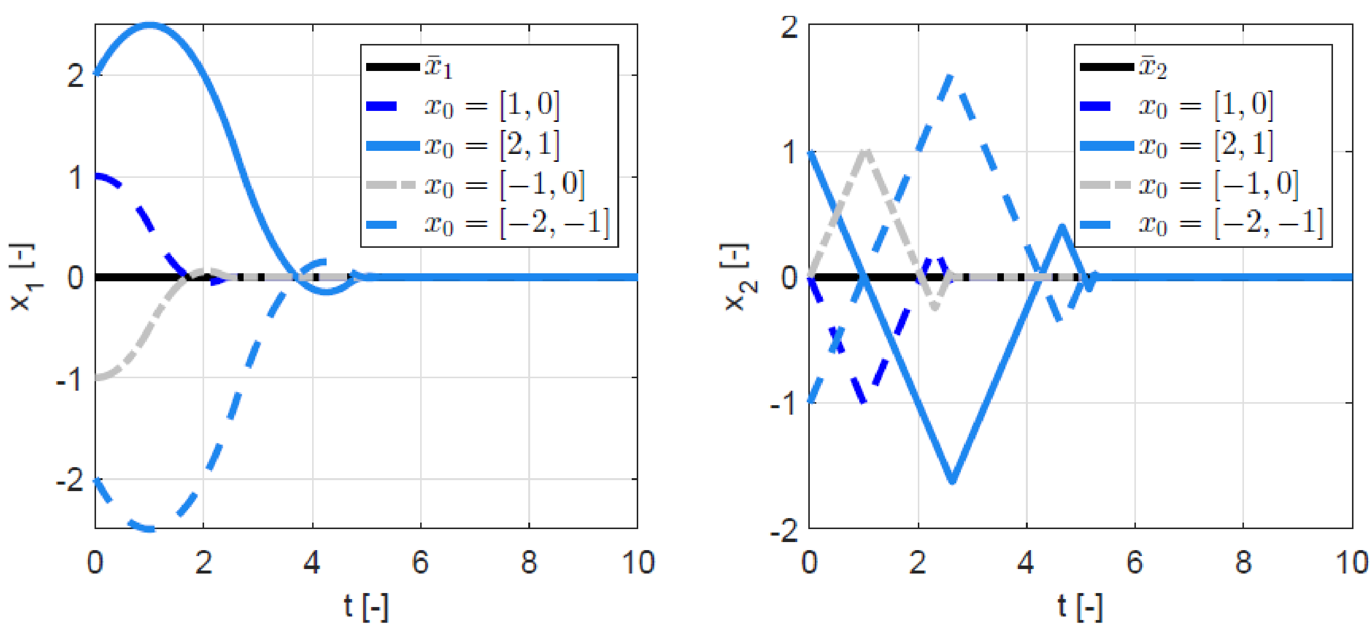
Example – Fuller‘s Problem

$$\min_{u(\cdot)} \quad \int_0^T (x_1(\tau))^2 d\tau$$

subject to

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u, \quad x(0) = x_0 \qquad \Rightarrow \qquad \det(s\tilde{E} - \tilde{A}) = -1$$

$$u(t) \in [-1, 1] \; a.e., \quad u(\cdot) \in \mathcal{L}^\infty$$



Back to Regular OCPs

Considered OCP

$$\min_{u(\cdot)} \int_0^T F(x(\tau), u(\tau))\, d\tau$$

$$\min_{u(\cdot)} \int_0^T F_{lq}(x(\tau), u(\tau))\, d\tau$$

subject to

$$\frac{d}{d\tau} x(\tau) = f(x(\tau), u(\tau)), \quad x(0) = x_0 \in \mathcal{X}_0$$

$$\frac{d}{d\tau} x(\tau) = Ax(\tau) + Bu(\tau), \quad x(0) = x_0 \in \mathcal{X}_0$$

$$u(\tau) \in \mathcal{U} \subset \mathbb{R}^{n_u}, \; x(\tau) \in \mathcal{X} \subseteq \mathbb{R}^{n_x}$$

$$F_{lq}(x, u) = \frac{1}{2} x^\top Q x + x^\top S u + \frac{1}{2} u^\top R u + q^\top x + r^\top u$$

Assumption. OCP is regular at turnpike $(\bar{x}, \bar{u})$, i.e. consider

$$H(x, u, \lambda) = F(x, u) + \lambda^\top f(x, u)$$

such that $H \in \mathcal{C}^2$ and

$$\det H_{uu}(\bar{x}, \bar{u}) = \det R \neq 0.$$

Back to Regular OCPs

Lemma (No exact turnpikes in regular OCPs).
Let

- the OCP be linear-quadratic and regular,
- let it exhibit a turnpike at $(\bar{x}, \bar{u}) \in \text{int}(\mathcal{X} \times \mathcal{U})$, and
- let A, B be controllable.

Then the turnpike property is approximate, i.e. it is not exact.

Sketch of proof:

- W.l.o.g. $(\bar{x}, \bar{u}) = 0$ and $x_0 = 0$.
- For $x_0 = 0$, we have $u^\star = -R^{-1}(r + B^\top \lambda^\star)$:
  $u^\star = 0 \quad \Leftrightarrow \quad r = -B^\top \lambda^\star$
- Starting from $x_0 = 0$:
  $\dot{\lambda}^\star = -A^\top \lambda^\star - q, \quad r = -B^\top \lambda^\star, \quad \lambda^\star(T) = 0$
- (A, B) controllable $\Rightarrow u^\star(\tau) \not\equiv 0$

Role of Adjoints in Turnpike Properties

NCO for $\mathcal{X} = \mathbb{R}^{n_x}$:

$$\begin{aligned}\frac{dx^\star(\tau)}{d\tau} &= \quad H_\lambda(\lambda_0^\star, \lambda^\star(\tau), x^\star(\tau), u^\star(\tau)), \quad x^\star(0) = x_0 \\ \frac{d\lambda^\star(\tau)}{d\tau} &= -H_x(\lambda_0^\star, \lambda^\star(\tau), x^\star(\tau), u^\star(\tau)), \quad \lambda^\star(T) = 0\end{aligned}$$

$$\forall \tau \in [0, T] \text{ and } \forall u \in \mathcal{U}$$

$$H(\lambda_0^\star, \lambda^\star(\tau), x^\star(\tau), u^\star(\tau)) \leq H(\lambda_0^\star, \lambda^\star(\tau), x^\star(\tau), u)$$

Turnpike $(\bar{x}, \bar{u}) \in \text{int}(\mathcal{X} \times \mathcal{U})$ corresponds to $\bar{\lambda}$ such that:

$$\begin{aligned}0 &= \quad H_\lambda(\lambda_0, \bar{\lambda}, \bar{x}, \bar{u}) \\ 0 &= -H_x(\lambda_0, \bar{\lambda}, \bar{x}, \bar{u}) \\ 0 &= \quad H_u(\lambda_0, \bar{\lambda}, \bar{x}, \bar{u})\end{aligned}$$

Observations

- Turnpike $(\bar{x}, \bar{u}) \in \text{int}(\mathcal{X} \times \mathcal{U})$ with $\bar{\lambda} \neq 0$ has a leaving arc.
- If turnpike is not exact, the leaving arc leads to practical convergence of NMPC.

Recovering Asymptotic Convergence in EMPC

- Add a terminal constraint, e.g. $x(t_k+T)=\bar{x}$
- Terminal penalty (Mayer term) $E(x)=-S(x)$ or rotate F by storage function:

$$F(x,u)\to\tilde{F}(x,u):=F(x,u)-\frac{\partial S}{\partial x}f(x,u)-F(\bar{x},\bar{u})$$
$$\tilde{F}(x,u)\geq\alpha(\|(x,u)-(\bar{x},\bar{u})\|)$$

$\Rightarrow$ Without terminal constraints open-loop solutions change due to rotation!

Adjoint interpretation of rotation

- Rotated stage costs imply that $\lambda^\star(\tau)\approx 0$ whenever $z^\star(\tau)\approx(\bar{x},\bar{u})$.
- $\lambda^\star(\tau)\approx 0=\lambda^\star(T)$ whenever $z^\star(\tau)\approx(\bar{x},\bar{u})$.

Linear Terminal Penalties in EMPC

$$\min_{u(\cdot)} \int_{t_k}^{t_k+T} F(x(\tau), u(\tau))d\tau + \bar{\lambda}^\top x(t_k+T)$$

subject to $\qquad (\mathsf{OCP}_{T,\bar{\lambda}}(\hat{x}(t_k)))$

$$\frac{d}{d\tau}x(\tau) = f(x(\tau), u(\tau)), \quad x(t_k) = \hat{x}(t_k)$$
$$u(\tau) \in \mathcal{U} \subset \mathbb{R}^{n_u}, x(\tau) \in \mathcal{X} \subset \mathbb{R}^{n_x}$$

Second variation at $\bar{x}, \bar{u}, \bar{\lambda}$:

$$A = f_x, B = f_u, Q = H_{xx}, S = H_{xu}, R = H_{uu}, q = F_x, r = F_u$$

$$\min_{u(\cdot)} \int_{t_k}^{t_k+T} F_{lq}(x(\tau), u(\tau))d\tau + \bar{\lambda}^\top x(t_k+T)$$

subject to $\qquad (\mathsf{LQR}_{T,\bar{\lambda}}(\hat{x}(t_k)))$

$$\frac{d}{d\tau}x(\tau) = Ax(\tau) + Bu(\tau), \; x(t_k) = \hat{x}(t_k)$$
$$F_{lq}(x,u) = \frac{1}{2}x^\top Qx + x^\top Su + \frac{1}{2}u^\top Ru + q^\top x + r^\top u$$

Linear Terminal Penalties in EMPC

$$\min_{u(\cdot)} \quad \int_{t_k}^{t_k+T} F(x(\tau), u(\tau))d\tau + \bar{\lambda}^\top x(t_k+T)$$

subject to $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $(\mathsf{OCP}_{T,\bar{\lambda}}(\hat{x}(t_k)))$

$$\frac{d}{d\tau}x(\tau) = f(x(\tau), u(\tau)), \quad x(t_k) = \hat{x}(t_k)$$

$$u(\tau) \in \mathcal{U} \subset \mathbb{R}^{n_u}, x(\tau) \in \mathcal{X} \subset \mathbb{R}^{n_x}$$

Theorem (Convergence of NMPC with linear end penalty).
Suppose that

- Σ is controlled via $\mathsf{OCP}_{T,\bar{\lambda}}(\hat{x}(t_k))$, Σ is locally controllable at $(\bar{x}, \bar{u})$,
- for all $\hat{x}(t_k) \in \mathcal{X}_0$, $\mathsf{OCP}_{T,\bar{\lambda}}(\hat{x}(t_k))$ is strictly dissipative w.r.t. $(\bar{x}, \bar{u}) \in \mathrm{int}(\mathcal{X} \times \mathcal{U})$,
- for some finite horizon $T > 0$, the solution to $\mathsf{LQR}_{T,\bar{\lambda}}(\hat{x}(t_k))$ is stabilizing.

Then there exists $T > 0$, $\delta > 0$ such that

- $\mathsf{OCP}_{T,\bar{\lambda}}(\hat{x}(t_k))$ is recursively feasible, and
- $\lim_{t\to\infty} \hat{x}(t, x_0, u^{mpc}(\cdot)) = \bar{x}$.

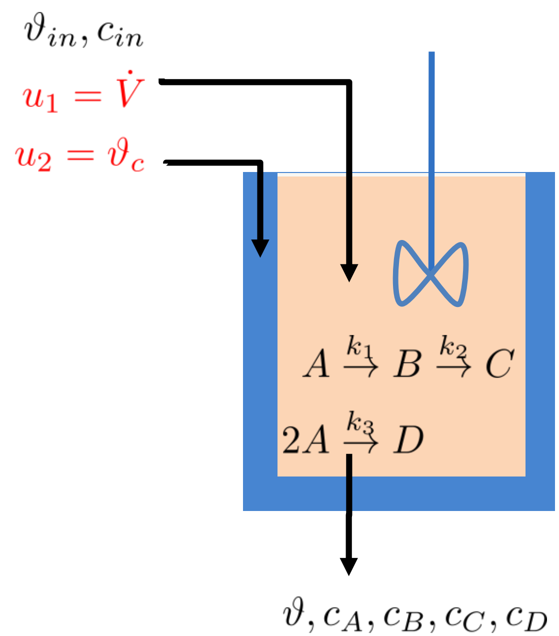
[Zanon & Faulwasser `17]

Example – Chemical Reactor

Van de Vusse Reactor $A \overset{k_1}{\rightarrow} B \overset{k_2}{\rightarrow} C, \qquad 2A \overset{k_3}{\rightarrow} D$

Dynamics (partial model)

$$\dot{c}_A = r_A(c_A, \vartheta) + (c_{in} - c_A)u_1$$
$$\dot{c}_B = r_B(c_A, c_B, \vartheta) - c_B u_1$$
$$\dot{\vartheta} = h(c_A, c_B, \vartheta) + \alpha(u_2 - \vartheta) + (\vartheta_{in} - \vartheta)u_1,$$
$$r_A(c_A, \vartheta) = -k_1(\vartheta)c_A - 2k_3(\vartheta)c_A^2$$
$$r_B(c_A, c_B, \vartheta) = \quad k_1(\vartheta)c_A - k_2(\vartheta)c_B$$
$$h(c_A, c_B, \vartheta) = -\delta\Big(k_1(\vartheta)c_A \Delta H_{AB} + k_2(\vartheta)c_B \Delta H_{BC} + 2k_3(\vartheta)c_A^2 \Delta H_{AD}\Big)$$
$$k_i(\vartheta) = \quad k_{i0} \exp \frac{-E_i}{\vartheta + \vartheta_0}, \quad i = 1, 2, 3.$$



Constraints

$$c_A \in [0, 6]\frac{mol}{l} \quad c_B \in [0, 4]\frac{mol}{l} \quad \vartheta \in [70, 150]°C$$
$$u_1 \in [3, 35]\frac{1}{h} \quad u_2 \in [0, 200]°C.$$

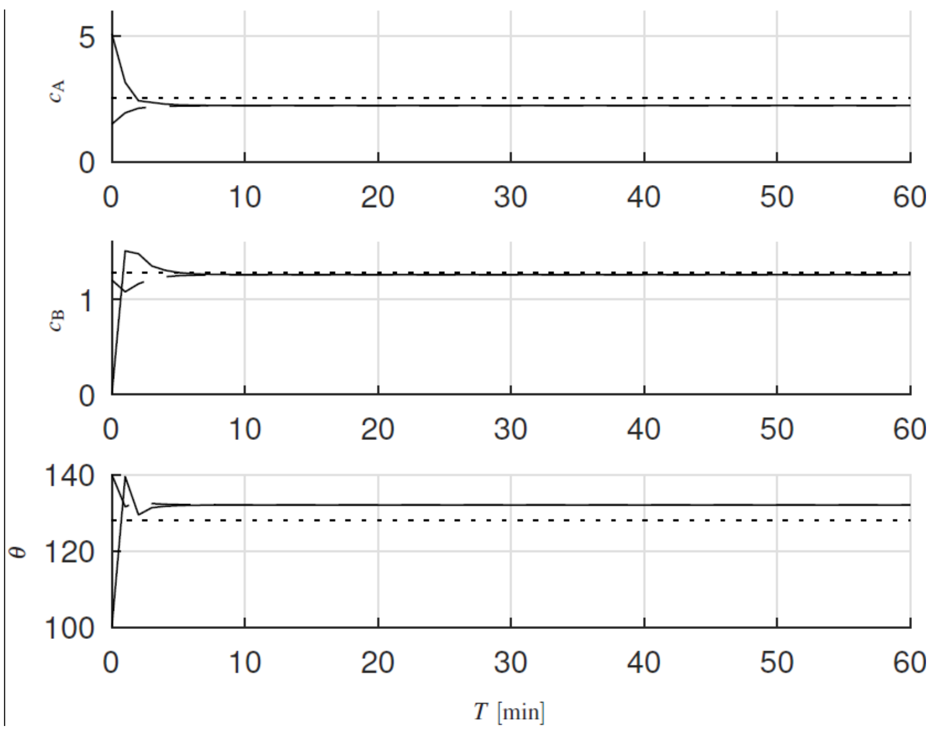
Objective = maximize produced amount of B

$$J_T(x_0, u(\cdot)) = \int_0^T -\beta c_B(t) u_1(t) dt, \qquad \beta > 0$$

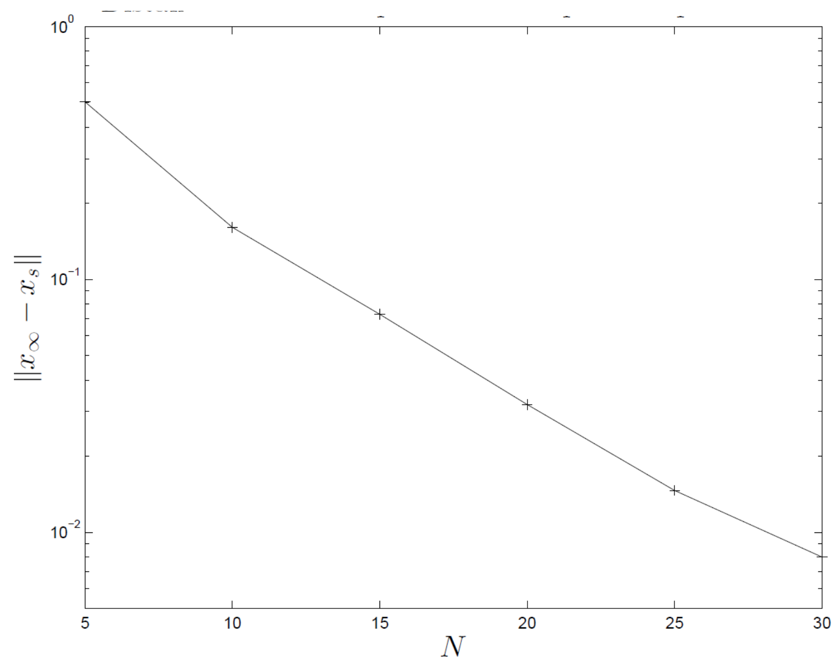
[Chen et al. `95; Rothfuß, Rudolph, Zeitz `96]

Example – Chemical Reactor

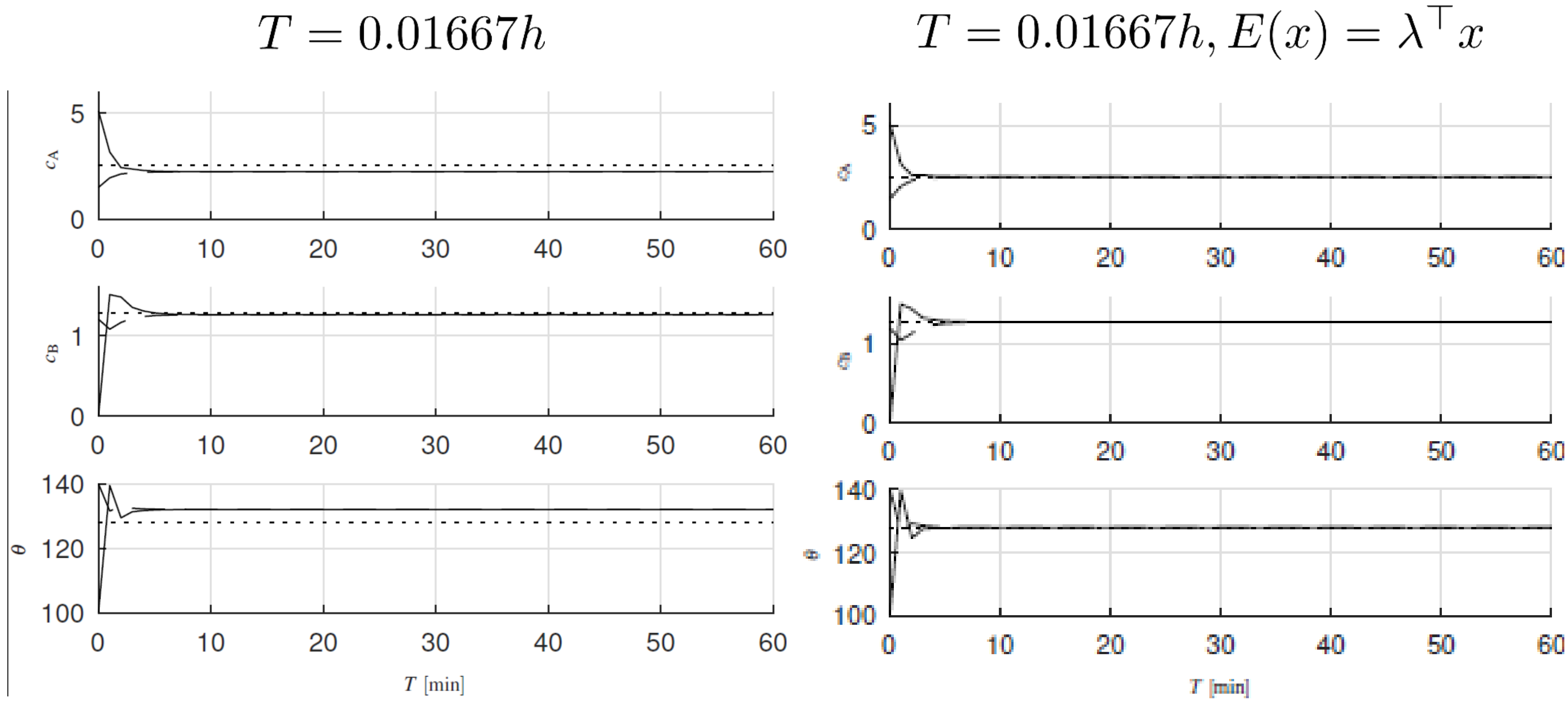
$T = 0.01667h$



Distance to equilibrium



Example – Chemical Reactor



Summary and Outlook

Turnpikes and dissipativity

- Suff. conditions for turnpikes via dissipativity (OCPs with or without terminal constraints).
- Suff. conditions for exact turnpikes.

Approximate versus exact turnpikes

- Linear-quadratic singular OCP → exactness of turnpikes via nilpotent DAE

↕

- Linear-quadratic regular OCP → approximate turnpikes (NCO = DAE with index 1)

EMPC with linear end penalty (gradient correction)

- Allows recovering asymptotic convergence/stability

Outlook

- Turnpikes with active constraints?
- Time-varying turnpikes? Classification thereof?
- …

Thank you! Questions?

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