Approaches for bilevel optimal control problems

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Universität der Bundeswehr München

Ingenieur Mathematik

Contents

Current Section

Bilevel Optimization Problem

Bilevel Optimization Problem

$$\left.\begin{array}{ll} \text{Minimize} & F(x,y) \\ \text{s.t.} & G(x,y) \in K \\ & H(x,y) = 0 \\ & y \in Y(x) \end{array}\right\} \quad \text{upper level problem}$$

where $Y(x)$ is the set of minimizers of

$$\left.\begin{array}{ll} \text{Minimize} & f(x,y) \\ \text{w.r.t.} & y \\ \text{s.t.} & g(x,y) \in C \\ & h(x,y) = 0 \end{array}\right\} \quad \text{lower level problem } NLP_L(x)$$

(optimistic viewpoint)

Applications

- locomotion and biomechanics

  [K. Hatz: Efficient numerical methods for hierarchical dynamic optimization with application to cerebral palsy gait modeling, Phd thesis, Uni Heidelberg, 2014]

  [K. Mombaur: Stability Optimization of Open-loop Controlled Walking Robots. PhD thesis, Uni Heidelberg, 2001.]

  [S. Albrecht: Modeling and numerical solution of inverse optimal control problems for the analysis of human motions, Phd thesis, TU München, 2013.]

- optimal control under safety constraints

  [M. Knauer: Bilevel-Optimalsteuerung mittels hybrider Lösungsmethoden am Beispiel eines deckengeführten Regalbediengerätes in einem Hochregallager. PhD thesis, Uni Bremen, 2009.]

- Red Bull Air Races (upper level : safety/fairness, lower level : minimize lap time)

  [F. Fisch: Development of a Framework for the Solution of High-Fidelity Trajectory Optimization Problems and Bilevel Optimal Control Problems, Phd thesis, FSD, TUM, 2011]

- Stackelberg dynamic games

  [H. Ehtamo, T. Raivio: On Applied Nonlinear and Bilevel Programming for Pursuit-Evasion Games, JOTA, 108 (1), pp. 65-96, 2001]

- optimization of mechanical multibody systems with contact and friction
- terminal aircraft scheduling

  [M. Sama, K. Palagachev, A. D'Ariano, M. Gerdts, D. Pacciarelli: Terminal Control Area Aircraft Scheduling and Trajectory Optimization Approaches, Proceedings of the Applied mathematical programming and Modelling (APMOD 2016) conference, Brno, Czech Republic, 2016]

- ...

Current Section

Overview on Approaches

General idea: Reduction to single level optimization problem

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Black Box Approach

View lower level problem as parametric optimal control problem:

$$\text{Minimize} \quad F(x, y(x)) \quad \text{s.t.} \quad G(x, y(x)) \in K, \; H(x, y(x)) = 0$$

issues: properties of map $x \mapsto y(x)$ (non-smooth, discontinuous, set-valued)?

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MPCC approach

Replace lower level problem by its first order necessary conditions and solve single level (mixed-integer) MPCC!

issues: not equivalent to bilevel problem, treatment of MPCC & state constraints

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Replace lower level problem by its first order necessary conditions and solve single level (mixed-integer) MPCC!

issues: not equivalent to bilevel problem, treatment of MPCC & state constraints

Value Function Approach

Use value function of lower level problem to obtain single level problem.

issues: computation of value function, properties of value function?

Bilevel Optimal Control Problem

Bilevel Optimal Control Problem

$$\left.\begin{array}{ll} \text{Minimize} & \Phi(x(t_f), y(t_f)) \\ \text{s.t.} & y'(t) = F(x(t), y(t), v(t)),\;\; y(t_0) = y_0 \\ & v(t) \in V \\ & (x, u) \in \mathcal{S}(y(t_f)) \end{array}\right\} \text{ upper level problem}$$

where $\mathcal{S}(y_f)$ is the set of minimizers of

$$\left.\begin{array}{ll} \text{Minimize} & \varphi(x(t_f)) \\ \text{w.r.t.} & (x, u) \\ \text{s.t.} & x'(t) = f(x(t), u(t)),\;\; x(t_0) = x_0 \\ & u(t) \in U \\ & \psi(x(t_f), y_f) = 0 \end{array}\right\} \begin{array}{l}\text{lower level problem} \\ \text{OCP}_L(y_f)\end{array}$$

Notion:

- upper level variables: $(y, v) \in W^{1,\infty}([t_0, t_f], \mathbb{R}^{n_y}) \times L^{\infty}([t_0, t_f], \mathbb{R}^{n_v})$
- lower level variables: $(x, u) \in W^{1,\infty}([t_0, t_f], \mathbb{R}^{n_x}) \times L^{\infty}([t_0, t_f], \mathbb{R}^{n_u})$

Bilevel Optimization Problem

Solution operator: For a given control u let $x_{x_0,u}(\cdot)$ denote a solution of the IVP

$$x'(t) = f(x(t), u(t)), \quad x(t_0) = x_0$$

Feasible set of $\text{OCP}_L(y)$: $(y \in \mathbb{R}^{n_y})$

$$\mathcal{A}(y) := \left\{ u \in L^\infty([t_0, t_f], U) \;\middle|\; \exists\, x_{x_0,u}(\cdot) \in W^{1,\infty}([t_0, t_f], \mathbb{R}^{n_x}) \;:\; \psi(x_{x_0,u}(t_f), y) = 0 \right\}$$

Value function of $\text{OCP}_L(y)$

$$\mathcal{V}(y) := \inf_{u \in \mathcal{A}(y)} \varphi(x_{x_0,u}(t_f))$$

(convention: $\inf \emptyset := +\infty$)

Bilevel Optimization Problem

Assumptions

(A_1) The functions Φ, F, φ and f are continuously differentiable and ψ is twice continuously differentiable with respect to all arguments.

(A_2) V and U are compact and convex subsets of $\mathbb{R}^{n_v}$ and $\mathbb{R}^{n_u}$ respectively.

(A_3) There exists an integrable function $k : [t_0, t_f] \to \mathbb{R}$ such that

$$\|f(x,u)\| \leq k(t)(1 + \|x\|) \qquad \forall\, (t,x,u) \in [0,T] \times \mathbb{R}^{n_x} \times U.$$

(A_4) $f(x,U)$ is a convex subset of $\mathbb{R}^{n_x}$ for every $x \in \mathbb{R}^{n_x}$.

(A_5) $\nabla_y \psi(x,y)$ has a full rank for every $(x,y) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_y}$.

Bilevel Optimal Control Problem

Theorem (Lipschitz continuity)

Let $(A_1) - (A_5)$ hold and let $y \in \mathbb{R}^{n_y}$ be such that there exists a neighborhood $B_\varepsilon(y)$ of y and a constant $C_y > 0$, such that for every $y' \in B_\varepsilon(y)$, $\mathcal{S}(y') \neq \emptyset$, and for every (x', u') solution of $OCP_L(y')$ with associated multipliers $(\boldsymbol{\lambda}_0', \boldsymbol{\lambda}', \boldsymbol{\sigma}')$, it holds $\boldsymbol{\lambda}_0' = 1$ and $\|\boldsymbol{\sigma}'\| \leq C_y$.

Then $\mathcal{V}$ is Lipschitz continuous in y and

$$\partial \mathcal{V}(y) \subseteq co \bigcup_{(x,u)\in \mathcal{S}(y)} \left\{ \zeta \in \mathbb{R}^{n_y} \;\middle|\; \begin{array}{l} \exists\, \boldsymbol{\lambda} \in W^{1,\infty}([t_0, t_f], \mathbb{R}^{n_x}),\ \boldsymbol{\sigma} \in \mathbb{R}^{n_\psi}: \\ \boldsymbol{\lambda}'(t) = -\nabla_x f(x(t), u(t))^\top \boldsymbol{\lambda}(t) \\ \boldsymbol{\lambda}(t_f) = \nabla \varphi(x(t_f)) + \nabla_x \psi(x(t_f), y)^\top \boldsymbol{\sigma} \\ \zeta = \nabla_y \psi(x(t_f), y)^\top \boldsymbol{\sigma} \end{array} \right\}.$$

Single Level Reformulation

Single level reformulation of bilevel OCP: ($\rightsquigarrow$ equivalent, nonsmooth, CQs typically fail)

SLOCP

$$
\begin{aligned}
\text{Minimize} \quad & \Phi(x(t_f), y(t_f)) \\
\text{s.t.} \quad & y'(t) = F(x(t), y(t), v(t)), \quad y(t_0) = y_0 \\
& x'(t) = f(x(t), u(t)), \quad x(t_0) = x_0 \\
& u(t) \in U, \qquad v(t) \in V \\
& \psi(x(t_f), y(t_f)) = 0 \\
& \boxed{\varphi(x(t_f)) \leq \mathcal{V}(y(t_f))}
\end{aligned}
$$

[J. V. Outrata: *On the numerical solution of a class of Stackelberg problems*, Z. Oper. Res. **34** (1990), 255–277]
[J. J. Ye: *Necessary conditions for bilevel dynamic optimization problems*. SICON **33** (1995), 1208–1223]
[J. J. Ye: *Optimal strategies for bilevel dynamic problems*, SICON **35** (1997), 512–531]

Single Level Reformulation

Definition (Calmness constraint qualification)

Let $(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ be an optimal solution for *SLOCP*. *SLOCP* is said to be partially calm in $(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ with modulus $\mu \geq 0$, if for every (x, y, u, v) satisfying

$$\begin{aligned} x'(t) &= f(x(t), u(t)), \quad x(t_0) = x_0 \\ y'(t) &= F(x(t), y(t), v(t)), \quad y(t_0) = y_0 \\ u(t) &\in U, \quad v(t) \in V \\ \psi(x(t_f), y(t_f)) &= 0 \end{aligned}$$

we have

$$\Phi(x(t_f), y(t_f)) - \Phi(\hat{x}(t_f), \hat{y}(t_f)) + \mu\,(\varphi(x(t_f)) - \mathcal{V}(y(t_f))) \geq 0.$$

[J. J. Ye and D. L. Zhu: *Optimality conditions for bilevel programming problems*, Optimization, **33** (1995), 9–27]
[J. J. Ye, D. L. Zhu and Q. J. Zhu: *Exact penalization and necessary optimality conditions for generalized bilevel programming problems*, SIOPT **7** (1997), 481–507]

Single Level Reformulation

Theorem (Necessary conditions I)

Let $(A_1) - (A_5)$ *hold and let* $(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ *be a local solution of SLOCP, such that it is partially calm in* $(\hat{x}, \hat{y}, \hat{u}, \hat{v})$ *with modulus* $\mu \geq 0$.

Then there exist $\lambda_0 \geq 0$, $p_x \in W^{1,\infty}([t_0, t_f], \mathbb{R}^{n_x})$, $p_y \in W^{1,\infty}([t_0, t_f], \mathbb{R}^{n_y})$, $\xi \in \mathbb{R}^{n_\psi}$, *and* $h \in \mathbb{R}$, *such that*

$$p_x'(t) = -\nabla_x f(\hat{x}(t), \hat{u}(t))^\top p_x(t) - \nabla_x F(\hat{x}(t), \hat{y}(t), \hat{v}(t))^\top p_y(t)$$

$$p_y'(t) = -\nabla_y F(\hat{x}(t), \hat{y}(t), \hat{v}(t))^\top p_y(t)$$

$$\min_{u \in U} \left\{ f(\hat{x}(t), u)^\top p_x(t) \right\} = f(\hat{x}(t), \hat{u}(t))^\top p_x(t)$$

$$\min_{v \in V} \left\{ F(\hat{x}(t), \hat{y}(t), v)^\top p_y(t) \right\} = F(\hat{x}(t), \hat{y}(t), \hat{v}(t))^\top p_y(t)$$

$$p_x(t_f) = \lambda_0 \nabla_x \Phi(\hat{x}(t_f), \hat{y}(t_f)) + \lambda_0 \mu \nabla_x \varphi(\hat{x}(t_f)) + \nabla_x \psi(\hat{x}(t_f), \hat{y}(t_f))^\top \xi$$

$$p_y(t_f) \in \lambda_0 \nabla_y \Phi(\hat{x}(t_f), \hat{y}(t_f)) - \lambda_0 \mu \partial \mathcal{V}(\hat{y}(t_f)) + \nabla_y \psi(\hat{x}(t_f), \hat{y}(t_f))^\top \xi$$

$$f(\hat{x}(t), \hat{u}(t))^\top p_x(t) + F(\hat{x}(t), \hat{y}(t), \hat{v}(t))^\top p_y(t) = h$$

Single Level Reformulation

Theorem (Necessary conditions II)

If the matrix

$$\begin{bmatrix} \nabla_u f(\hat{x}(t), \hat{u}(t)) & 0 \\ 0 & \nabla_v F(\hat{x}(t), \hat{y}(t), \hat{v}(t)) \end{bmatrix}$$

is of full rank a.e. in (t_0, t_f) *and there exist a solution* $\hat{d} = (\hat{d}_x, \hat{d}_y, \hat{d}_u, \hat{d}_v)$ *of the system*

$$\begin{aligned} d_x'(t) &= \nabla_x f(\hat{x}, \hat{u}) d_x(t) + \nabla_u f(\hat{x}, \hat{u}) d_u(t), \\ d_y'(t) &= \nabla_x F(\hat{x}, \hat{y}, \hat{v}) d_x(t) + \nabla_y F(\hat{x}, \hat{y}, \hat{v}) d_y(t) + \nabla_v F(\hat{x}, \hat{y}, \hat{v}) d_v(t), \\ d_x(0) &= 0, \qquad d_y(0) = 0 \\ \nabla_x \psi(\hat{x}(t_f), \hat{y}(t_f)) & d_x(t_f) + \nabla_y \psi(\hat{x}(t_f), \hat{y}(t_f)) d_y(t_f) = 0, \end{aligned}$$

such that $\hat{u}(t) + \hat{d}_u(t) \in int(U)$ *and* $\hat{v}(t) + \hat{d}_v(t) \in int(V)$ *almost everywhere in* (t_0, t_f)*, then* $\boldsymbol{\lambda}_0 = 1$.

Proof Idea

Outline of proof:

- calmness CQ allows to shift $\varphi(x(t_f)) - \mathcal{V}(y(t_f)) \leq 0$ as a penalty term into objective
- apply necessary conditions of Fritz John type from non-smooth analysis in [R. Vinter: *Optimal control*, Birkhäuser, Bostson, 2010]
- show normality of multiplier λ_0 by MFCQ condition and contradiction

Details:

[K. Palagachev, M. Gerdts: *Necessary conditions for a class of bilevel optimal control problems exploiting the value function*, Pure and Applied Functional Analysis, Vol. 1(4), pp. 505-524, 2016]

Lower Level Problem (Pursuer)

Motivation: pursuit-evasion dynamic Stackelberg game in the 2d plane

lower level problem $OCP_L(z_{P,f})$

Minimize

$$t_f = \int_0^{t_f} 1 dt$$

subject to the constraints

$$\begin{aligned} z_P'(t) &= v_P(t), \quad z_P(0) = z_{P,0}, \quad z_P(t_f) = z_{P,f}, \\ v_P'(t) &= u_P(t), \quad v_P(0) = v_P(t_f) = 0, \\ u_{P,i}(t) &\in [-u_{max}, u_{max}], \quad i = 1, 2. \end{aligned}$$

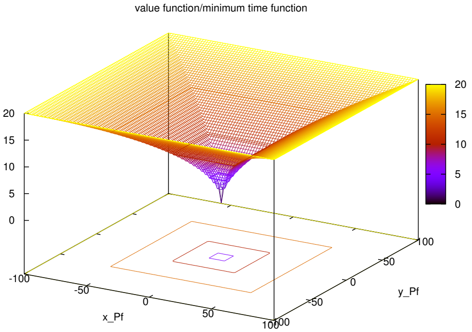
Notion:

- $z_P = (x_P, y_P)^\top$: position of pursuer
- $z_{P,f}$ is an input parameter from an upper level problem.

Value Function / Minimum Time Function

Optimal value function of OCP_L: (=minimum time function)

$$\mathcal{V}(x_{P,f}, y_{P,f}) = \max \left\{ 2\sqrt{\frac{|x_{P,0} - x_{P,f}|}{u_{max}}}, 2\sqrt{\frac{|y_{P,0} - y_{P,f}|}{u_{max}}} \right\}$$



$(x_{P,0} = y_{P,0} = 0, u_{max} = 1)$

Upper Level Problem (Evader)

upper level problem (OCP-U)

Minimize

$$-t_f + \int_0^{t_f} \frac{\alpha_1}{2} w(t)^2 + \frac{\alpha_2}{2} a(t)^2 dt$$

subject to the constraints

$$\begin{aligned}
x_E'(t) &= v_E(t)\cos\psi(t), & x_E(0) &= x_{E,0}, \quad x_E(t_f) = x_P(t_f),\\
y_E'(t) &= v_E(t)\sin\psi(t), & y_E(0) &= y_{E,0}, \quad y_E(t_f) = y_P(t_f),\\
\psi'(t) &= \frac{v_E(t)}{\ell}\tan\delta(t), & \psi(0) &= \psi_0,\\
\delta'(t) &= w(t), & \delta(0) &= \delta_0,\\
v_E'(t) &= a(t), & v_E(0) &= v_{E,0},\\
v_E(t) &\in [0, v_{E,max}],\\
w(t) &\in [-w_{max}, w_{max}],\\
a(t) &\in [a_{min}, a_{max}].
\end{aligned}$$

Reformulation

Equivalent single level problem

Minimize

$$-t_f + \int_0^{t_f} \frac{\alpha_1}{2} w(t)^2 + \frac{\alpha_2}{2} a(t)^2 dt$$

subject to the constraints

$$\begin{aligned}
x_E'(t) &= v_E(t)\cos\psi(t), & x_E(0) &= x_{E,0},\\
y_E'(t) &= v_E(t)\sin\psi(t), & y_E(0) &= y_{E,0},\\
\psi'(t) &= \frac{v_E(t)}{\ell}\tan\delta(t), & \psi(0) &= \psi_0,\\
\delta'(t) &= w(t), & \delta(0) &= \delta_0,\\
v_E'(t) &= a(t), & v_E(0) &= v_{E,0},\\
z_P'(t) &= v_P(t), & z_P(0) &= z_{P,0},\ z_P(t_f) = z_E(t_f),\\
v_P'(t) &= u_P(t), & v_P(0) &= v_P(t_f) = 0,\\
v_E(t) &\in [0, v_{E,max}],\quad w(t) \in [-w_{max}, w_{max}], & a(t) &\in [a_{min}, a_{max}],\\
u_P(t) &\in [-u_{max}, u_{max}]^2, & t_f &\leq \mathcal{V}(x_E(t_f), y_E(t_f))
\end{aligned}$$

Numerical Issue

Caution!

The constraint

$$t_f \leq \mathcal{V}(x_E(t_f), y_E(t_f))$$

with value function V of continuous OCP may become infeasible, if it is used in the discretized problem!

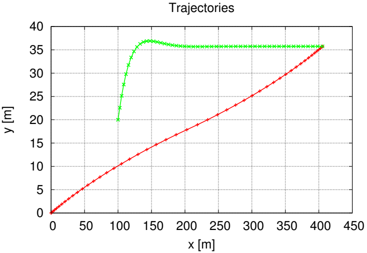
$\leadsto$ the value function V_h of the discretized problem should be used

$\leadsto$ or use relaxation

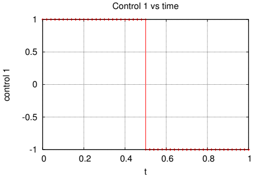
$$t_f \leq \mathcal{V}(x_E(t_f), y_E(t_f)) + \varepsilon, \qquad \varepsilon > 0$$

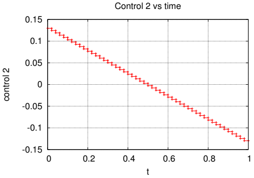
Numerical Results

Trajectories
(green=upper level, red=lower level)

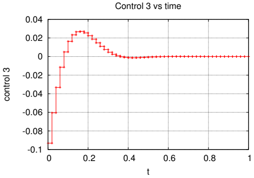


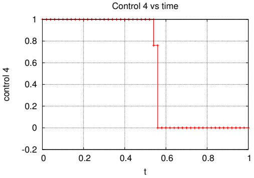
Lower level player:





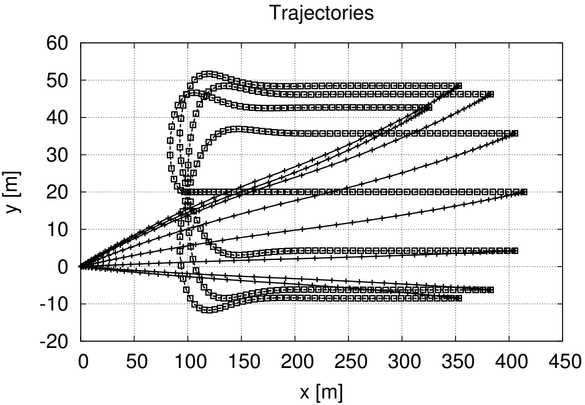
Upper level player:





$v_{E,0} = 10$, $\psi_E(0) = \pi/4$, $\alpha_1 = 10$, $\alpha_2 = 0$, $w_{max} = 0.5$, $v_{E,max} = 20$, $a_{min} = -5$, $a_{max} = 1$, $u_{max} = 5$, $N = 50$,
$t_f \approx 18.01$

Numerical Results



Value Function

How to compute the value function?

- analytically
- Hamilton-Jacobi-Bellman theory (continuous case) / dynamic programming (discrete case) [Grüne, Zidani, Bokanowski, Bardi/Capuzzo-Dolcetta, Falcone, Kalise, Mitchell, Turova/Botkin, ...]
  $\rightsquigarrow$ software ROC-HJ [Bokanowski/Zidani]
- in case of minimum time function: reachable sets [Baier/Le 2015, Colombo/Le 2015, ...]
- pointwise evaluation at $(x(t_0), x(t_f), p)$ using suitable optimal control software
  $\rightsquigarrow$ typically only local minima are obtained

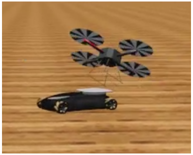
Current Section

Interaction of Robots

Scenario:

Interaction of two robots in multiple phases, e.g.

- phase 1: approach
- phase 2: interaction/rendezvous/docking
- phase 3: separation


Interaction of Robots

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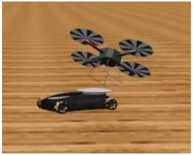
⇝ multiphase optimal control problem


Interaction of Robots

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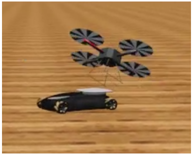
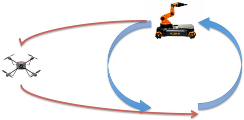
$\rightsquigarrow$ multiphase optimal control problem

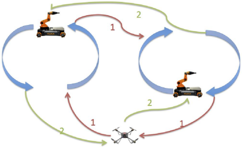
$\rightsquigarrow$ can be transformed to standard optimal control problem



Interaction of Robots

One step further:

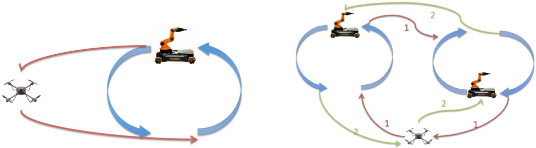




- Interaction of many robots to fulfill predefined jobs
- Each job can have multiple phases, e.g.
  - phase 1: approach
  - phase 2: interaction/rendezvous/docking
  - phase 3: separation

Interaction of Robots

One step further:



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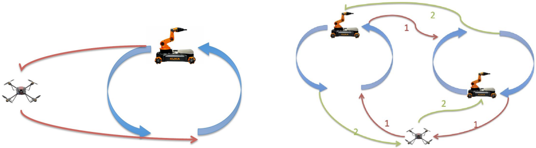
  - phase 3: separation

If sequence of jobs is fixed ...

$\rightsquigarrow$ multiphase optimal control problem with free initial and final time

Interaction of Robots

One step further:



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- Each job can have multiple phases, e.g.
  - phase 1: approach
  - phase 2: interaction/rendezvous/docking
  - phase 3: separation

If sequence of jobs is fixed ...

$\rightsquigarrow$ multiphase optimal control problem with free initial and final time

If sequence of jobs and starting times of jobs are not fixed ...

$\rightsquigarrow$ coupling of scheduling problem and optimal control problem

Bilevel Optimization Problem

Mixed-integer bilevel optimization problem

Bilevel structure:

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- upper level: scheduling problem

  ⇝ find starting times of jobs such that total time is minimized

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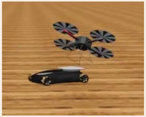
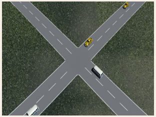
  ⇝ for a given starting time find optimal trajectory


Bilevel Optimization Problem

Mixed-integer bilevel optimization problem

Bilevel structure:

- upper level: scheduling problem

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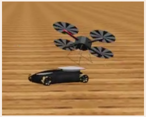
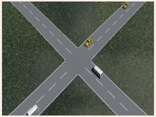


Notation: (single phase)

- jobs $1, \ldots, J$
- t_j = starting time of job j
- $d_j(t)$ = duration of job j with starting time t $\rightsquigarrow$ solution of optimal control problem

Bilevel Optimization Problem

Scheduling Problem (mixed-integer optimization problem)

Minimize

$$\max_{j=1,\ldots,J} \{t_j + d_j(t_j)\}$$

w.r.t.

$$t_j \qquad (j = 1, \ldots, J), \qquad w_{jk} \qquad (j, k = 1, \ldots, J, j \neq k)$$

s.t.

$$t_j + d_j(t_j) - t_k \leq M \cdot w_{jk} \qquad (j \neq k)$$

$$t_k + d_k(t_k) - t_j \leq M \cdot (1 - w_{jk}) \qquad (j \neq k)$$

$$t_j \geq 0$$

$$w_{jk} \in \{0, 1\} \qquad (j, k \in \{1, \ldots, J\})$$

($M > 0$ sufficiently large number)

Bilevel Optimization Problem

Input parameter: initial times t_j for jobs $j = 1, \ldots, J$

Parametric Optimal Control Problem OCP(t_j) for Job j

Minimize

$$\int_{t_j}^{t_j+T_j} f_0(t, x(t), u(t))dt$$

w.r.t. (x, u, T_j) s.t.

$$\begin{aligned} x'(t) - f(t, x(t), u(t)) &= 0 & t_j \le t \le t_j + T_j \\ c(t, x(t), u(t)) &\le 0 & t_j \le t \le t_j + T_j \\ s(t, x(t)) &\le 0 & t_j \le t \le t_j + T_j \\ \psi(x(t_j), x(t_j + T_j)) &= 0 & \end{aligned}$$

Output: phase durations $d_j(t_j) = T_j$, for jobs $j = 1, \ldots, J$

MPCC Approach

Approach:

- replace lower level OCP by its necessary conditions

  $\rightsquigarrow$ discretize first vs. stay in continuous formulation

- solve resulting mixed-integer MPCC

MPCC Approach

Approach:

- replace lower level OCP by its necessary conditions

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Standard OCP

$$
\begin{array}{llcll}
\text{Minimize} & \Phi(x(0), x(1)) & & & \\
\text{s.t.} & x'(t) - f(x(t), u(t)) & = & 0 & \text{a.e. in } [0,1] \\
 & c(x(t), u(t)) & \leq & 0 & \text{a.e. in } [0,1] \\
 & s(x(t)) & \leq & 0 & \text{in } [0,1] \\
 & \psi(x(0), x(1)) & = & 0 &
\end{array}
$$

Local Minimum Principle (for Standard OCP)

Augmented Hamiltonian:

$$H := \boldsymbol{\lambda}^\top f(x, u) + \boldsymbol{\eta}^\top c(x, u)$$

Necessary Conditions

(constraint qualification $\Rightarrow \boldsymbol{\ell}_0 = 1$)

Local Minimum Principle (for Standard OCP)

Augmented Hamiltonian:

$$H := \boldsymbol{\lambda}^\top f(x,u) + \boldsymbol{\eta}^\top c(x,u)$$

Necessary Conditions

Adjoint equation: $(\boldsymbol{\lambda}, \boldsymbol{\mu} \in BV)$

$$\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}(1) + \int_t^1 \boldsymbol{\nabla}_x H[\boldsymbol{\tau}] d\boldsymbol{\tau} + \int_t^1 \boldsymbol{\nabla}_x s[\boldsymbol{\tau}] d\boldsymbol{\mu}(\boldsymbol{\tau})$$

(constraint qualification $\Rightarrow \boldsymbol{\ell}_0 = 1$)

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Transversality conditions:

$$\boldsymbol{\lambda}(0)^\top = -\ell_0 \Phi'_{x_0} - \psi'_{x_0}{}^\top \boldsymbol{\sigma}, \qquad \boldsymbol{\lambda}(1)^\top = \ell_0 \Phi'_{x_1} + \psi'_{x_1}{}^\top \boldsymbol{\sigma}$$

(constraint qualification $\Rightarrow \ell_0 = 1$)

Local Minimum Principle (for Standard OCP)

Augmented Hamiltonian:

$$H := \boldsymbol{\lambda}^\top f(x,u) + \boldsymbol{\eta}^\top c(x,u)$$

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Transversality conditions:

$$\boldsymbol{\lambda}(0)^\top = -\boldsymbol{\ell}_0 \Phi'_{x_0} - \boldsymbol{\psi}'_{x_0}{}^\top \boldsymbol{\sigma}, \qquad \boldsymbol{\lambda}(1)^\top = \boldsymbol{\ell}_0 \Phi'_{x_1} + \boldsymbol{\psi}'_{x_1}{}^\top \boldsymbol{\sigma}$$

Stationarity of Hamiltonian:

$$\nabla_u H[t] = 0$$

(constraint qualification $\Rightarrow \boldsymbol{\ell}_0 = 1$)

Local Minimum Principle (for Standard OCP)

Augmented Hamiltonian:

$$H := \boldsymbol{\lambda}^\top f(x,u) + \boldsymbol{\eta}^\top c(x,u)$$

Necessary Conditions

Adjoint equation: $(\boldsymbol{\lambda}, \boldsymbol{\mu} \in BV)$

$$\boldsymbol{\lambda}(t) = \boldsymbol{\lambda}(1) + \int_t^1 \nabla_x H[\boldsymbol{\tau}] d\boldsymbol{\tau} + \int_t^1 \nabla_x s[\boldsymbol{\tau}] d\boldsymbol{\mu}(\boldsymbol{\tau})$$

Transversality conditions:

$$\boldsymbol{\lambda}(0)^\top = -\ell_0 \Phi'_{x_0} - {\psi'_{x_0}}^\top \boldsymbol{\sigma}, \qquad \boldsymbol{\lambda}(1)^\top = \ell_0 \Phi'_{x_1} + {\psi'_{x_1}}^\top \boldsymbol{\sigma}$$

Stationarity of Hamiltonian:

$$\nabla_u H[t] = 0$$

Complementarity conditions:

- $0 \leq \boldsymbol{\eta}(t) \quad \perp \quad -c[t] \geq 0$
- $\boldsymbol{\mu}$ is monotonically increasing and constant on inactive parts.

(constraint qualification $\Rightarrow \ell_0 = 1$)

Local Minimum Principle (for Standard OCP)

If order of pure state constraint $s(x) \leq 0$ is one ...

$\boldsymbol{\lambda}$ is in $W^{1,\infty}([0,1],\mathbb{R}^{n_x})$ and adjoint equation reduces to

$$\boldsymbol{\lambda}'(t) = -\nabla_x H[t] - \nabla_x s[t]\boldsymbol{\mu}'(t)$$

and complementarity condition reads

$$0 \leq \boldsymbol{\mu}'(t) \quad \perp \quad -s[t] \geq 0$$

but: often the order of a pure state constraint is greater than one!

Virtual Control Regularization

If order of pure state constraint is greater than one ...

Virtual Control Regularization of Standard OCP

$$
\begin{array}{llcll}
\text{Minimize} & \Phi(x(0),x(1)) + \dfrac{\kappa(\alpha)}{2}\displaystyle\int_0^1 \|v(t)\|^2 dt & & & \\
\text{s.t.} & x'(t) - f(x(t),u(t)) & = & 0 & \text{a.e. in } [0,1] \\
 & c(x(t),u(t)) & \leq & 0 & \text{a.e. in } [0,1] \\
 & s(x(t)) - \gamma(\alpha)v(t) & \leq & 0 & \text{a.e. in } [0,1] \rightarrow \boxed{\text{mixed constraint}} \\
 & \psi(x(0),x(1)) & = & 0 &
\end{array}
$$

[Krumbiegel/Cherednichenko/Rösch'08, G./Hüpping'12]

Note: basically equivalent to penalty method, since $v^*(t) = \frac{1}{\gamma(\alpha)} \max\{0, s(x^*(t))\}$

Bilevel Optimization Problem – MPCC Formulation

MPCC Formulation (single phase, standard problem)

Minimize

$$\max_{j=1,\ldots,J} t_j + d_j$$

s.t.

$$
\begin{aligned}
t_j + d_j - t_k &\leq M \cdot w_{jk} && (j \neq k) \\
t_k + d_k - t_j &\leq M \cdot (1 - w_{jk}) && (j \neq k) \\
t_j &\geq 0, \; w_{jk} \in \{0, 1\} && (j, k \in \{1, \ldots, J\}) \\
x^{(j)\prime}(t) &= f^{(j)}(x^{(j)}(t), u^{(j)}(t)) && t \in [t_j, t_j + d^{(j)}] \\
0 &= \psi(x^{(j)}(t_j), x^{(j)}(t_j + d_j)) \\
\boldsymbol{\lambda}^{(j)\prime}(t) &= -\nabla_x H^{(j)}[t] \quad (+ \quad \text{transversality conditions}) \\
0 &= \quad \nabla_u H^{(j)}[t] \\
0 \leq \boldsymbol{\eta}^{(j)}(t) \quad &\perp \quad - c^{(j)}(x^{(j)}(t), u^{(j)}(t)) \geq 0
\end{aligned}
$$

Numerical Solution

- solve $\nabla_u H^{(j)}[t] = 0$ for u, i.e. $u = U(x^{(j)}, \boldsymbol{\lambda}^{(j)}, \boldsymbol{\eta}^{(j)})$
- apply branch & bound method and direct shooting technique with SQP

Treatment of MPCC:

- relaxation/regularization approaches
  [Steffensen, Fletcher/Leyffer/Scholtes/Ralph, Kanzow/Schwartz, ...]
- penalty approaches
  [Luo/Pang/Ralph, ...]
- best results by applying NCP function (Fischer-Burmeister) and relaxation

$$-\varepsilon \leq \varphi_{FB}(\boldsymbol{\eta}^{(j)}(t), -c^{(j)}(x^{(j)}(t), u^{(j)}(t))) \leq \varepsilon$$

resp.

$$\varphi_{FB}(\boldsymbol{\eta}^{(j)}(t), -c^{(j)}(x^{(j)}(t), u^{(j)}(t)))^2 \leq \varepsilon^2$$

$\rightsquigarrow$ small violation of constraints permitted

Example

Optimal control problem for job j

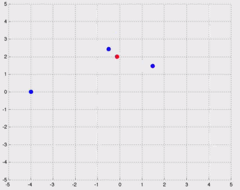
Minimize

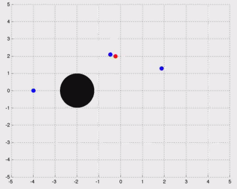
$$d_1^{(j)} + d_2^{(j)} + \frac{c}{2} \int_{t_j}^{t_j + d_1^{(j)} + d_2^{(j)}} \|u(t)\|^2 dt$$

s.t.

$$\begin{aligned}
x_1{}'(t) &= x_3(t) \\
x_2{}'(t) &= x_4(t) \\
x_3{}'(t) &= u_1(t) \\
x_4{}'(t) &= u_2(t) \\
x(t_j) &= [x_{0,1}^{(j)}, x_{0,2}^{(j)}, 0, 0]^T \\
x(t_j + d_1^{(j)}) &= (x_{T,1}(t_j + d_1^{(j)}), x_{T,2}(t_j + d_1^{(j)}), free, free)^\top \\
x(t_j + d_1^{(j)} + d_2^{(j)}) &= (x_{0,1}^{(j)}, x_{0,2}^{(j)}, 0, 0)^\top
\end{aligned}$$

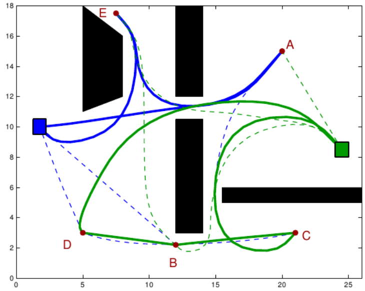
+ state constraint to avoid ball-shaped obstacle

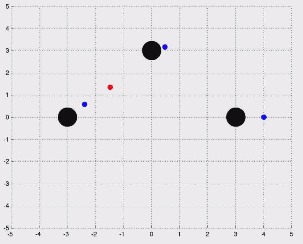


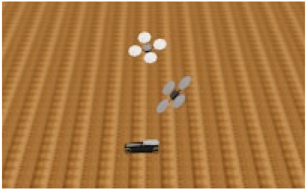


($J = 3$ jobs, $P = 2$ phases per job)

Motion Planning for Robots



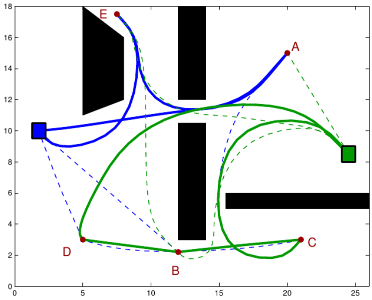




[joint work with C. Landry, W. Welz, D. Hömberg, R. Henrion]

Vehicle Routing

1. Compute the approximated trajectories

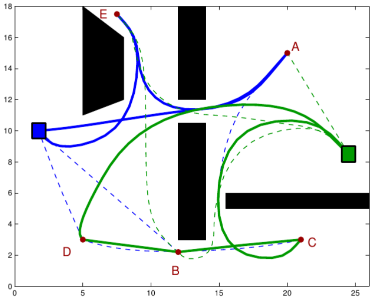


Details:
[C. Landry, W. Welz, M. Gerdts: Combining discrete and continuous optimization to solve kinodynamic motion planning problems, Optimization and Engineering, Vol. 17(3), pp. 533-556, 2016]

[C. Landry, M. Gerdts, R. Henrion, D. Hömberg: Path planning and Collision avoidance for robots, Numerical Algebra, Control and Optimization, Vol. 2(3), pp. 437-463, 2012]

Vehicle Routing

1. Compute the approximated trajectories
2. Find optimal sequences [Skutella/Welz]



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Vehicle Routing

1. Compute the approximated trajectories
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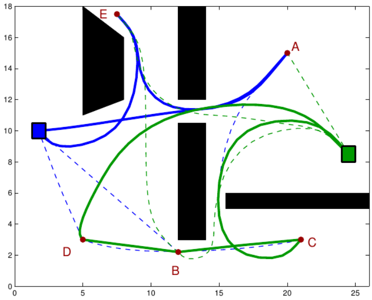

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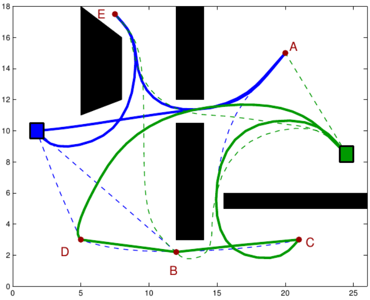
Vehicle Routing

1. Compute the approximated trajectories
2. Find optimal sequences [Skutella/Welz]
3. Compute exact trajectories that have not been computed yet
4. **If** collision **then**
   (a) add new constraint:
   these two trajectories cannot be used simultaneously
   (b) goto 2.

   else
   (c) Compute the optimal sequences with the exact trajectories
   (d) **If** same output **then** return **else** goto 3.

   endif



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Summary

Approaches towards bilevel problems:

- black-box approach
- MPCC approach
- value function approach

Common issues:

- nonsmoothness, discontinuities, local minima
- equivalence to bilevel problem?
- numerical stability $\rightsquigarrow$ initial guess generation?
- branch & bound $\rightsquigarrow$ local minima? convex underestimators?
- ...

Thanks for your attention!

Questions?

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