On Model Predictive Control for the Fokker-Planck Equation

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Numerical methods for optimal control problems:
algorithms, analysis and applications
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Outline

1. Model Predictive Control (MPC) ...
    - Introduction
    - Stability of the MPC Closed Loop System
    - Exponential Controllability w.r.t. ℓ
2. ... for the Fokker-Planck Equation
    - Introduction
    - Minimal Stabilizing Horizon Length
    - Numerical Simulations
3. Conclusion

Basic idea of MPC

Common tasks in optimal control

- Steer the state to a desired equilibrium and keep it there.
- Follow a reference trajectory.

$\Rightarrow$ Infinite horizon optimal control problem; difficult to solve.

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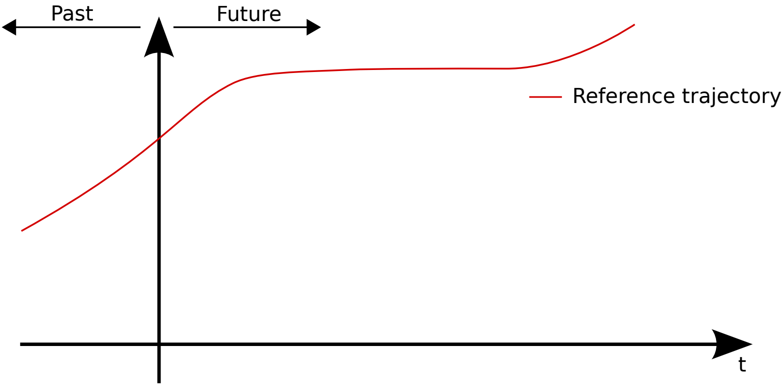
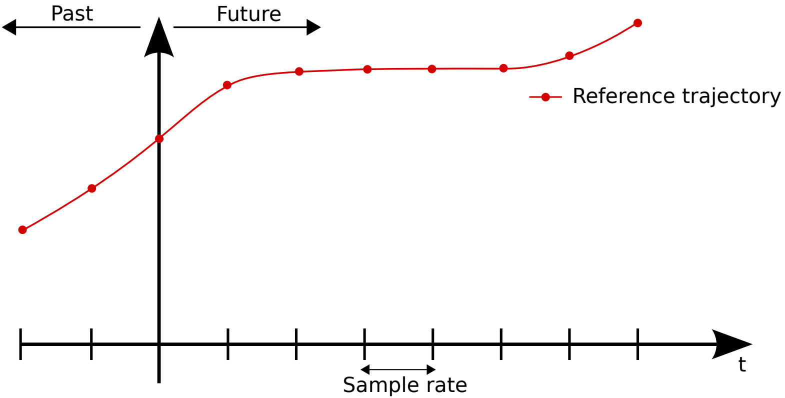
Idea

Split up the problem into several iterative OCPs on (shorter and) finite time horizons.



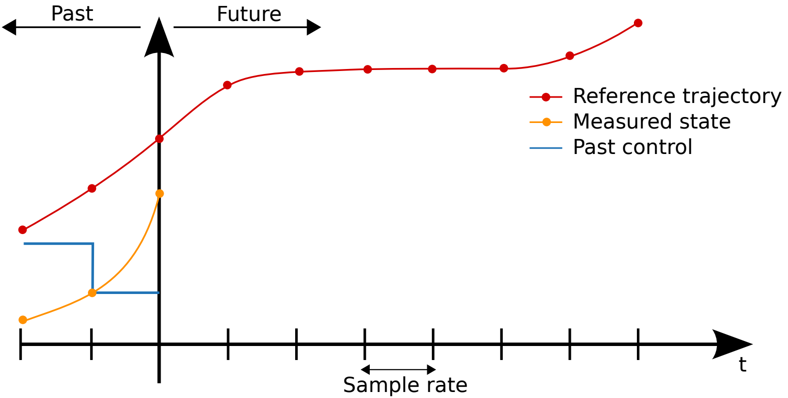
Consider an OCP on $[0, \infty)$



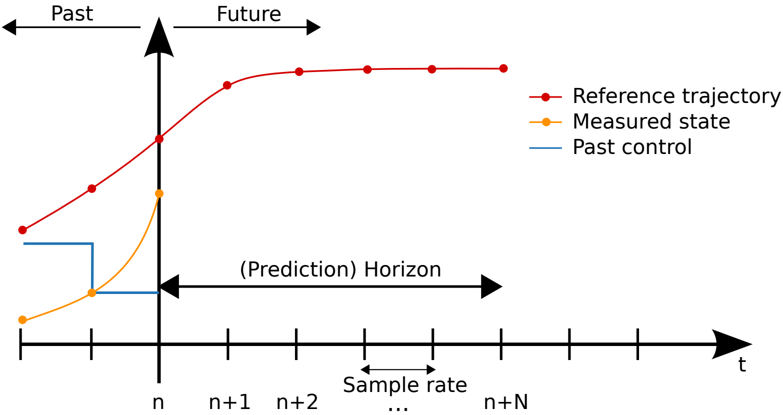
Consider an OCP on $[0, \infty)$ in a discrete time setting:

$$y(k+1) = f(y(k), u(k)), \; y(0) = y_0$$

where $y(k) \in Y$, $u(k) \in U$, Y and U being metric spaces.

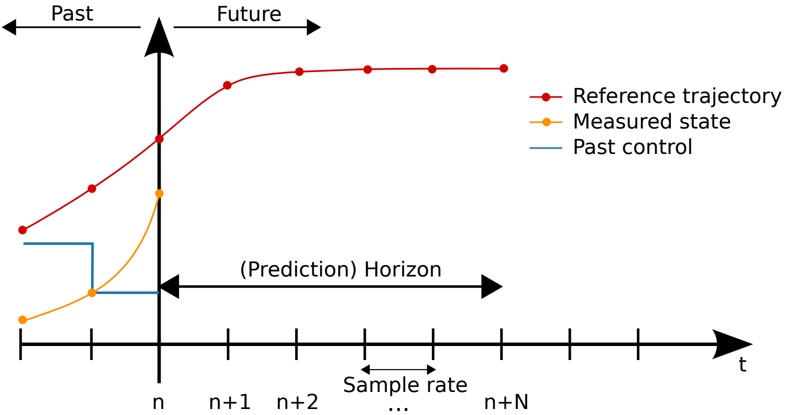


Instead of minimizing $J_\infty(y_0, u) := \sum_{k=0}^{\infty} \ell(y_u(k; y_0), u(k))$, where $\ell\colon Y \times U \to \mathbb{R}_{\geq 0}$ is a continuous stage cost function and $y_u(\cdot; y_0)$ is the solution trajectory for a given control sequence $(u(k))_{k\in\mathbb{N}_0}$ and an initial state y_0,

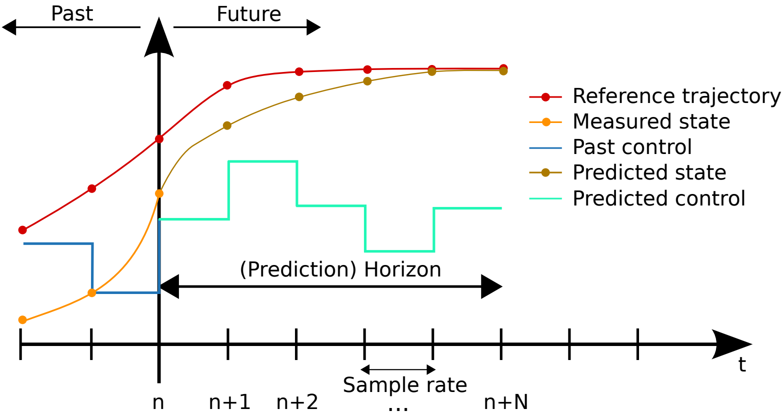


we choose a horizon $N \geq 2$ and compute a feedback law $\mathcal{F}$ via the following steps. For each time t_n, $n = 0, 1, 2, \ldots$:

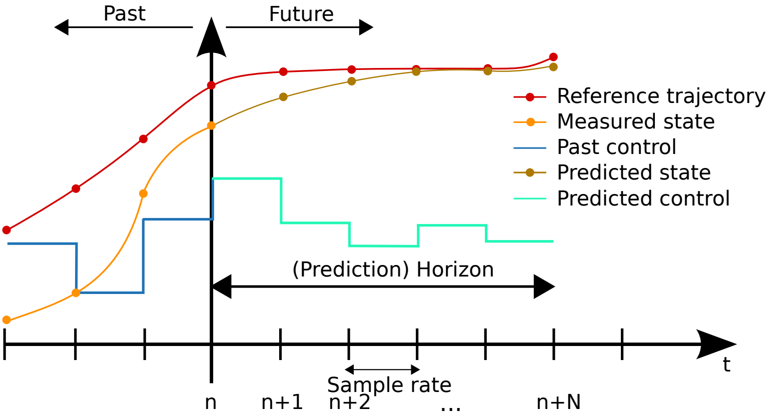
1. Measure the current state $y(n)$ and set $y_0 := y(n)$.



2. Solve the OCP on the current time horizon $[t_n, t_{n+N}]$:
$u^* = \arg\min_{u \in U^N} J_N(y_0, u)$, where
$J_N(y_0, u) := \sum_{k=0}^{N-1} \ell(y_u(k; y_0), u(k))$.



3. Apply the first value of the calculated optimal control sequence on $[t_n, t_{n+1}]$ and set $\mathcal{F}(n) := u^*(0)$.



4. Set $n := n + 1$ and go to 1.

$\Rightarrow$ Resulting MPC closed-loop: $y_{\mathcal{F}}(k+1) = f(y_{\mathcal{F}}(k), \mathcal{F}(y_{\mathcal{F}}(k)))$.

Adaption of http://en.wikipedia.org/wiki/File:MPC_scheme_basic.svg (CC BY-SA 3.0)

Question

How to guarantee stability of the MPC closed loop system

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Reminder: Cost functional J_N

$$\min_{u \in U^N} J_N(y_0, u) := \sum_{k=0}^{N-1} \ell(y_u(k; y_0), u(k))$$

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Definition (Exponential Controllability w.r.t. stage costs ℓ)

The system

$$y(k+1) = f(y(k),u(k)), \quad y(0) = y_0$$

is called exponentially controllable w.r.t. stage costs ℓ iff there exist an overshoot bound $C \geq 1$ and a decay rate $\rho \in (0,1)$ such that for each state $\mathring{y} \in Y$ there is a control $u_{\mathring{y}} \in U$ satisfying

$$\ell(y_{u_{\mathring{y}}}(k;\mathring{y}),u_{\mathring{y}}(k)) \leq C\rho^k \min_{u\in U} \ell(\mathring{y},u)$$

for all $k \in \mathbb{N}_0$.

Theorem (Grüne, Pannek, 2011, Thm 6.18 and Section 6.6)

Let $(\bar{y}, \bar{u})$ be an equilibrium, i.e., $f(\bar{y}, \bar{u}) = \bar{y}$. Consider the MPC scheme with stage costs

$$\ell(y(k), u(k)) = \frac{1}{2}\|y(k) - \bar{y}\|^2 + \frac{\lambda}{2}\|u(k) - \bar{u}\|^2$$

for some norm $\|\cdot\|$ and $\lambda > 0$. (In particular, we have $\ell(\bar{y}, \bar{u}) = 0$ and $\ell(y, u) > 0$ for $(y, u) \neq (\bar{y}, \bar{u})$.)

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- *Let the exponential controllability property be satisfied for the above stage costs. Then there exists $N_0 \geq 2$ such that the equilibrium $(\bar{y}, \bar{u})$ is globally asymptotically stable for the MPC closed loop for any optimization horizon $N \geq N_0$.*

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- *If, in addition, the exponential controllability property holds with $C = 1$, then $N_0 = 2$ (instantaneous control).*

Itô SDE:

$$dX_t = b(X_t, t; u)dt + \sigma(X_t, t)dW_t$$
$$X_t(t = 0) = X_0$$

Fokker-Planck Equation

$$\partial_t y(x,t) - \sum_{i,j=1}^{d} \partial_{ij}^2 \big(a_{ij}(x,t)y(x,t)\big) + \sum_{i=1}^{d} \partial_i \big(b_i(x,t;u)y(x,t)\big) = 0 \text{ in } Q$$
$$y(\cdot, 0) = y_0 \text{ in } \mathbb{R}^d$$

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where $Q := \mathbb{R}^d \times (0, T)$
$y \colon \mathbb{R}^d \times [0, \infty[\to \mathbb{R}_{\geq 0}$ is the PDF ($\int_{\mathbb{R}^d} y(x,t)\, dx = 1$),
$y_0 \colon \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ is the initial PDF ($\int_{\mathbb{R}^d} y_0(x)\, dx = 1$),
$a = \sigma\sigma^T/2$ is a symmetric positive definite matrix,
$b_i \colon \mathbb{R}^d \times [0, \infty[\times U \to \mathbb{R}, i = 1, ..., d$, and
$\partial_i z$ is the partial derivative of z w.r.t. x_i.

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Aim

Apply the above theorem, i.e., prove exponential controllability w.r.t. stage costs

$$\ell(y(k), u(k)) = \frac{1}{2} \|y(k) - \bar{y}\|^2 + \frac{\lambda}{2} \|u(k) - \bar{u}\|^2$$

(or other suitable stage costs), where $(\bar{y}, \bar{u})$ is an equilibrium.

Observation [Risken '96, Annunziato, Borzì '10]

For the controlled (multi-dim.) Ornstein-Uhlenbeck process given by

- $a_{ij} := \delta_{ij}\sigma_i^2/2$ where σ_i are positive constants, $i, j = 1, \ldots, d$,
- $b_i(x, t; u) := -\nu_i x_i + u_i$ with $\nu_i > 0, i = 1, \ldots, d$,

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the (asymptotically stable) equilibrium solution is given by

$$\bar{y}(x; \bar{u}) := \lim_{t\to\infty} y(x, t; \bar{u}) = \left((2\pi)^d \prod_{i=1}^{d} \frac{\sigma_i^2}{2\nu_i}\right)^{-1/2} \exp\left(-\sum_{i=1}^{d} \frac{(x_i - \frac{\bar{u}_i}{\nu_i})^2}{\frac{\sigma_i^2}{\nu_i}}\right)$$

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$\Rightarrow$ Controlling ν_i and u_i, any multivariate Gaussian distribution can be reached.

Proposition (2016)

Consider the 1D OU process with a control $u \in \mathbb{R}$ and stage cost

$$\ell(y(k), u(k)) = \frac{1}{2} \|y(k) - \bar{y}\|^2_{L^2(\mathbb{R})} + \frac{\lambda}{2} |u(k) - \bar{u}|^2 .$$

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Therefore, we start with the following stage cost:

$$\ell(y(k), u(k), v(k)) = \frac{1}{2} \|y(k) - \bar{y}\|^2_{L^2(\mathbb{R})} + \frac{\lambda}{2} |u(k) - \bar{u}|^2 + \frac{\lambda}{2} |v(k) - \bar{v}|^2 .$$

Example (1)

Simulation (no optimization) of the 1D OU process with a (suboptimal) "control" $(u, v) = (\bar{u}, \bar{v})$. The parameters of the dynamics are:

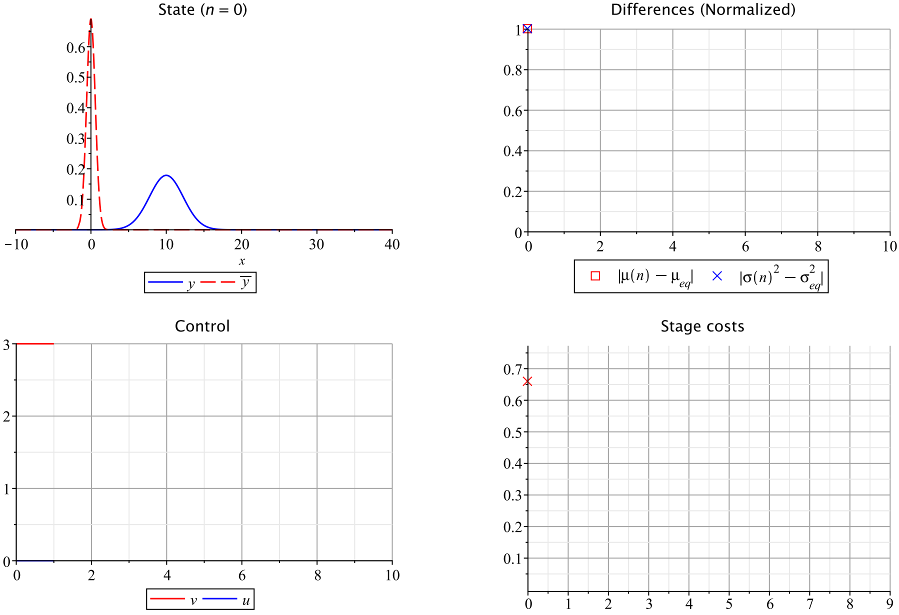
$$a_{11} = \sigma^2/2 = 1,$$
$$(\bar{u}, \bar{v}) = (0, 3).$$

The initial condition at $t_0 = 0$ is given by

$$\mathring{\mu} = 10,$$
$$\mathring{\sigma}^2 = 5.$$

The simulation is carried out until the final time $T = 2$.

Minimal Stabilizing Horizon Length



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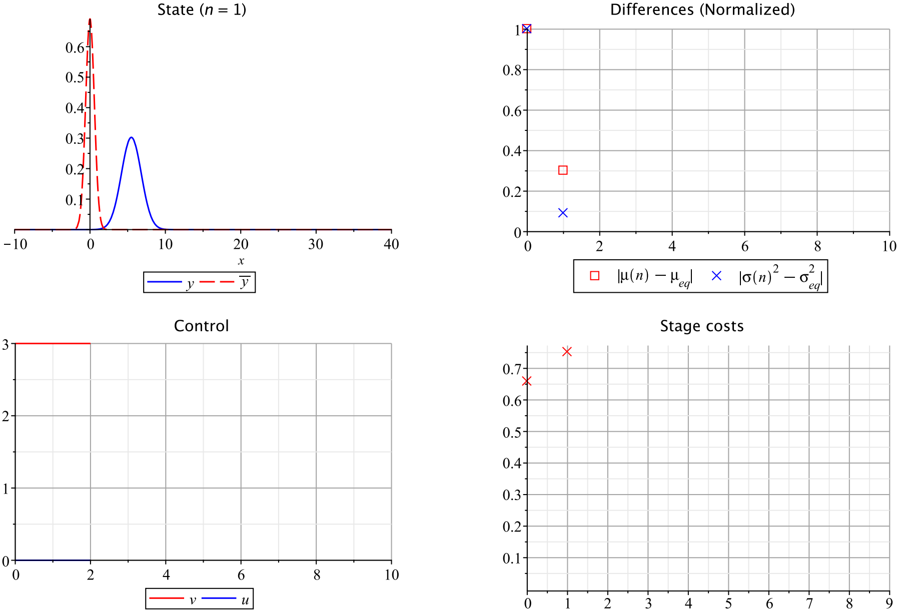

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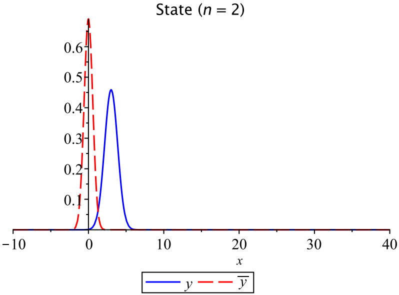

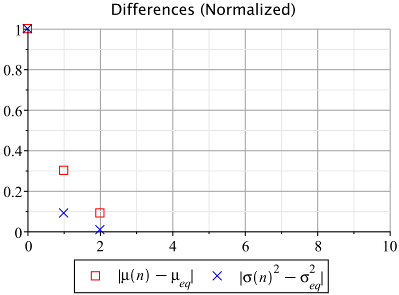

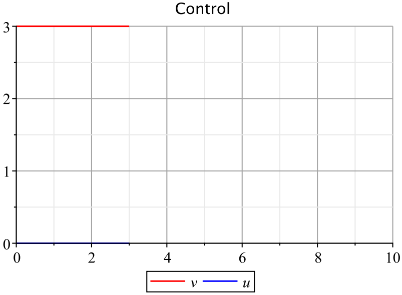

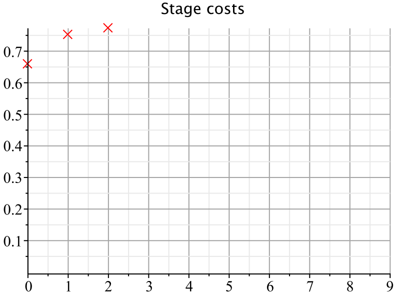

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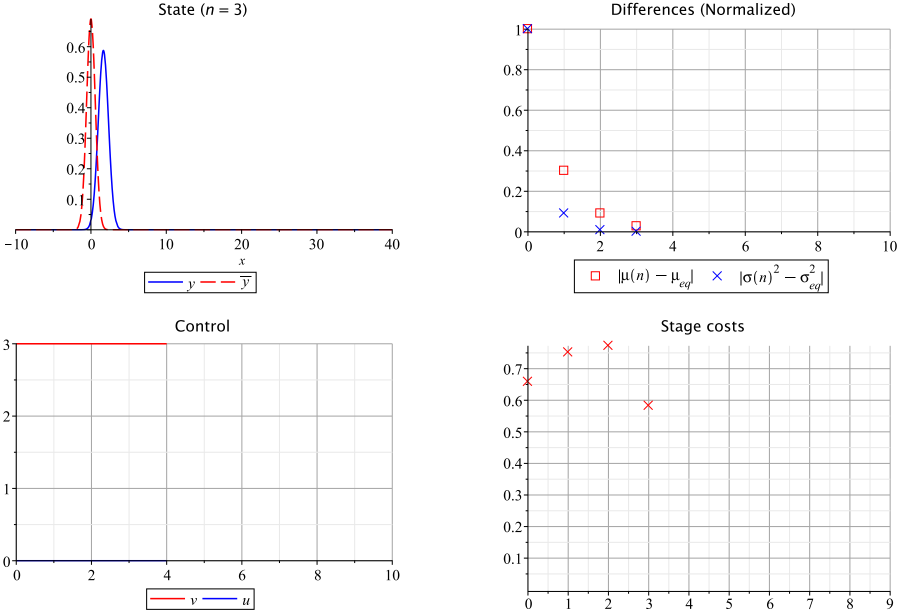
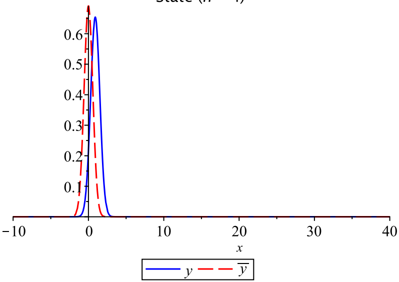
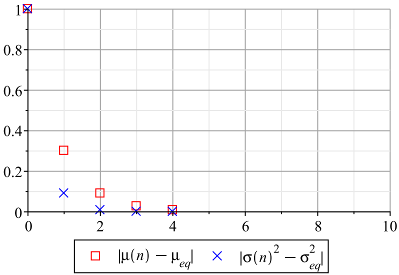


Minimal Stabilizing Horizon Length

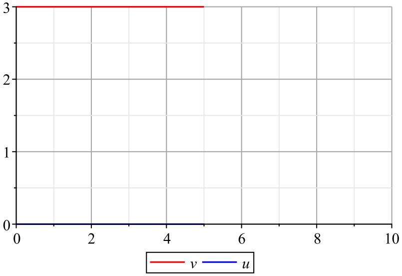
State ($n = 4$)



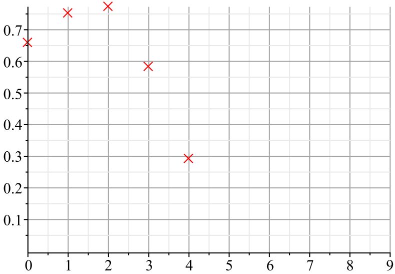
Differences (Normalized)



Control



Stage costs



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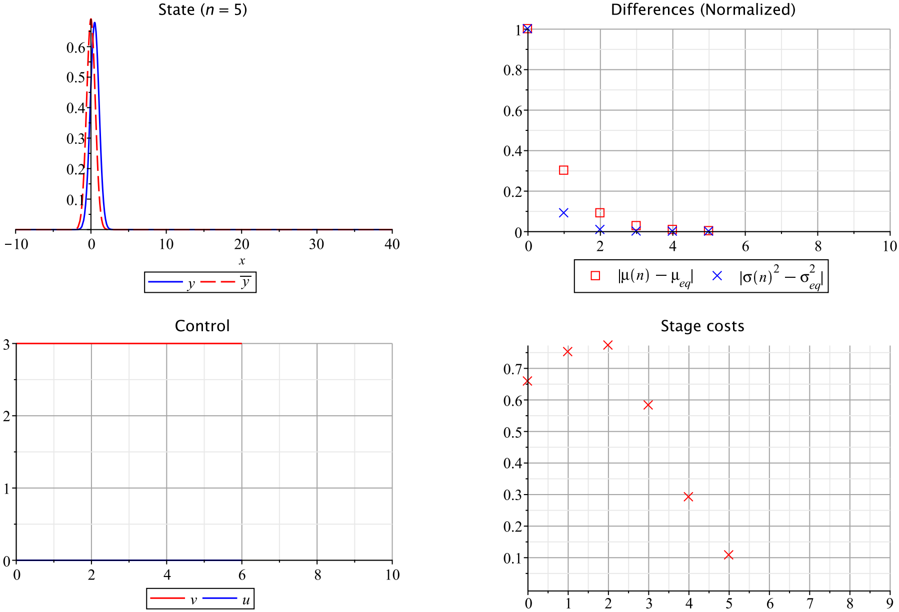

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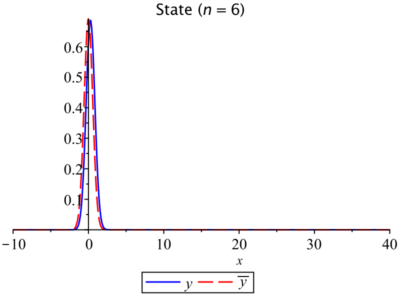

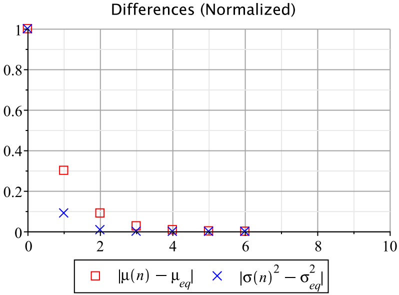

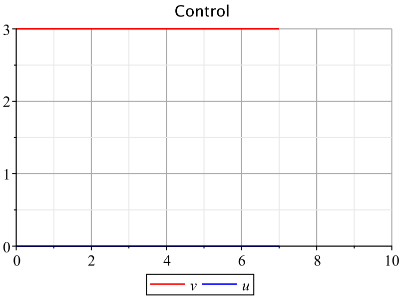

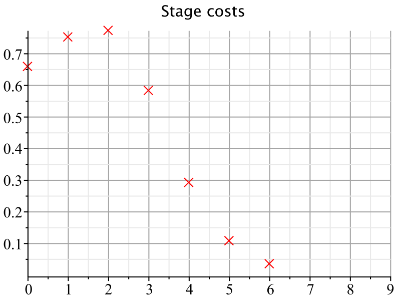

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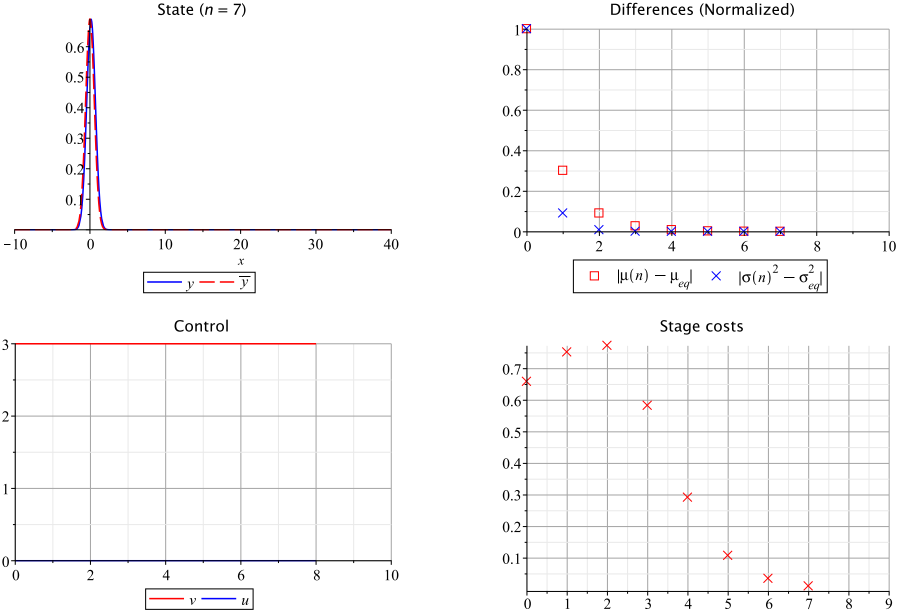

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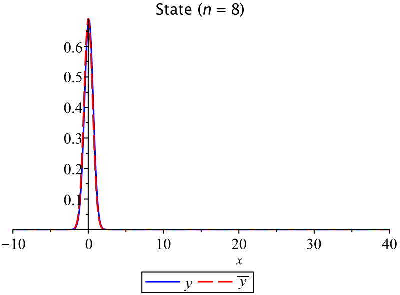
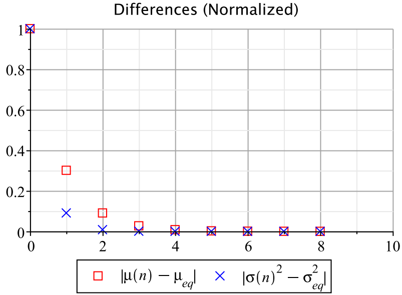
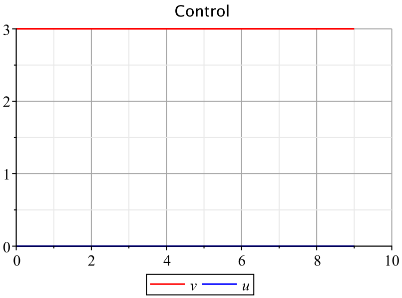
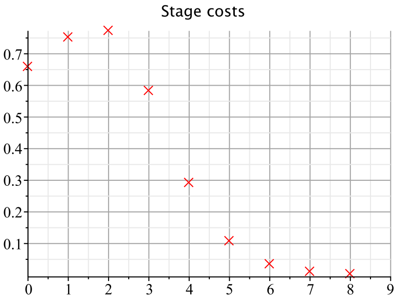
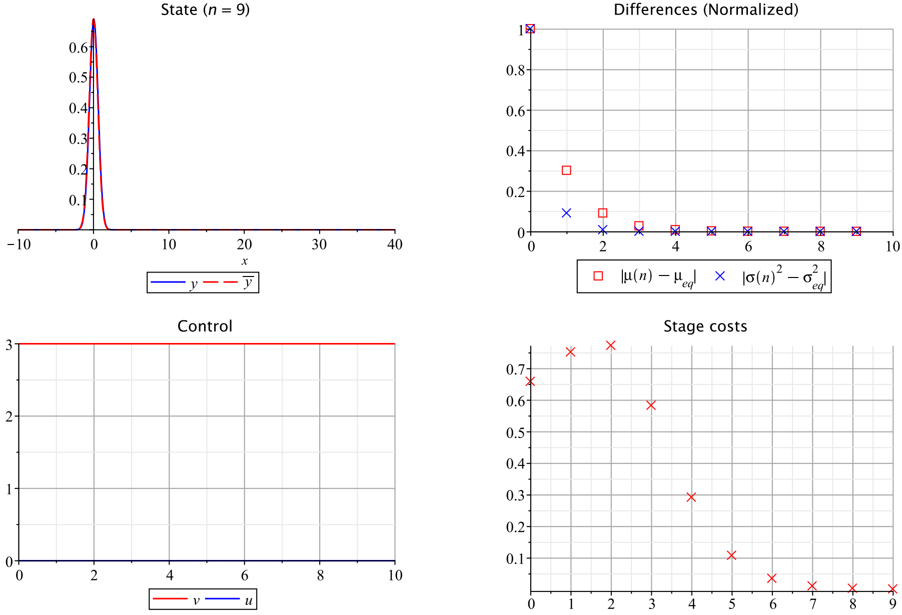








Minimal Stabilizing Horizon Length



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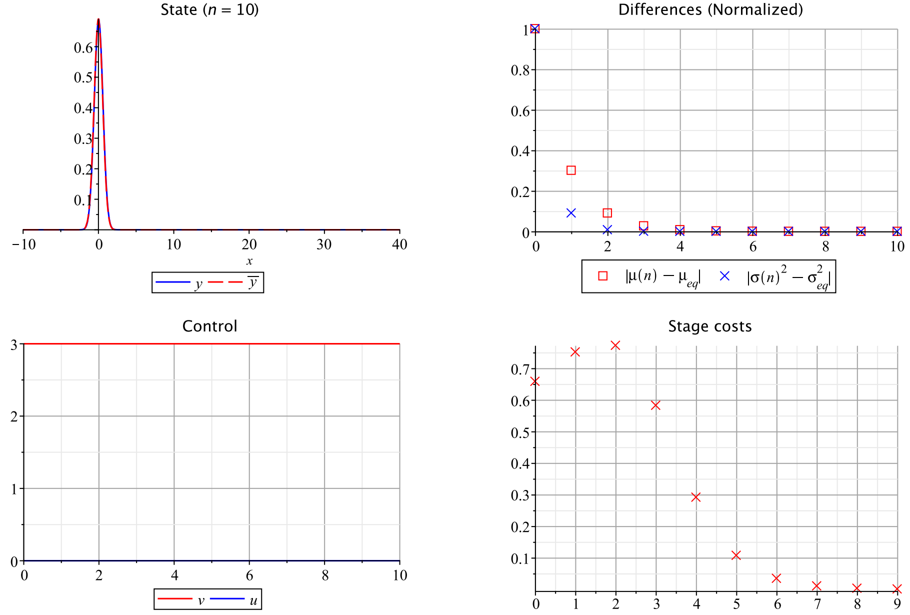

Observation

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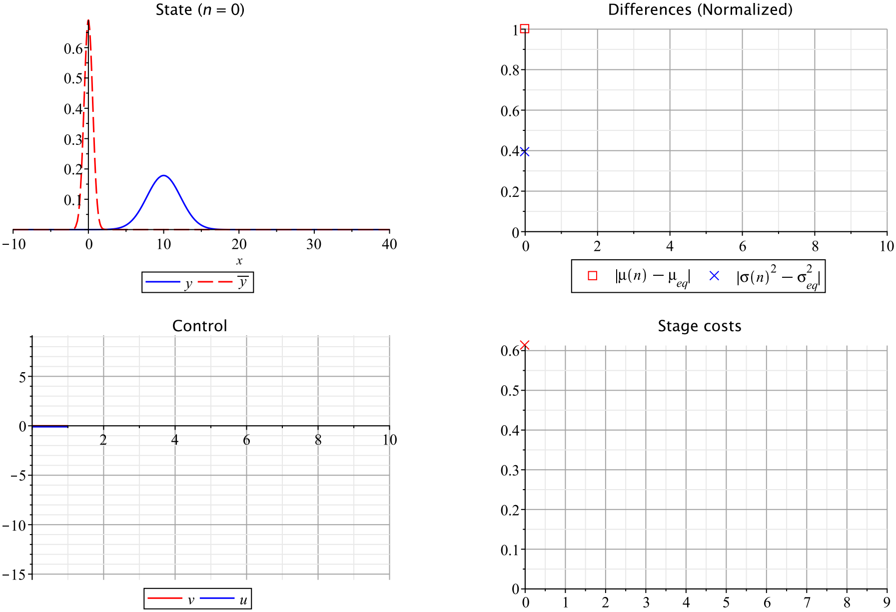
Example (2)

1 Optimal control of the 1D OU process using MPC, with parameters given by Example 1. Additional parameters:

MPC sampling time T_s	0.2
λ in the cost functional	1e-4

Other than $v > 0$, which is required by the OU process, there are no control constraints.

Minimal Stabilizing Horizon Length



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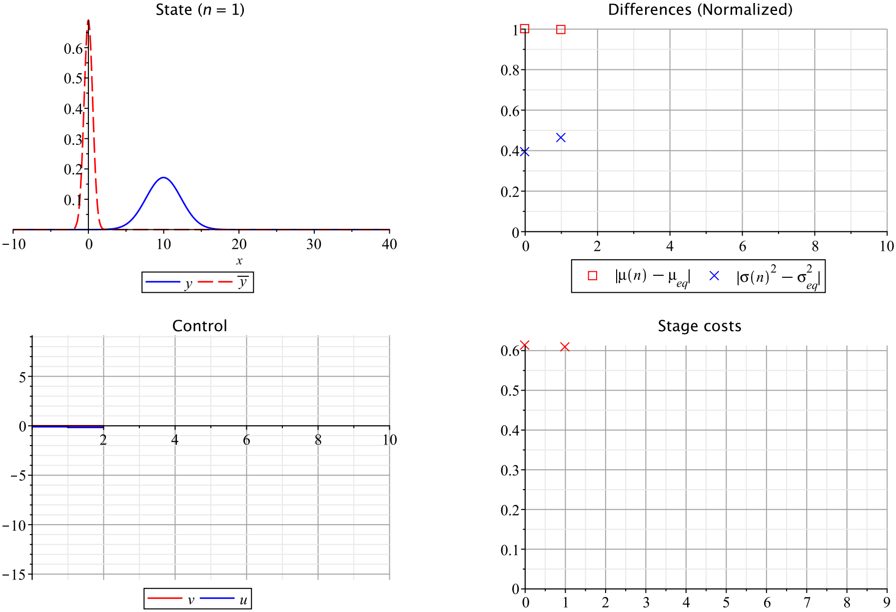

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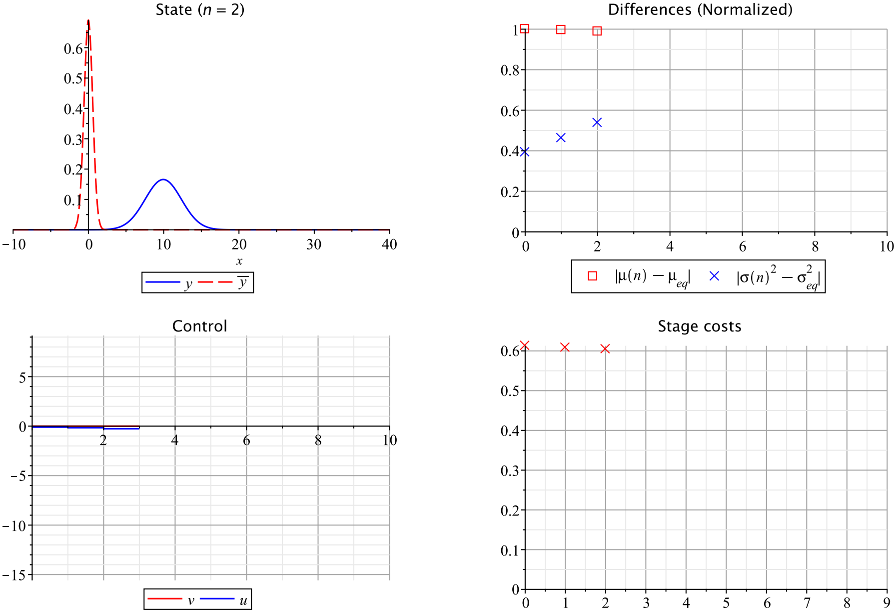

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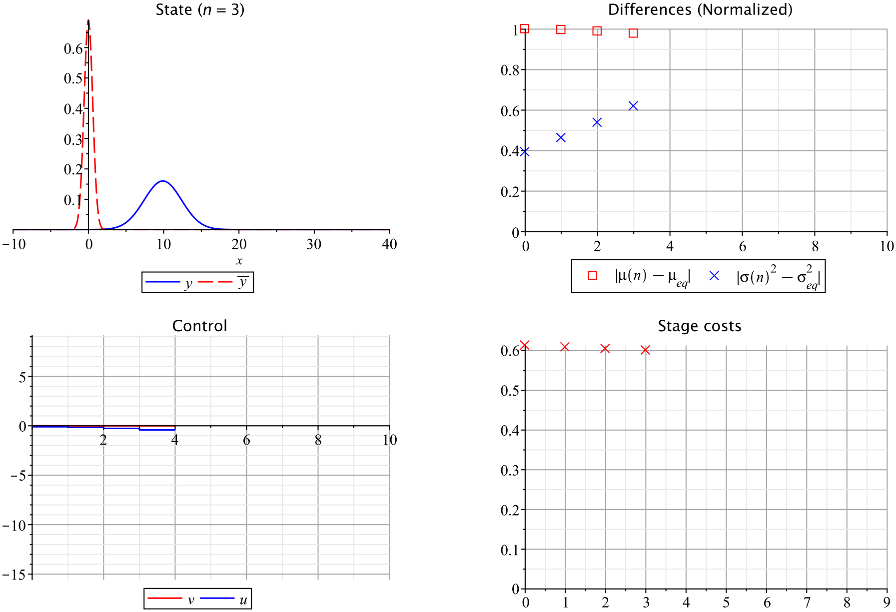

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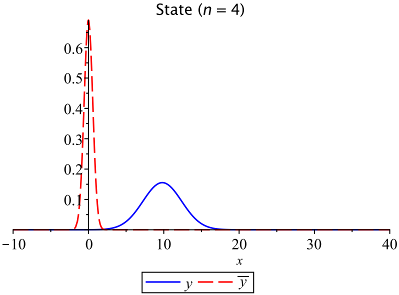

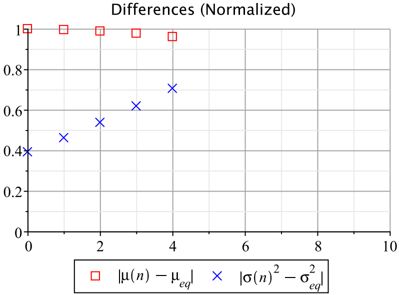

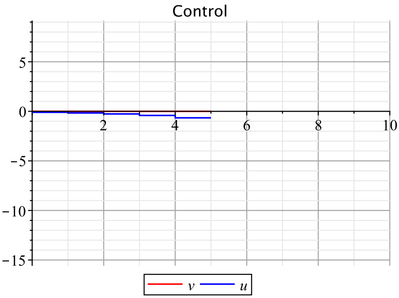

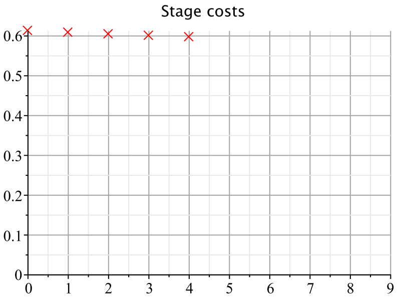

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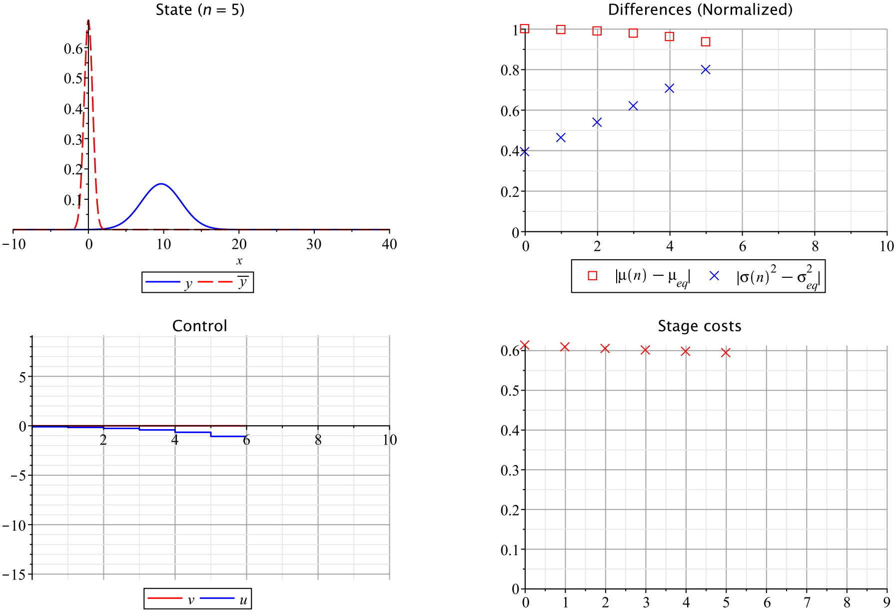

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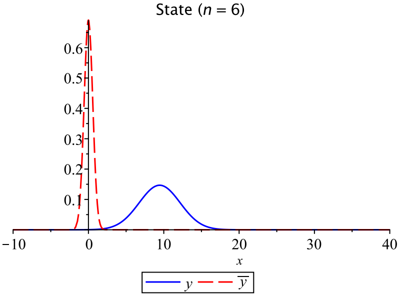

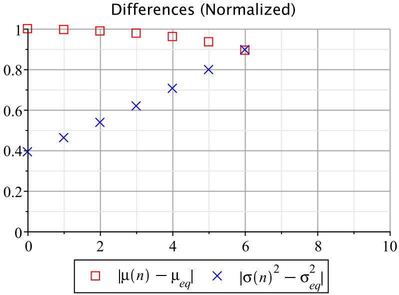

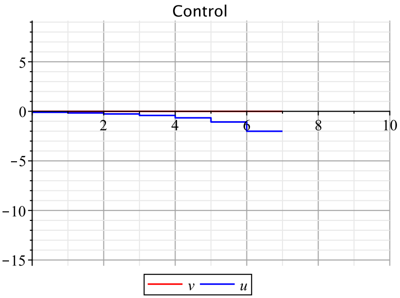

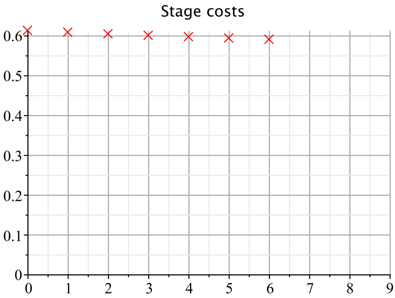

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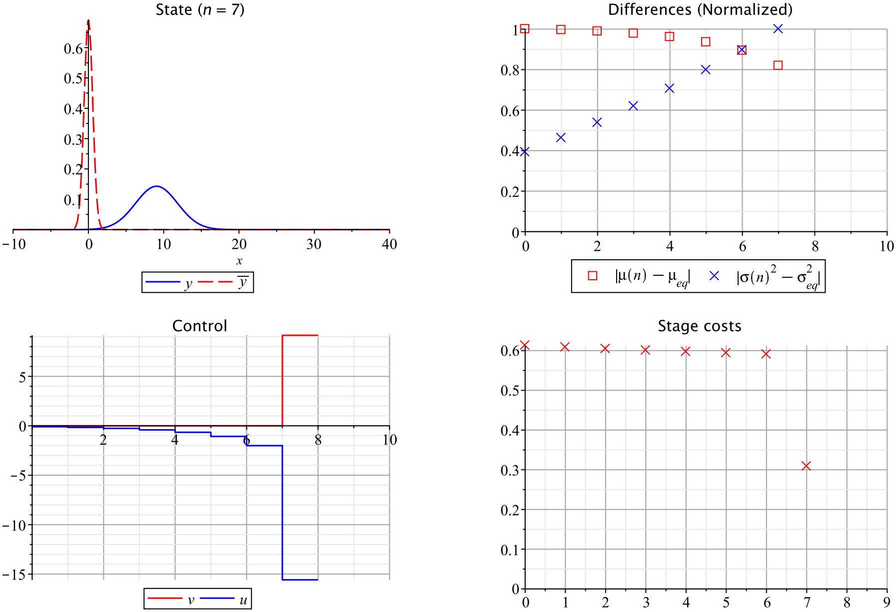

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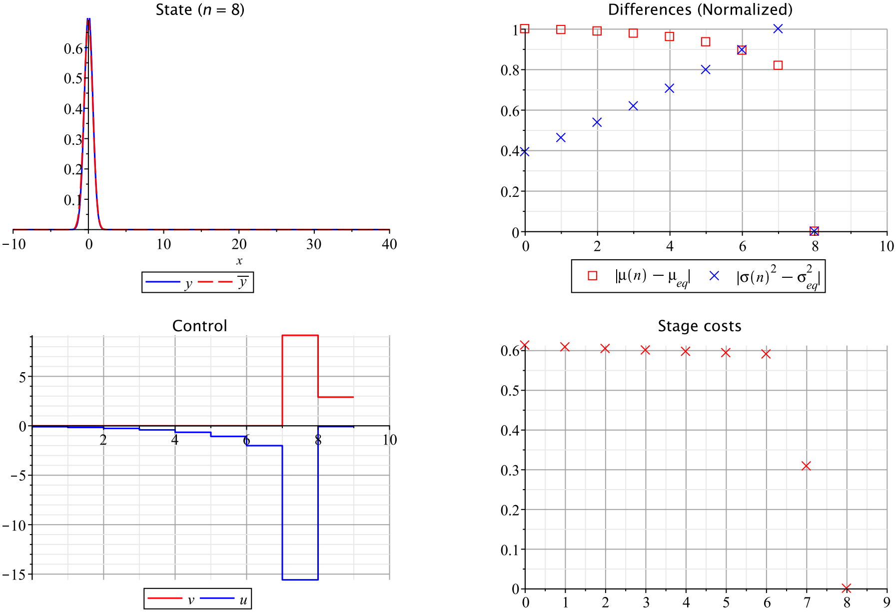

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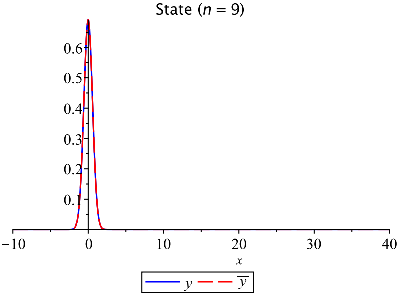

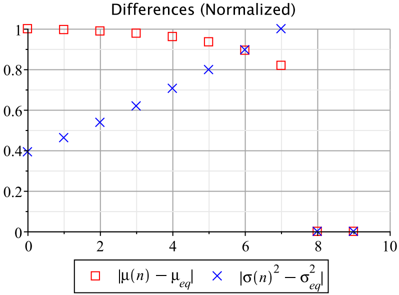

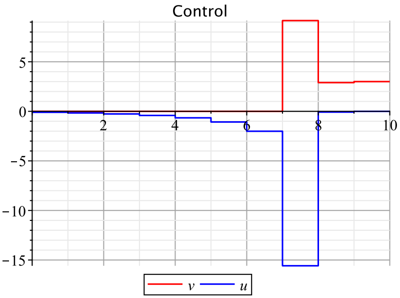

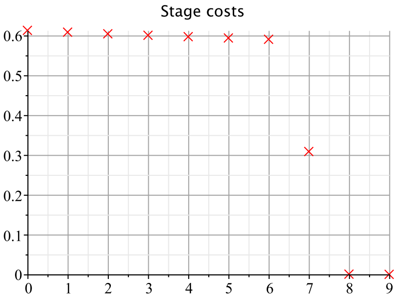

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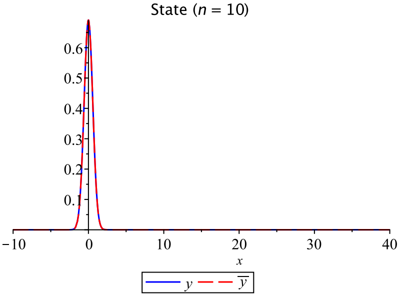

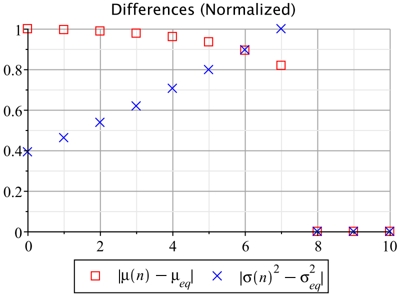

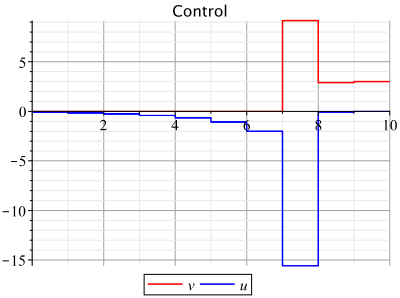

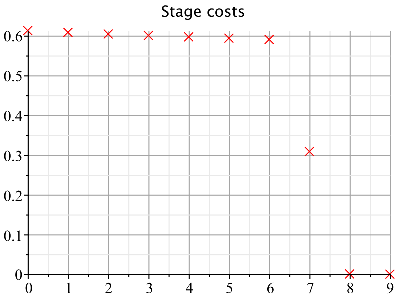

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Assuming $\mu(n) \neq \mu_{eq} = \frac{\bar{u}}{\bar{v}}$, we would like to have monotone convergence of the mean, i.e.,

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Using the explicit solution formula [Risken '96, Annunziato and Borzì '10], we get

$$\mu(n+1) = \frac{u(n)}{v(n)}(1 - e^{-v(n)T_s}) + \mu(n)e^{-v(n)T_s},$$

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$\Rightarrow$ Two cases:

1. $\mu(n) > \mu_{eq} \quad \Rightarrow \quad \mu(n+1) \geq \mu_{eq}$
2. $\mu(n) < \mu_{eq} \quad \Rightarrow \quad \mu(n+1) \leq \mu_{eq}$

Consider case 1 (case 2 analogous). Then we have monotone convergence of the mean iff

$$\begin{aligned} 0 &\overset{!}{>} |\mu(n+1) - \mu_{eq}| - |\mu(n) - \mu_{eq}| \\ &= \mu(n+1) - \mu(n) \\ &= \left(\frac{u(n)}{v(n)} - \mu(n)\right) \underbrace{\left(1 - e^{-v(n)T_s}\right)}_{>0} \end{aligned}$$

$$\Leftrightarrow \frac{u(n)}{v(n)} < \mu(n).$$

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The last inequality can be guaranteed for the special case $0 = \mu_{eq} = \frac{\bar{u}}{\bar{\nu}}$: We always have $v(n) > 0$. Furthermore, in this case, $\mu(n) > 0$ and $u(n) \leq 0$ (otherwise, higher state and control costs occur).

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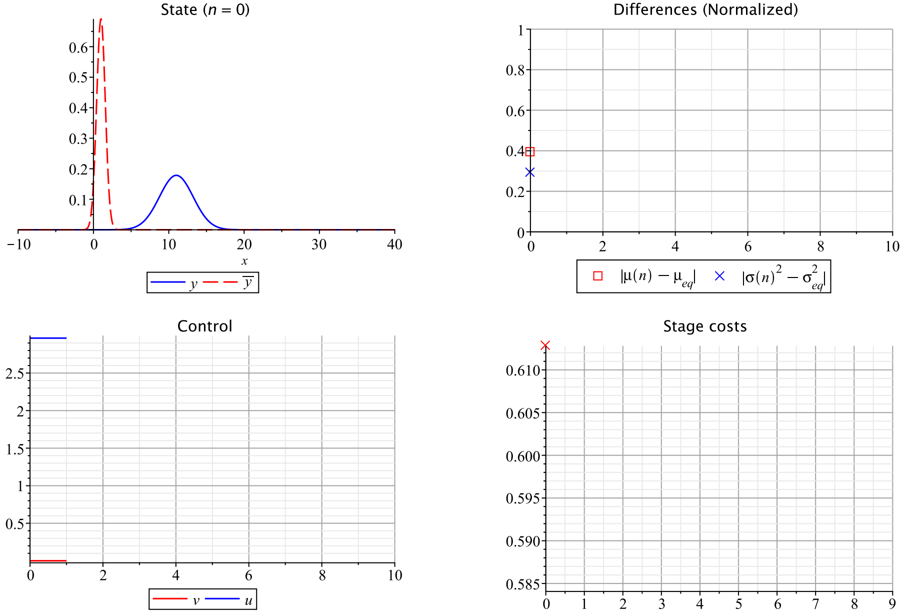
In addition, if $\mu(n) = \mu_{eq} = 0$, then $\mu(k) = \mu_{eq} = 0$ for all $k \geq n$.

Example (3)

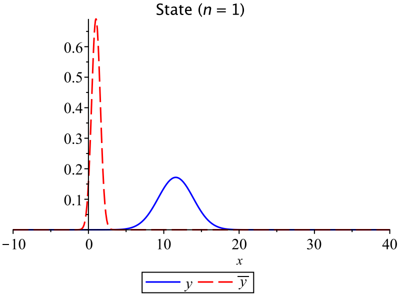
Optimal control of the 1D OU process using MPC, where the initial and target distribution have been shifted one unit to the right, i.e.

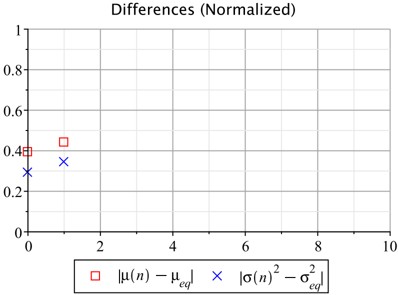
$$\mathring{\mu} = 11,$$
$$(\bar{u}, \bar{v}) = (3, 3).$$

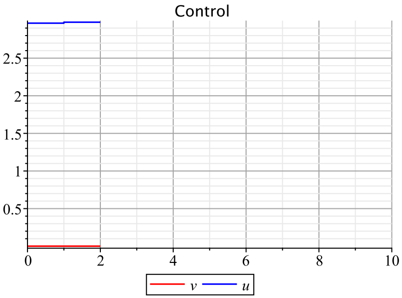
Minimal Stabilizing Horizon Length

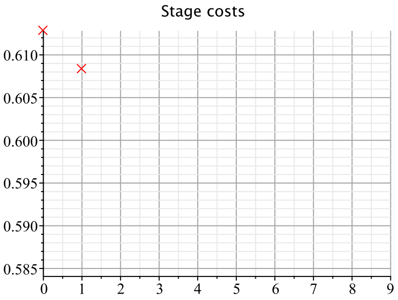


Minimal Stabilizing Horizon Length

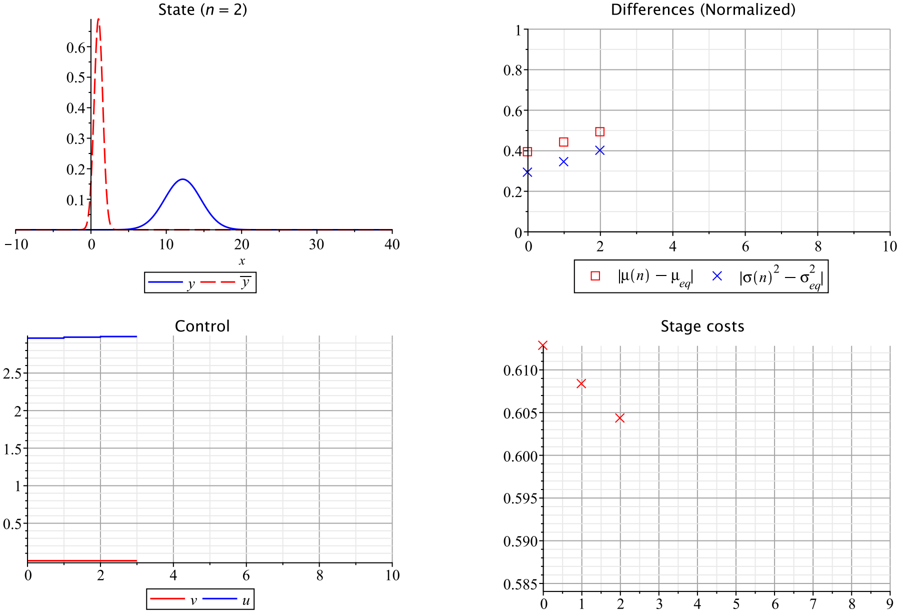








Minimal Stabilizing Horizon Length



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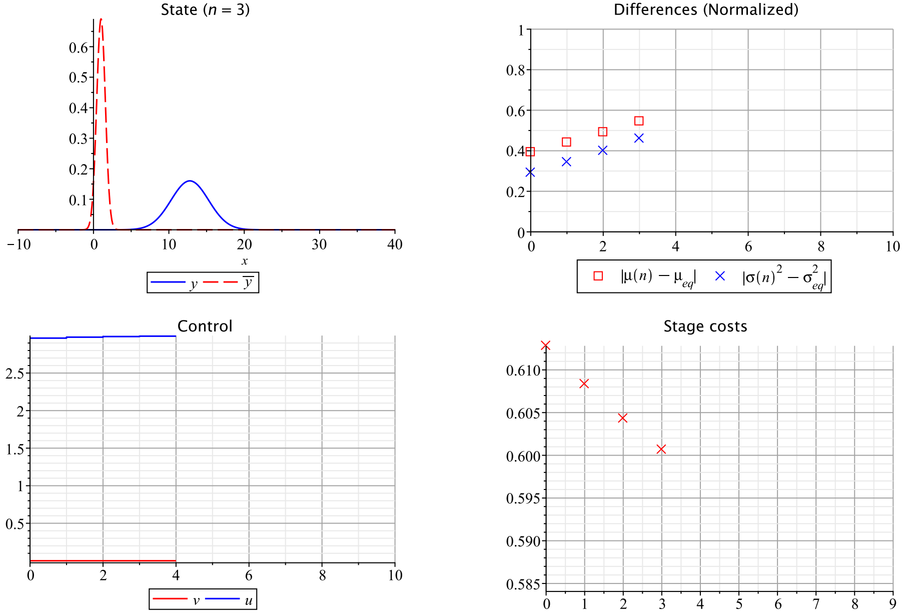

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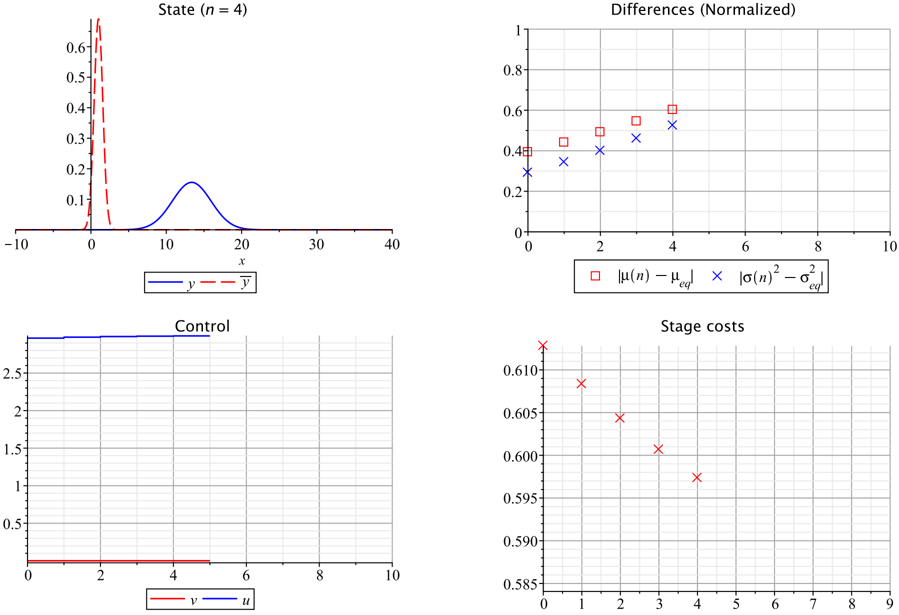

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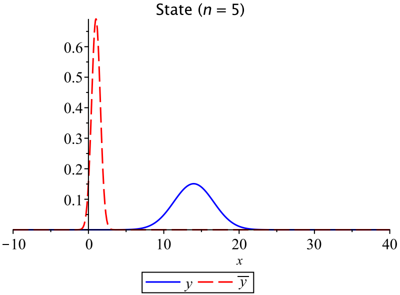

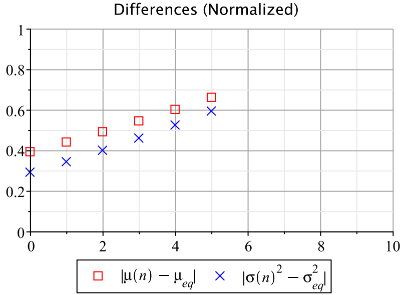

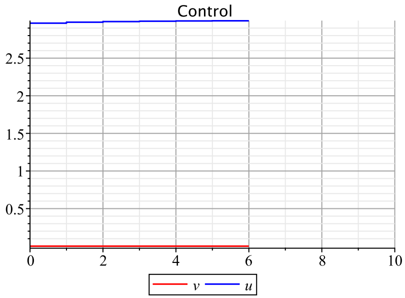

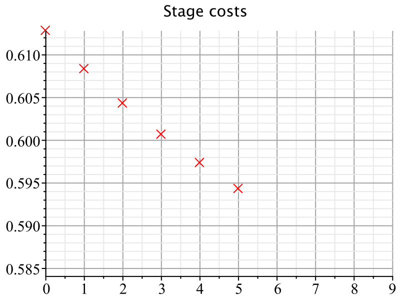

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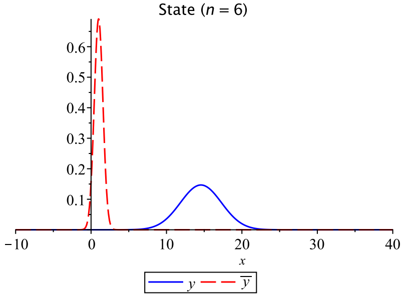

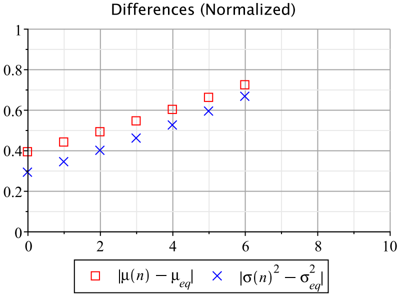

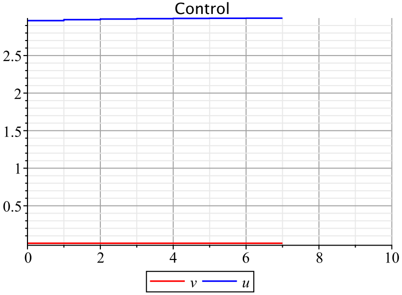

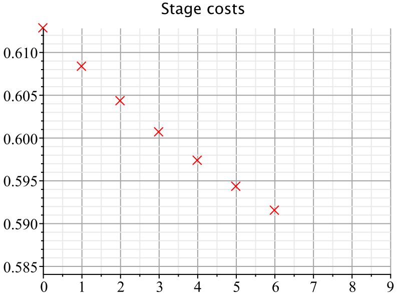

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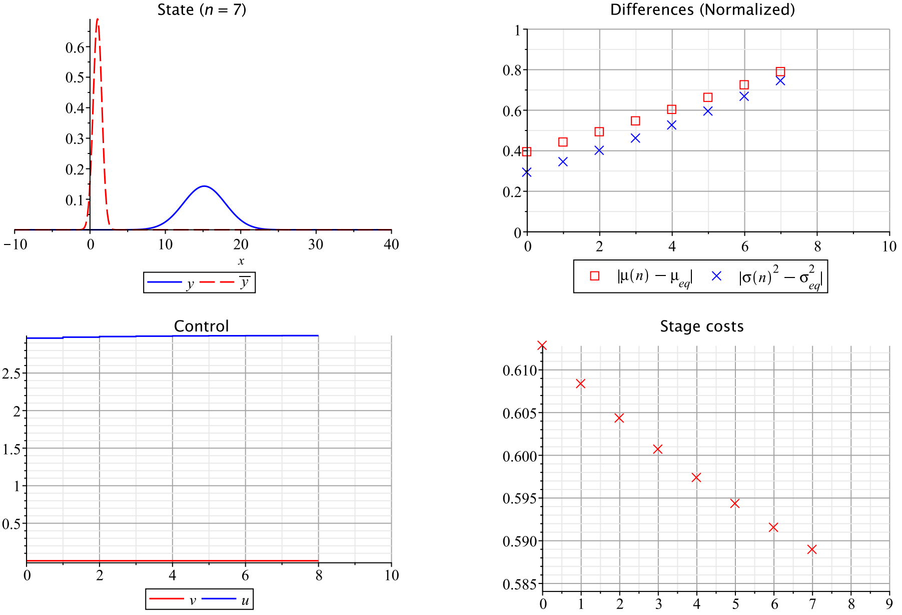

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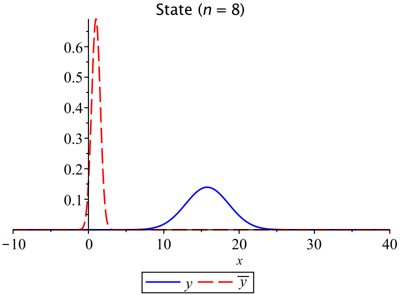

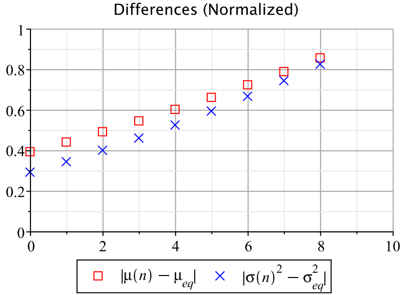

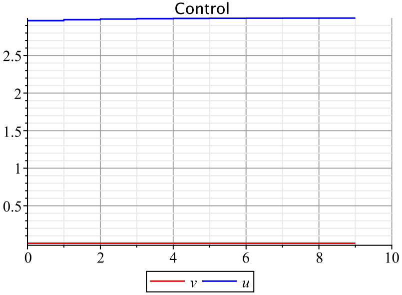

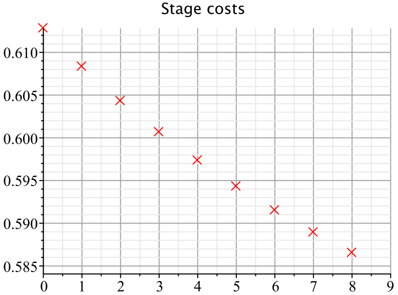

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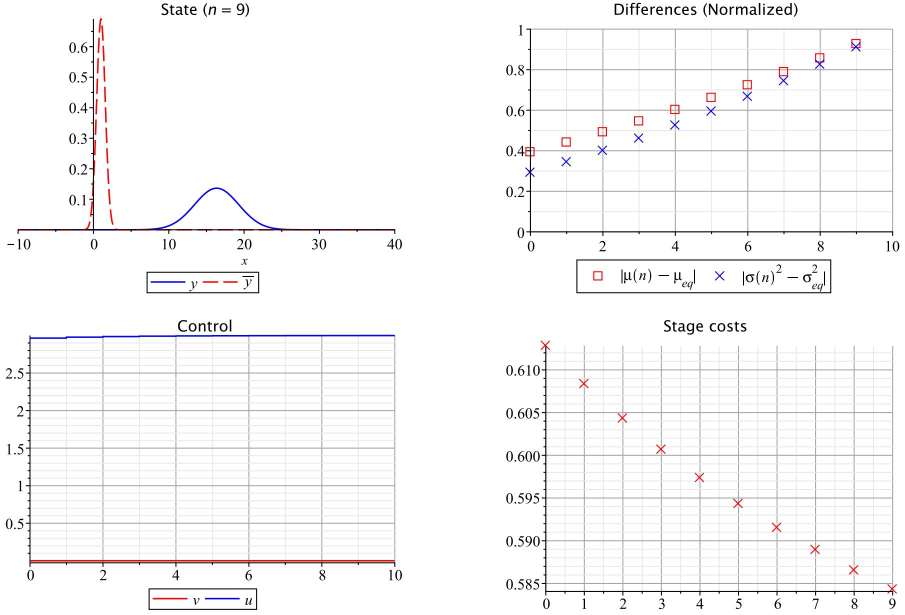

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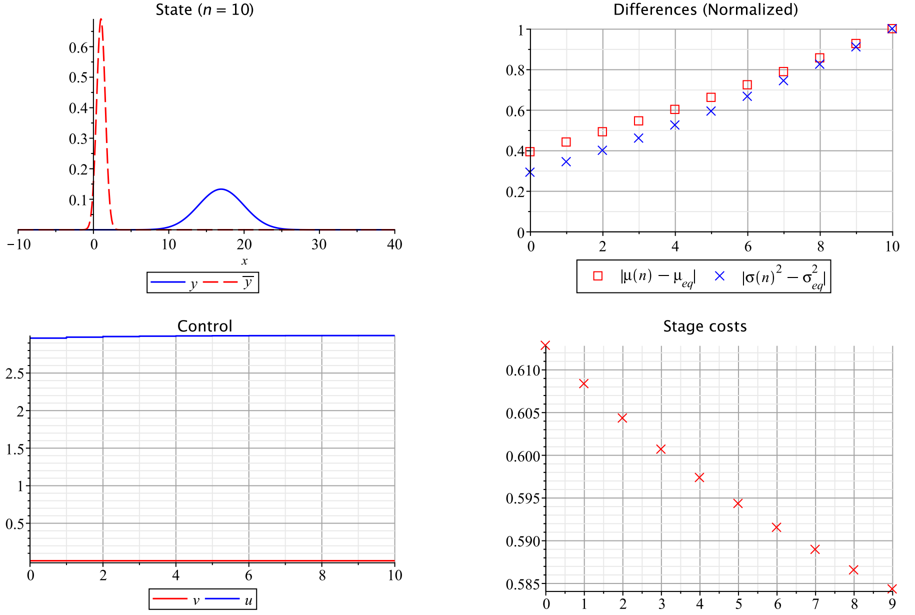

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Transformation of coordinates: $\tilde{x} := x + \frac{\bar{u}}{\bar{v}}$

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It holds:

$$y(\tilde{x}, t) - \bar{y}(\tilde{x}) = y(x, t) - \bar{y}(x),$$

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Control costs: Instead of $b(x, t; u, v) = -v(n)x + u(n)$ we now have:

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$\Rightarrow$ New offset in the control, leading to

$$\ell_2(y(k), u(k), v(k)) := \frac{1}{2}\,\|y(k) - \bar{y}\|^2_{L^2(\mathbb{R})} + \frac{\lambda}{2}|v(k) - \bar{v}|^2 + \frac{\lambda}{2}|u(k) - \frac{\bar{u}}{\bar{v}}v(k)|^2.$$

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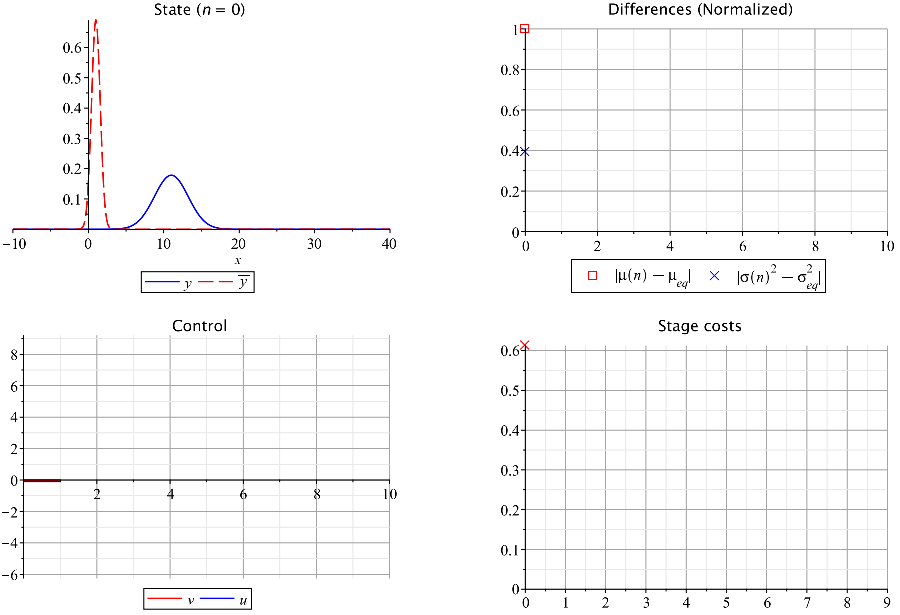

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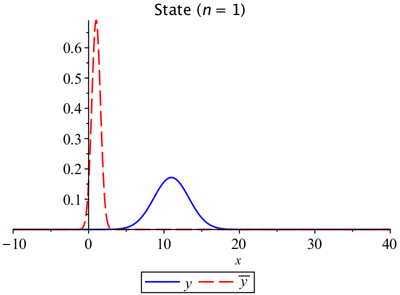
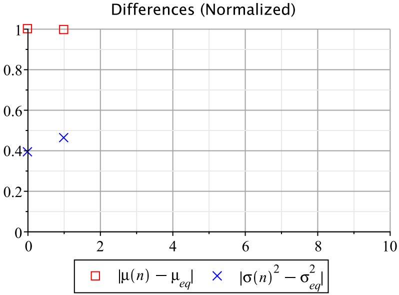
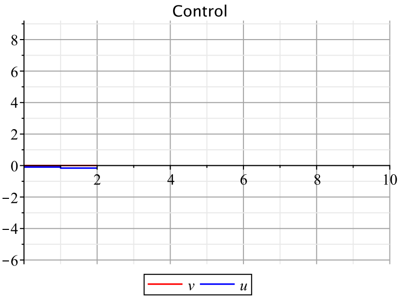
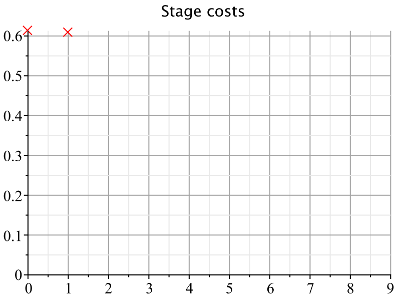
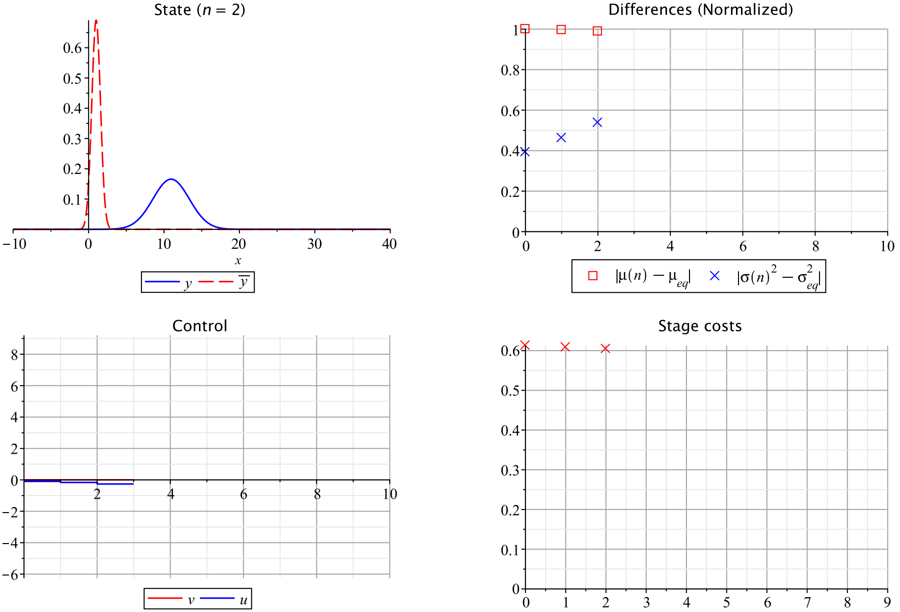




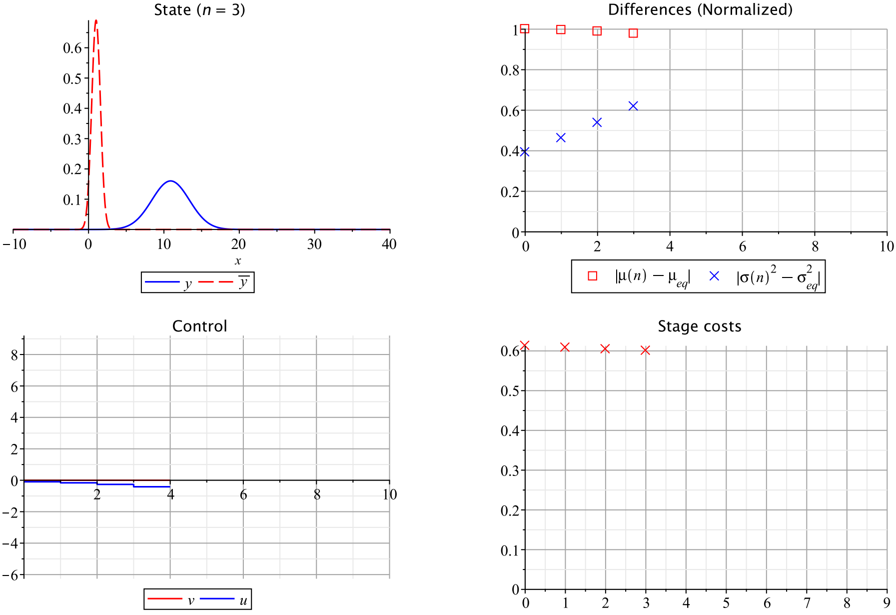




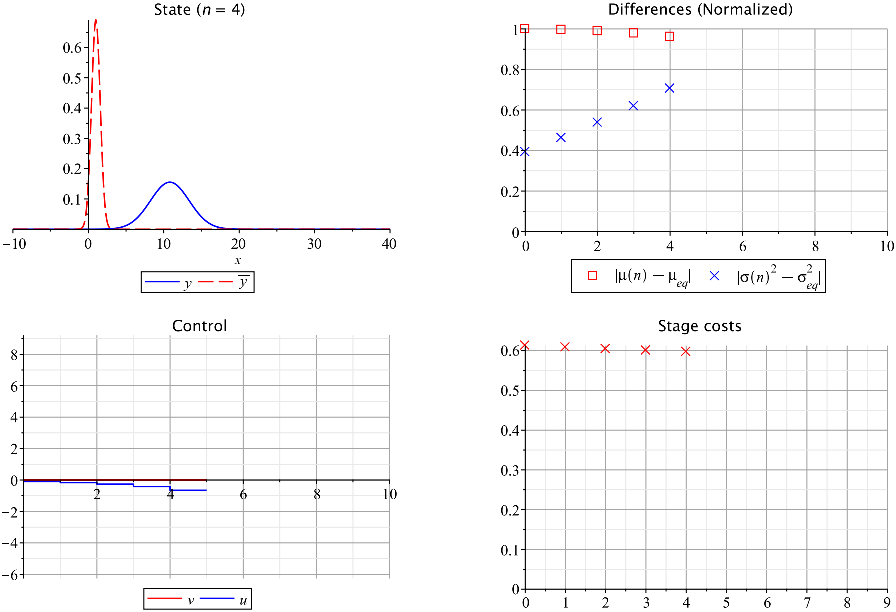
Minimal Stabilizing Horizon Length



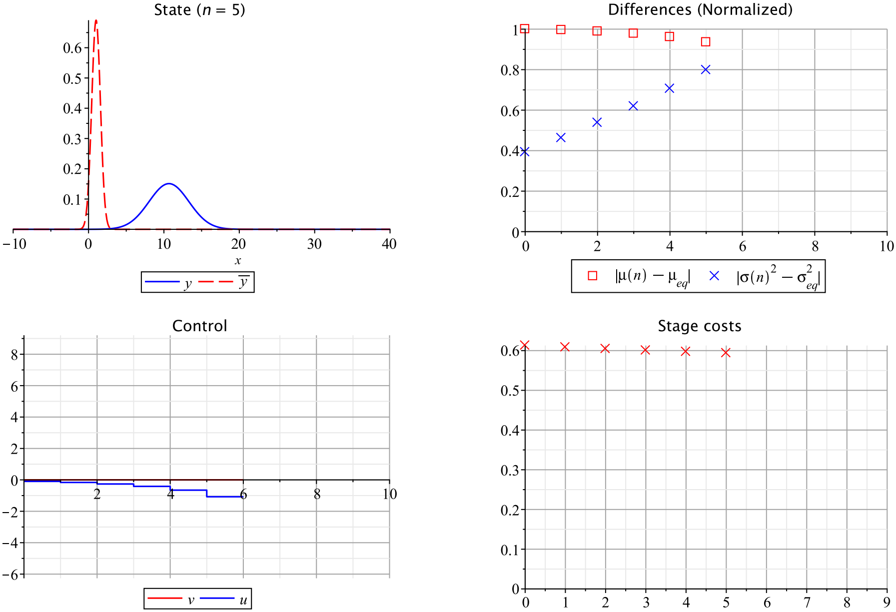
Minimal Stabilizing Horizon Length



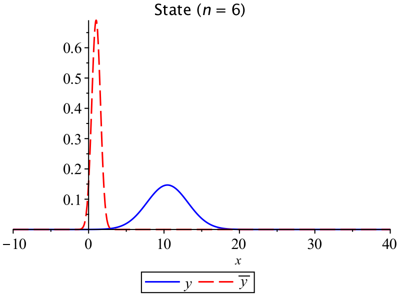
Minimal Stabilizing Horizon Length

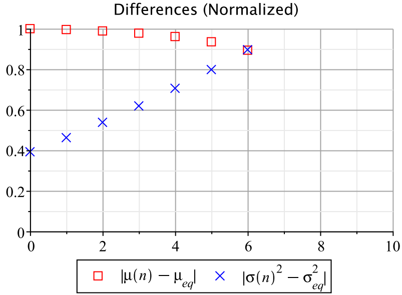


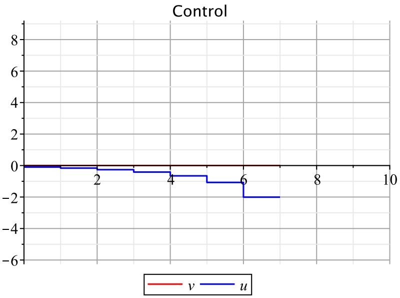
Minimal Stabilizing Horizon Length

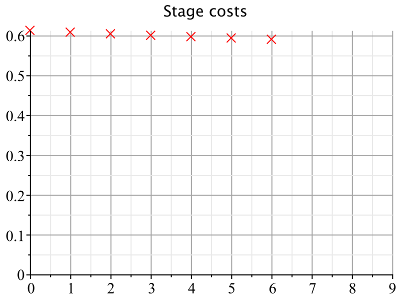


Minimal Stabilizing Horizon Length



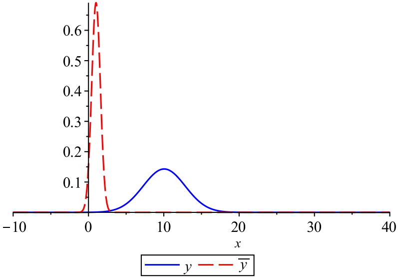




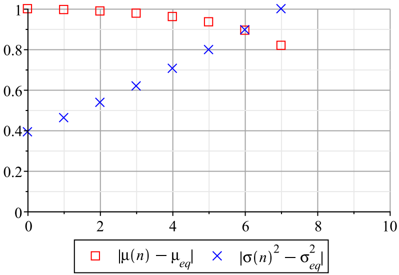


Minimal Stabilizing Horizon Length

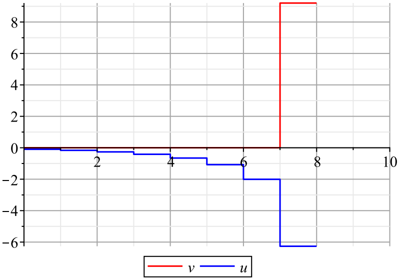
State ($n = 7$)



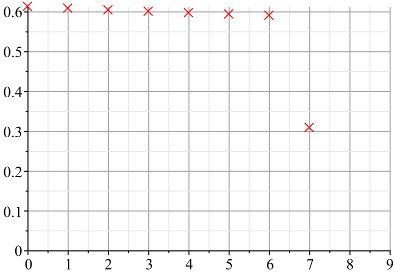
Differences (Normalized)



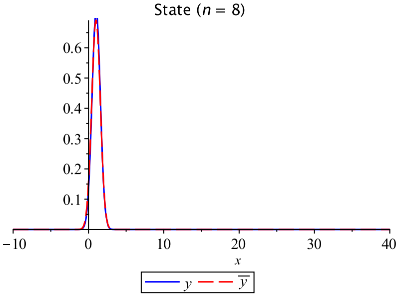
Control

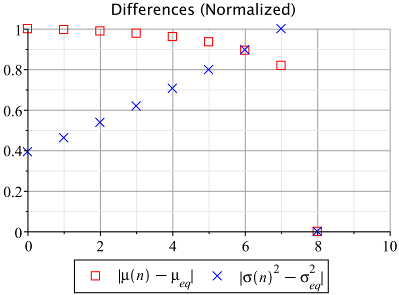


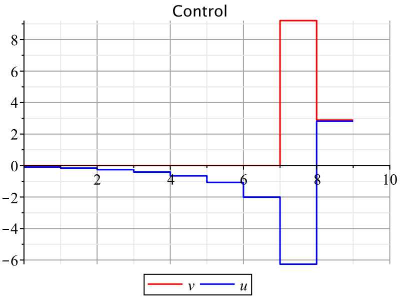
Stage costs

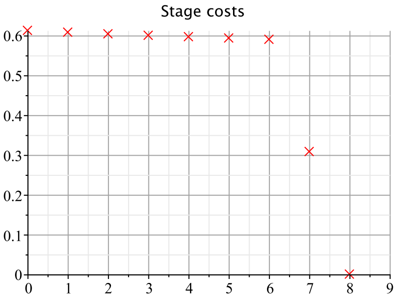


Minimal Stabilizing Horizon Length



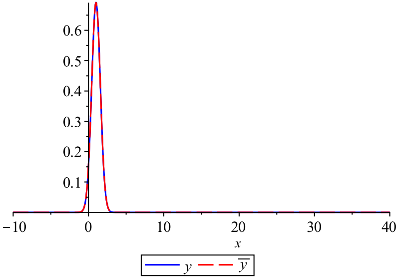




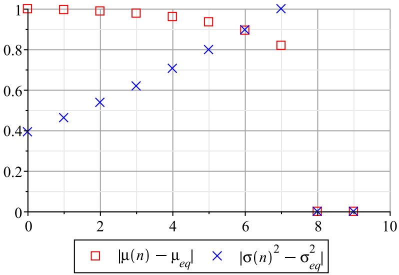


Minimal Stabilizing Horizon Length

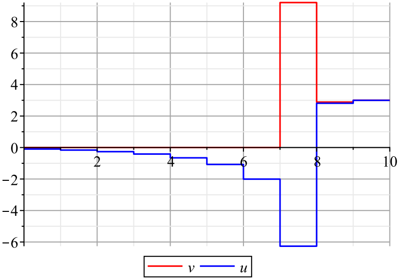
State ($n = 9$)



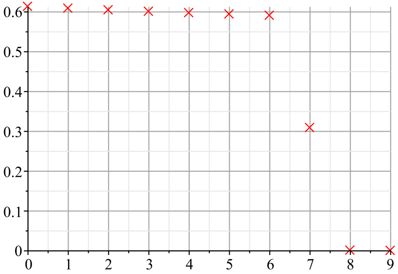
Differences (Normalized)



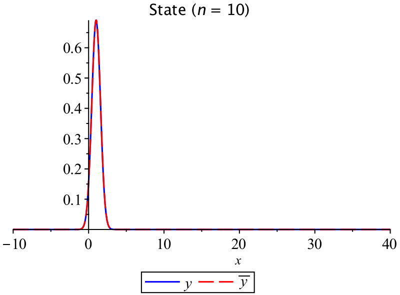
Control

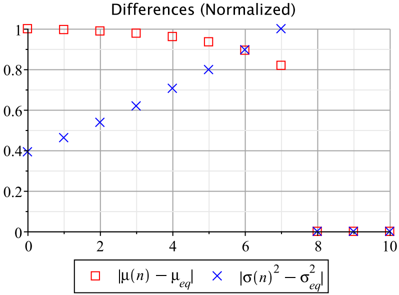


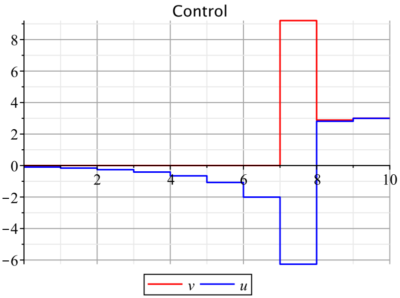
Stage costs

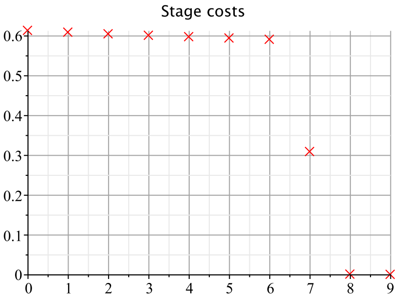


Minimal Stabilizing Horizon Length









Proposition

Consider the 1D OU process with a drift term

$$b(x,t;u,v) = -vx + u$$

and stage cost

$$\ell_2(y(k),u(k),v(k)) = \frac{1}{2}\|y(k)-\bar{y}\|^2_{L^2(\mathbb{R})} + \frac{\lambda}{2}|v(k)-\bar{v}|^2 + \frac{\lambda}{2}\left|u(k)-\frac{\bar{u}}{\bar{v}}v(k)\right|^2.$$

Then the equilibrium $(\bar{y},\bar{u},\bar{v})$ is globally asymptotically stable for the MPC closed loop for any optimization horizon $N \geq 2$.

Proposition

Consider the 1D OU process with a drift term

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$$\ell_2(y(k),u(k),v(k)) = \frac{1}{2}\,\|y(k)-\bar{y}\|^2_{L^2(\mathbb{R})} + \frac{\lambda}{2}|v(k)-\bar{v}|^2 + \frac{\lambda}{2}\left|u(k)-\frac{\bar{u}}{\bar{v}}v(k)\right|^2.$$

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Note

$$\ell_2(y(k),\bar{u},\bar{v}) = \frac{1}{2}\,\|y(k)-\bar{y}\|^2_{L^2(\mathbb{R})}$$

Proof (conceptual)

1. Monotone convergence to the target mean
   $\Rightarrow \exists \tilde{n} \in \mathbb{N}_0 \ \forall n \geq \tilde{n} : \mu(n) = \mu_{eq}$.
2. Once $\mu(n) = \mu_{eq}$, we prove that the system is exponentially controllable w.r.t. stage costs ℓ_2 with $C = 1$ using the suboptimal control $(u, v) = (\bar{u}, \bar{v})$, cf. Lemma below.

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Lemma

The function $V(t) := \frac{1}{2} \|y(t) - \bar{y}\|^2_{L^2(\mathbb{R})}$ fulfills $V(t) \leq e^{-Kt} V(0)$ with some decay rate $K > 0$.

Conclusion

MPC closed loop stability is guaranteed for the Ornstein-Uhlenbeck process with Gaussian initial condition and a control function that is linear in space, even for the shortest possible horizon.

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Open questions:

- Other stochastic processes and other distributions?
- Economic MPC?

Conclusion

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Thank you for your attention!