A Hybrid control approach to Optimal Routing for Sailing Boats.

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RÉGION NORMANDIE

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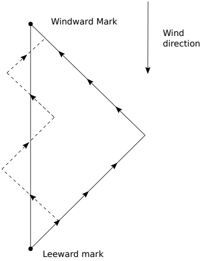
Basic Goal

Our objective is to find the optimal trajectory to move from A to B.

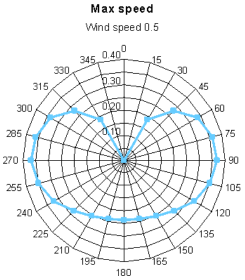
A typical case of interest is when the path A-B is (more or less) *aligned* with the wind direction. In that case the optimal trajectory is not trivially a straight line.

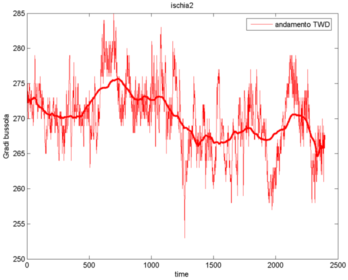


Challenges and motivations

- Sailing boat dynamics - *Complex behaviors*
- Presence of Discontinuous/non convex dynamics
- Presence of noisy data (*Wind*)

Polar plot of the dynamics and wind direction during a race.

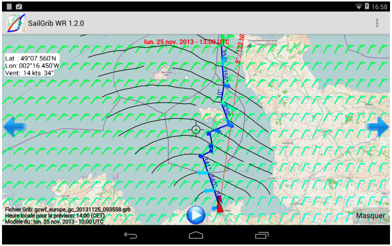


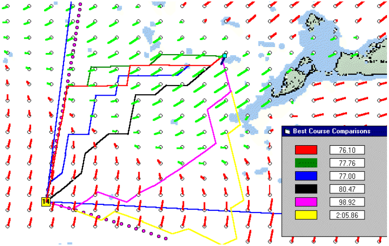


Other challenging aspects

- Change of dynamics - *Change of sails*
- Presence of constraints (islands, etc.)
- Presence of competitors/moving obstacles

Example: *Route planning* some available software





A Hybrid framework for the route planning problem

Hybrid control

Let $\mathcal{I} = \{1, 2, \ldots, N_{\mathcal{I}}\}$ be finite, and consider the controlled system (X, Q) described by:

$$\begin{cases} dX(t) = f(X(t), Q(t), u(t))dt + \sigma(X(t), Q(t))\, dW_t, \\ X(0) = x, \;\; Q(0^+) = q. \end{cases}$$

$X(t)$ and $Q(t)$ are the continuous and the discrete component of the state at time t. The dynamics $f : \mathbb{R}^d \times \mathcal{I} \times U \to \mathbb{R}^d$ are controlled in:

$$\mathcal{U} = \{u : (0, \infty) \to U \mid u \text{ measurable}, \;\; U \text{ compact}\},$$

with f and σ globally bounded and uniformly Lipschitz continuous w.r.t. x.

Switch function

The term $Q(t)$ models the possibility to switch between the various dynamics of the system, and takes values in the set of discrete controls $\mathbb{Q}$, that is:

$$\mathbb{Q} = \{Q(\cdot) : (0, \infty) \to \mathcal{I} | Q(t) = \sum_{i}^{N} w_i \chi_{t_i}(t) \text{ (piecewise constant)}\}.$$

where $\chi_{t_i}(t) = 1$ if $t \in [t_i, t_{i+1})$ and 0 otherwise, $\{t_i\}_{i=1,\ldots,N}$ are the (ordered) times at which a switch occurs, and $\{w_i\}_{i=1,\ldots,N}$ are values in $\mathcal{I}$.

Cost functional

The trajectory starts from $(x, q) \in \mathbb{R}^d \times \mathcal{I}$. The control strategy $\Theta := (u, \{t_i\}, \{w_i\})$ minimizes the cost functional:

$$J(x, q; \Theta) := \mathbb{E}\left(\int_0^{\tau_{x,q}} e^{-\lambda t} dt + \sum_{i=0}^{N} C\left(X(t_i), Q(t_i^-), Q(t_i^+)\right) e^{-\lambda t_i}\right)$$

where $\lambda > 0$ is the discount factor, and $C : \mathbb{R}^d \times \mathcal{I} \times \mathcal{I} \to \mathbb{R}_+$ is the switching cost between the dynamics.

The latter is bounded, strictly positive and Lipschitz continuous w.r.t. x and to satisfy the further condition

$$C(x, q_1, q_2) < C(x, q_1, q_3) + C(x, q_3, q_2),$$

for any triple of indices q_1, q_2 and q_3.

Value function

The term $\tau_{x,q}$ is the first time of arrival in the compact target set $\mathcal{T} \subset \mathbb{R}^d$, i.e.,

$$\tau_{x,q} := \min_{t \in [0,+\infty)} \{t \mid X(t) \in \mathcal{T}\}.$$

The value function v of the problem is then defined, for $\Theta \in \mathcal{U} \times \mathbb{R}_+^{\mathbb{N}} \times \mathcal{I}^{\mathbb{N}}$, as:

$$v(x,q) := \inf_{\Theta} J(x,q;\Theta),$$

and is characterized via a suitable Hamilton–Jacobi–Bellman (HJB) equation.

Differential characterization

Defining for $x, p \in \mathbb{R}^d$ and $q \in \mathcal{I}$ the Hamiltonian function by

$$H(x,q,p) := \sup_{u \in U} \{-f(x,q,u) \cdot p - 1\}$$

and the controlled switching operator $\mathcal{N}$ by:

$$\mathcal{N}\phi(x,q) := \inf_{w \in \mathcal{I}} \{\phi(x,w) + C(x,q,w)\},$$

we have a Bellman equation of the following form:

$$\max\left(v - \mathcal{N}v, \lambda v + H(x,q,Dv) + \frac{1}{2}\operatorname{tr}\left(\sigma\sigma^t D^2 v\right)\right) = 0,$$

defined on $(\Omega \setminus \mathcal{T}) \times \mathcal{I}$, i.e., a system of Quasi-Variational Inequalities, complemented with the boundary condition

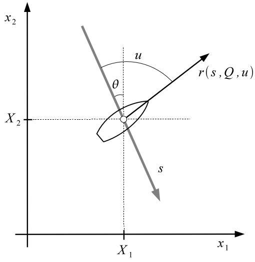
$$v(x,q) = 0 \quad x \in \partial\mathcal{T}.$$

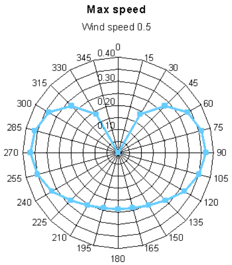
Practical models for route planning

Let be x_1 and x_2 (position of the boat) x_3 (evolution of the wind) and the control u the (unsigned) angle between the boat direction and the wind, so that $u \in U = [0, \pi]$. The motion of the boat is by

$$\begin{cases} \dot{X}_1(t) = r(s(X,t), Q(t), u(t)) \sin(-\theta(X,t) \pm u(t)) \\ \dot{X}_2(t) = r(s(X,t), Q(t), u(t)) \cos(\theta(X,t) \pm u(t)), \end{cases}$$

where $+$ (starboard tack) and $-$ (port tack). The function $r : \mathbb{R}_+ \times \mathcal{I} \times [0, \pi] \to \mathbb{R}_+$ models the *polar plot* of the boat.





Polar plot

The choice of the function $r(s, q, u)$ is related to the technical characteristics of the craft. It differs from one boat to another, but we can detect some general features:

- $r(s, q, u)$ is continuous w.r.t. both variables s and u;
- For given q, it depends on the wind speed s and the relative angle u, but neither on time nor on position; moreover, the dependence on s is monotone;
- $r(s, q, 0) = 0$, which means that the boat has always zero speed when ‘pointing directly against the wind’;
- $r(s, q, \cdot)$ has typically (but not necessarily) a single maximum point inside $[0, \pi]$.

Wind modelling

The wind is characterized by direction θ and speed s. We assume that they evolve in time according to a lumped parameter model, i.e., the system of SDEs

$$\begin{cases} ds(x,t) = g_1(x, s(x,t), \theta(x,t))dt + g_2(x, s(x,t), \theta(x,t))dW_t^{(1)} \\ d\theta(x,t) = h_1(x, s(x,t), \theta(x,t))dt + h_2(x, s(x,t), \theta(x,t))dW_t^{(2)}, \end{cases}$$

in which x is considered as a parameter, $dW_t^{(i)}$ $(i = 1, 2)$ denotes the differential of a standard Brownian process.

We will discuss now in detail a simplified model:

$$\begin{cases} \dot{X}_1 = r(s,u)\sin\left(-\theta + (-1)^Q u\right) \\ \dot{X}_2 = r(s,u)\cos\left(\theta + (-1)^Q u\right). \end{cases}$$

We assume the wind speed s is constant $s(x,t) \equiv \bar{s}$ and the direction evolves according to

$$d\theta = a(\theta)dt + \bar{\sigma} dW_t.$$

We assume that the cost of tacking is constant, $C(x,1,2) = C(x,2,1) = \bar{C}$, and use a positive λ in order to obtain a convergent value iteration.

$$\begin{cases} f_1(x,q,u) = r(\bar{s},u)\sin\left(-x_3 + (-1)^q u\right) \\ f_2(x,q,u) = r(\bar{s},u)\cos\left(x_3 + (-1)^q u\right) \\ f_3(x,q,u) = a(x_3), \end{cases}$$

which results in a Hamiltonian function in the form

$$H(x,q,p) = \sup_{u\in U}\Big\{ - r(\bar{s},u)\left[\sin\left(-x_3 + (-1)^q u\right) p_1 \right. \\ \left. + \cos\left(x_3 + (-1)^q u\right) p_2\right] \Big\} - a(x_3)p_3 - 1$$

The term related to the stochastic component of the evolution reads in turn as

$$\frac{1}{2}\operatorname{tr}\left(\sigma\sigma^t D^2 v\right) = \frac{\bar{\sigma}^2}{2} v_{x_3x_3}.$$

The switching operator $\mathcal{N}$ is given by:

$$\mathcal{N}\phi(x,q) = \begin{cases} \min\{\phi(x,1), \phi(x,2) + \bar{C}\} & \text{if } q = 1, \\ \min\{\phi(x,2), \phi(x,1) + \bar{C}\} & \text{if } q = 2. \end{cases}$$

Then, the Bellman equation takes the form of the following system of two Quasi-Variational Inequalities:

$$\max\Big(v(x,1) - \min\left\{v(x,1), v(x,2) + \bar{C}\right\},$$
$$\lambda v(x,1) + H(x, \nabla v(x,1), 1) \quad + \tfrac{\bar{\sigma}}{2} v_{x_3 x_3}(x,1)\Big) = 0,$$
$$\max\Big(v(x,2) - \min\left\{v(x,2), v(x,1) + \bar{C}\right\},$$
$$\lambda v(x,2) + H(x, \nabla v(x,2), 2) \quad + \tfrac{\bar{\sigma}}{2} v_{x_3 x_3}(x,2)\Big) = 0.$$

Numerical Resolution via Semilagrangian Methods

Consider a discrete grid of nodes (x_j, q) of discretization steps $\Delta = (\Delta t, \Delta x)$ and the approximate value function by V^Δ.
The scheme at (x_j, q) in fixed point (or value iteration) form is

$$V^\Delta(x_j, q) = \min\left(NV^\Delta(x_j, q), \Sigma\left(x_j, q, V^\Delta\right)\right).$$

The numerical operator Σ is the approximation of the Hamiltonian function. The discrete switch operator N at (x_j, q) is

$$NV^\Delta(x_j, q) := \min_{q' \in \mathcal{I}}\left\{V^\Delta(x_j, q') + C(x_j, q, q')\right\},$$

and, in fact, this corresponds to the exact definition.
With this notation, a standard semi-Lagrangian discretization is

$$\Sigma\left(x_j, q, V^\Delta\right) = \frac{1}{2d}\sum_{i=1}^{d}\min_{u \in U}\left\{\Delta t + e^{-\lambda \Delta t}\left(\mathbb{I}\left[V^\Delta\right](\delta_i^+, q) + \mathbb{I}\left[V^\Delta\right](\delta_i^-, q)\right)\right\},$$

where $\delta_i^\pm := x_j + \Delta t\, f(x_j, q, u) \pm \sqrt{d\Delta t}\,\sigma(x_j, q)e_i$.

Test 1

Let us fix some of the parameters of the problem: $\mathcal{T} := B(\hat{x}, 0.04)$ with $\hat{x} := (0, 1.7)$. $\ell \equiv 0.5$, $C \equiv 0.6$ (the cost to switch between dynamics).

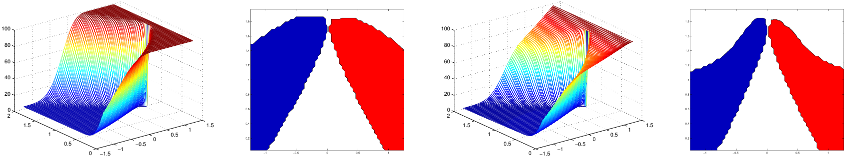


Figure: Test 1: comparison between the solution $(q = 1)$ in the case of $\bar{\sigma} = 0$ (above/left) and $\bar{\sigma} = 0.02$ (above/right) and optimal switching map for a wind of direction $\theta = 0$ for $\bar{\sigma} = 0$ (below/left) and $\bar{\sigma} = 0.02$ (below/right) both the comparisons are made for $x_3 = 0$. The space between the two switching regions is generally called *tacking triangle*

Comparison with various diffusion coefficients

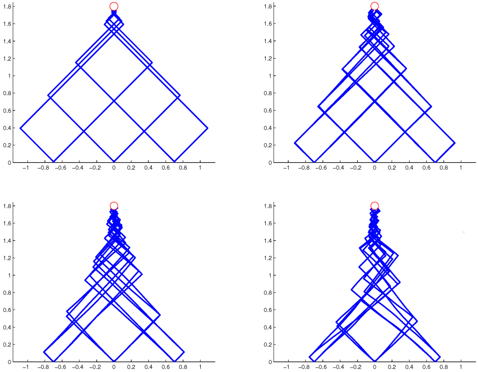
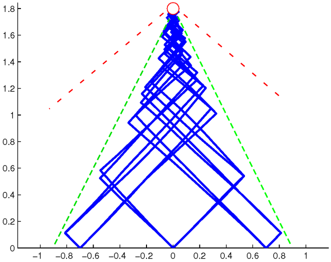


Figure: Test 1: some optimal trajectories for different diffusion coefficients. The target is identified by the red circle (above/left $\bar{\sigma} = 0$, above/right $\bar{\sigma} = 0.01$, bottom/left $\bar{\sigma} = 0.05$, bottom/right $\bar{\sigma} = 0.1$)

Comparison with racing tactics



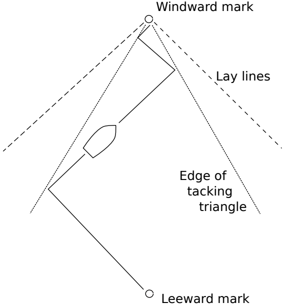


Figure: Test 1: The results of our tests compared with classic sailing tactics.

Test 2: the wind rotates

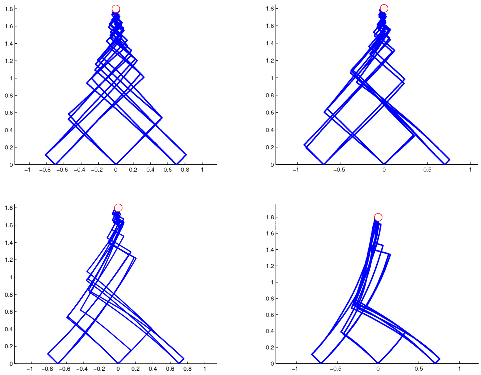


Figure: Some optimal trajectories for different drift values. $\bar{\sigma} = 0.05$ (above/left $a = 0$, above/right $a = 0.05$, bottom/left $a = 0.15$, bottom/right $a = 0.3$)

Test 2: rotation and diffusion

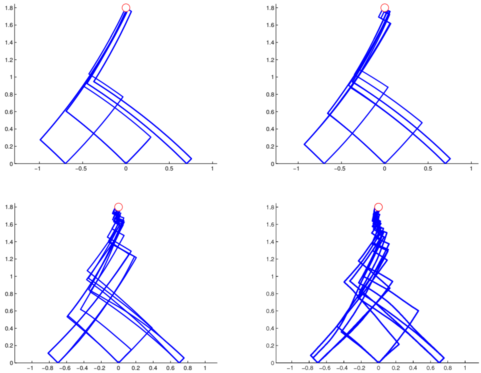
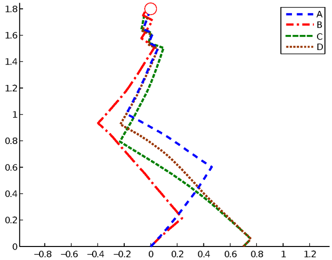


Figure: Some optimal trajectories for the wind turning anti-clockwise (drift $a = 0.15$) for different diffusion coefficients (above/left $\bar{\sigma} = 0$, above/right $\bar{\sigma} = 0.01$, bottom/left $\bar{\sigma} = 0.05$, bottom/right $\bar{\sigma} = 0.1$)

Test 2: Tacking on a lift



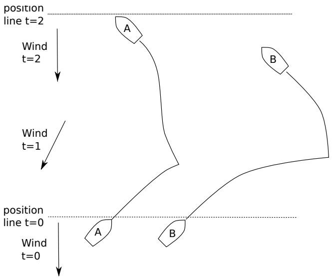


Figure: Test 2: (left) automatic evaluation of the optimal tacking in presence of wind perturbations (right) a classical tactic strategy of 'tacking on a lift'.

Test 3: Coastal route planning

We include the presence of obstacles with a simple *penalization procedure* of the speed of evolution of the dynamics. Considered then a constraint subset $\Gamma \subset \mathbb{R}^d$ we substitute the function $r(s,u)$ with

$$r(x,s,u) := \begin{cases} r(s,u), & \text{if } x \in \mathbb{R}^d \setminus \Gamma, \\ 0, & \text{if } x \in \Gamma. \end{cases}$$

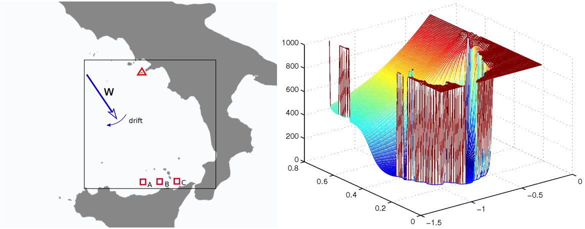


Figure: (left) the area of interest (Right) The value function (left dynamic $Q=1$) for $x_3 = -0.5$, $\bar{\sigma} = 0.005$.

Test 3: Coastal route planning

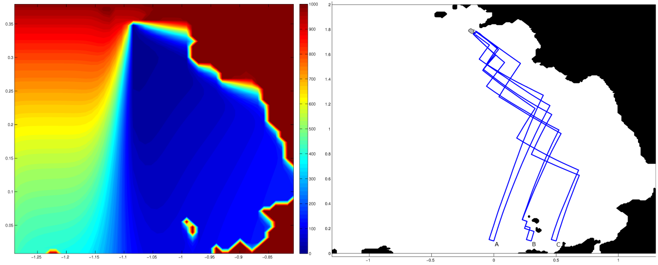


Figure: Test 3: (left) contour plot of the value function (left dynamic $Q = 1$) for $x_3 = -0.5$, $\sigma = 0.005$ and (right) some optiml trjectories starting from some points of interest.

Further developments

We have a certain validation of the effectiveness of our approach. Several points are still open and can be explored. In particular

- Improve the accuracy and the speed of our tools, in order to consider more complex cases (multiple dynamics, more complex diffusion parameters).
- Consider the presence of other competitors (moving constraints) on the domain.
- Consider a mean field framework to model very crowded environments.

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Thank you.