Performance Verification and Optimal Synthesis of Optimization-based Controllers

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BuildNet

SEVENTH FRAMEWORK PROGRAMME

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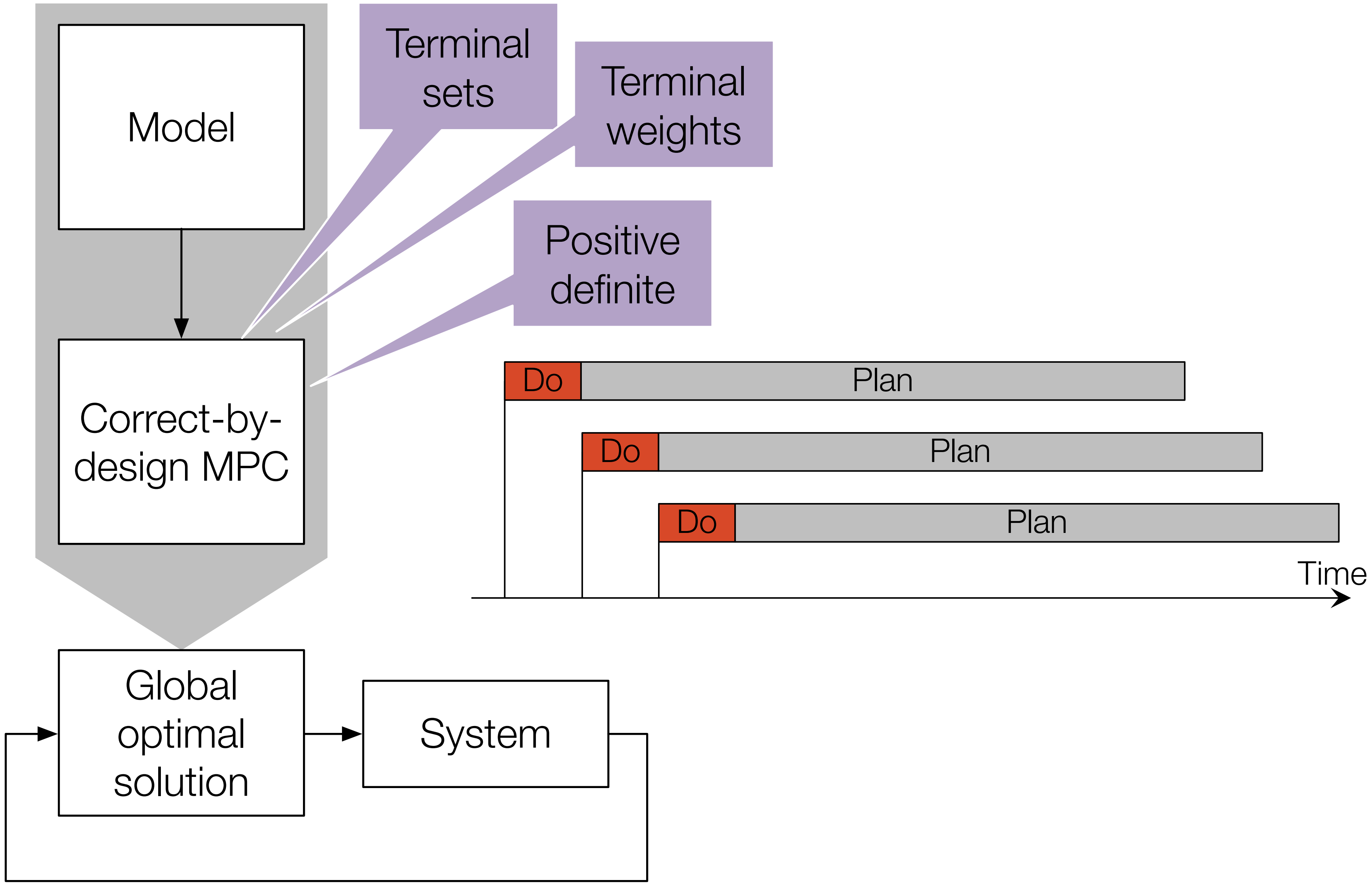
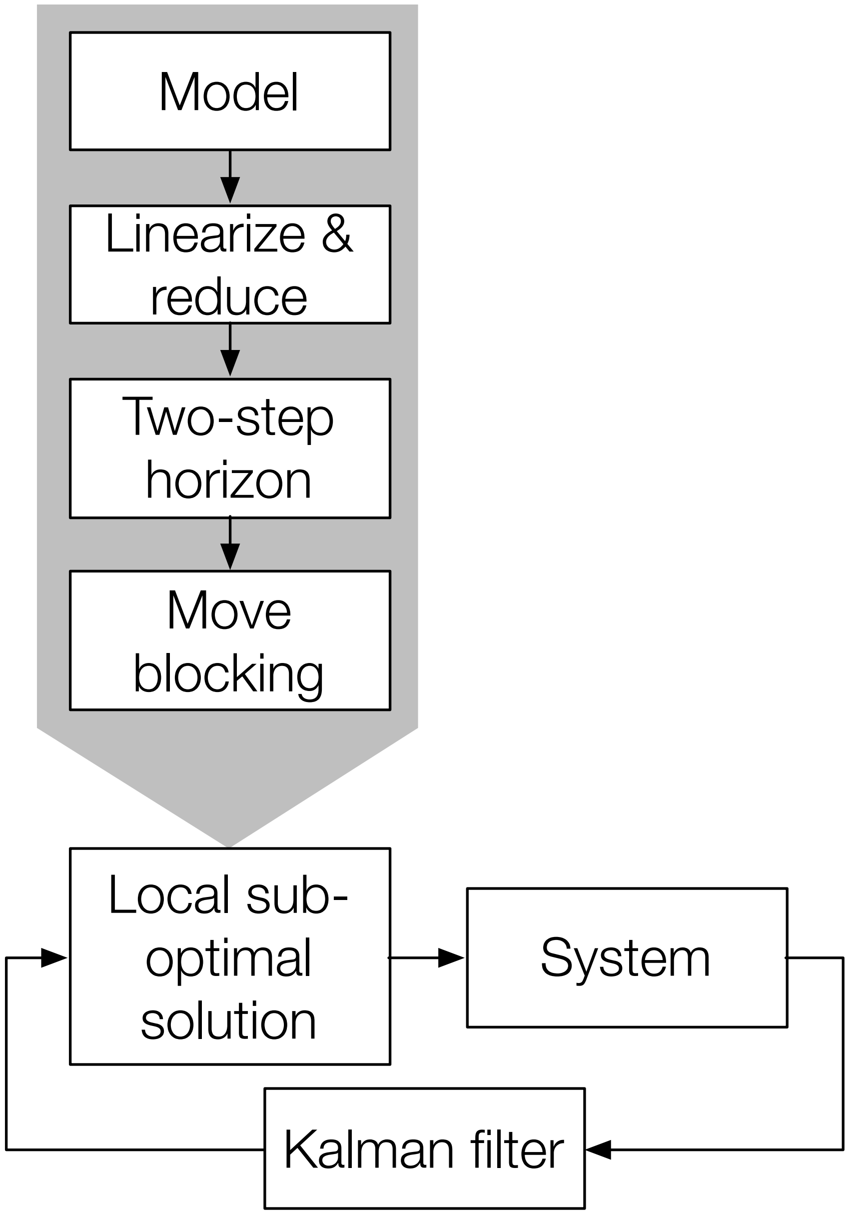
EPFL
ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

“Correct” MPC



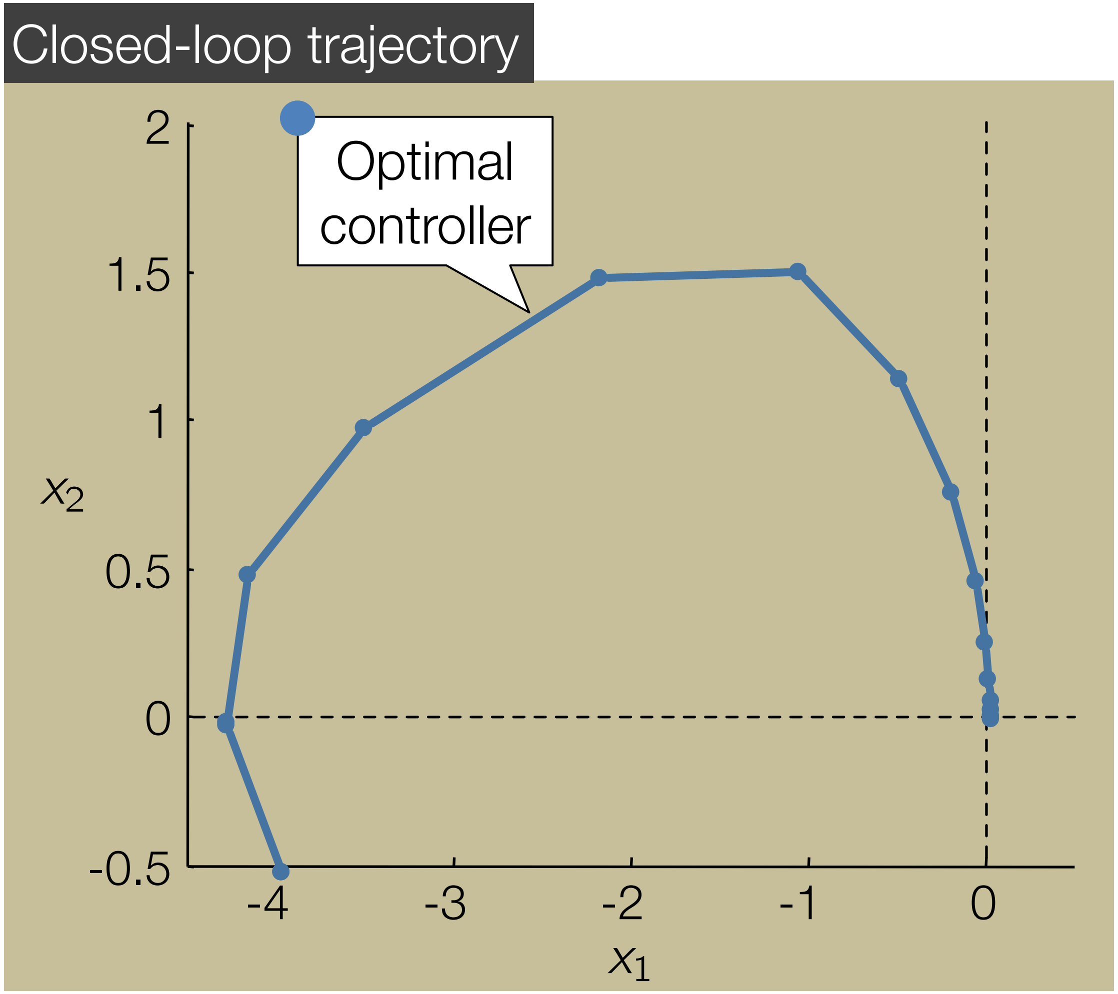
Model
Terminal sets
Terminal weights
Positive definite
Correct-by-design MPC
Do
Plan
Do
Plan
Do
Plan
Time
Global optimal solution
System

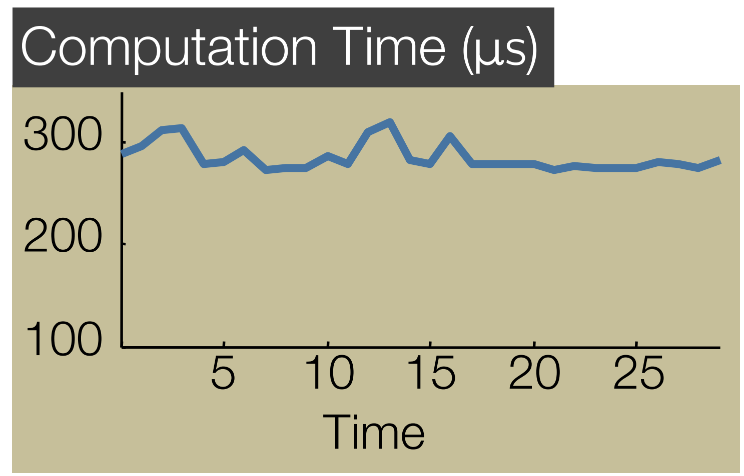
“Fast” MPC



Great results! … but ***extreme tuning***

Truncated Computation ⟹ Unstable Behavior





Toy Example:

$$x^+ = \begin{bmatrix} 1.2 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u$$

$$|x_1| \leq 5, -5 \leq x_2 \leq 1$$

$$|u| \leq 1, N = 5, Q = I, R = 1$$

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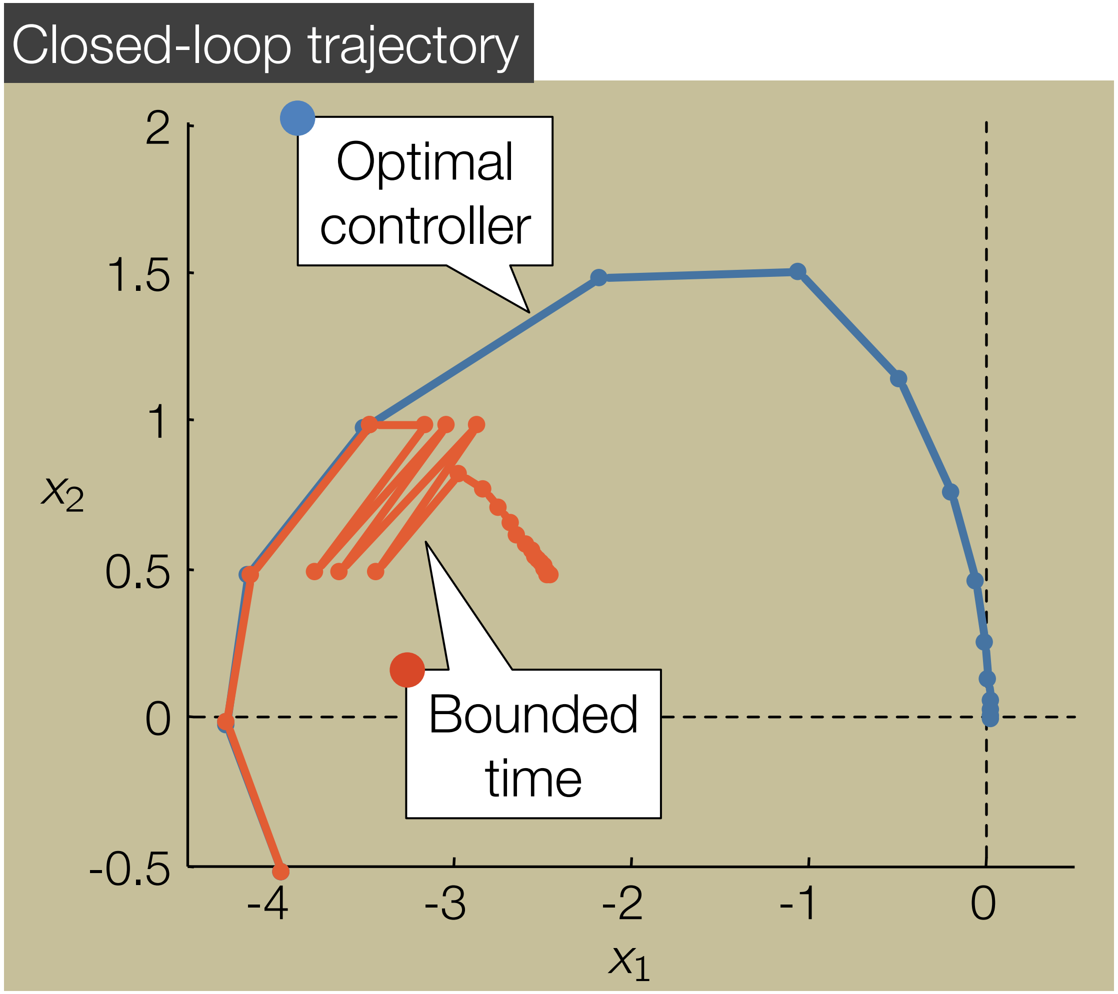

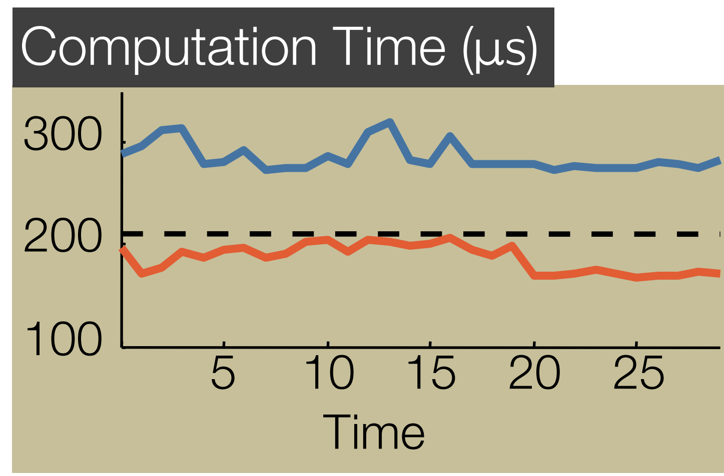

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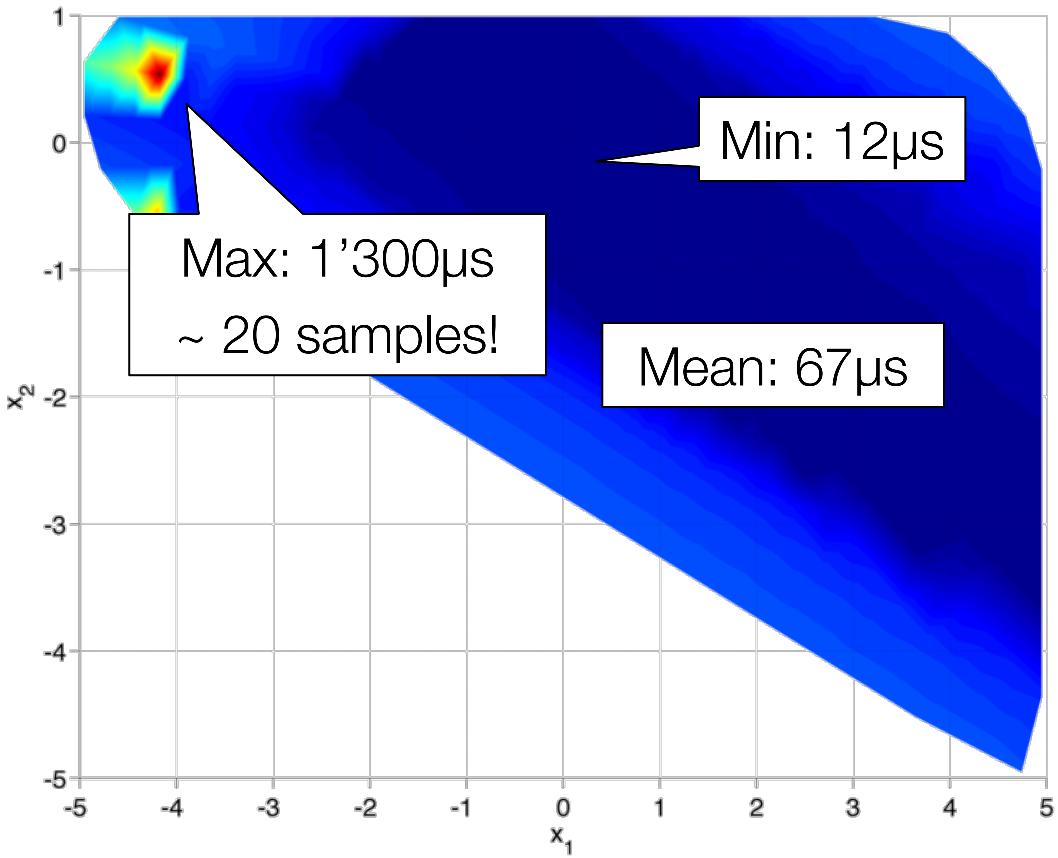
$|u| \leq 1, N = 5, Q = I, R = 1$

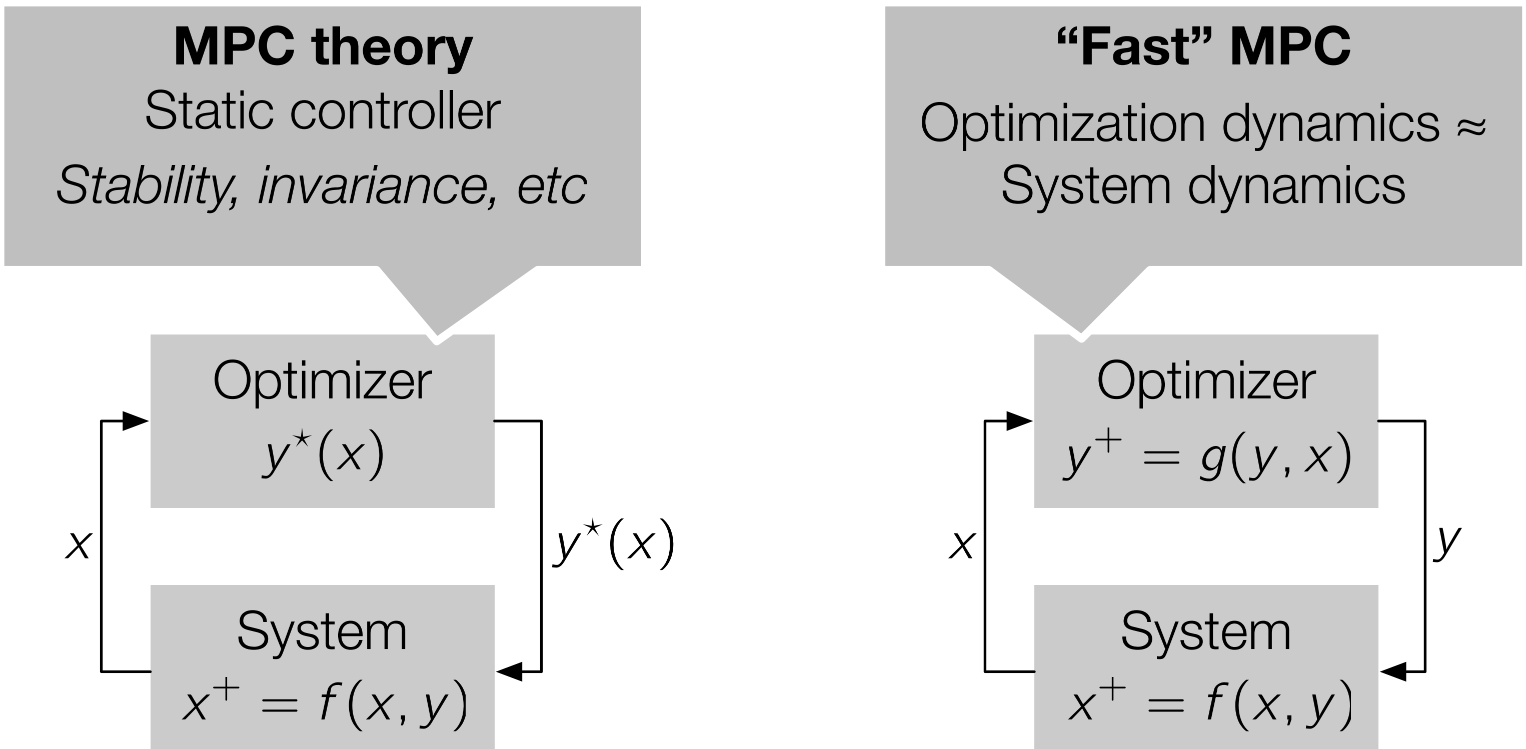
High Variability in Computation Times

Time to 1% sub-optimality



Worst-case occurs in rare, but extremely important conditions

Today: Real-time Certification of MPC Controllers

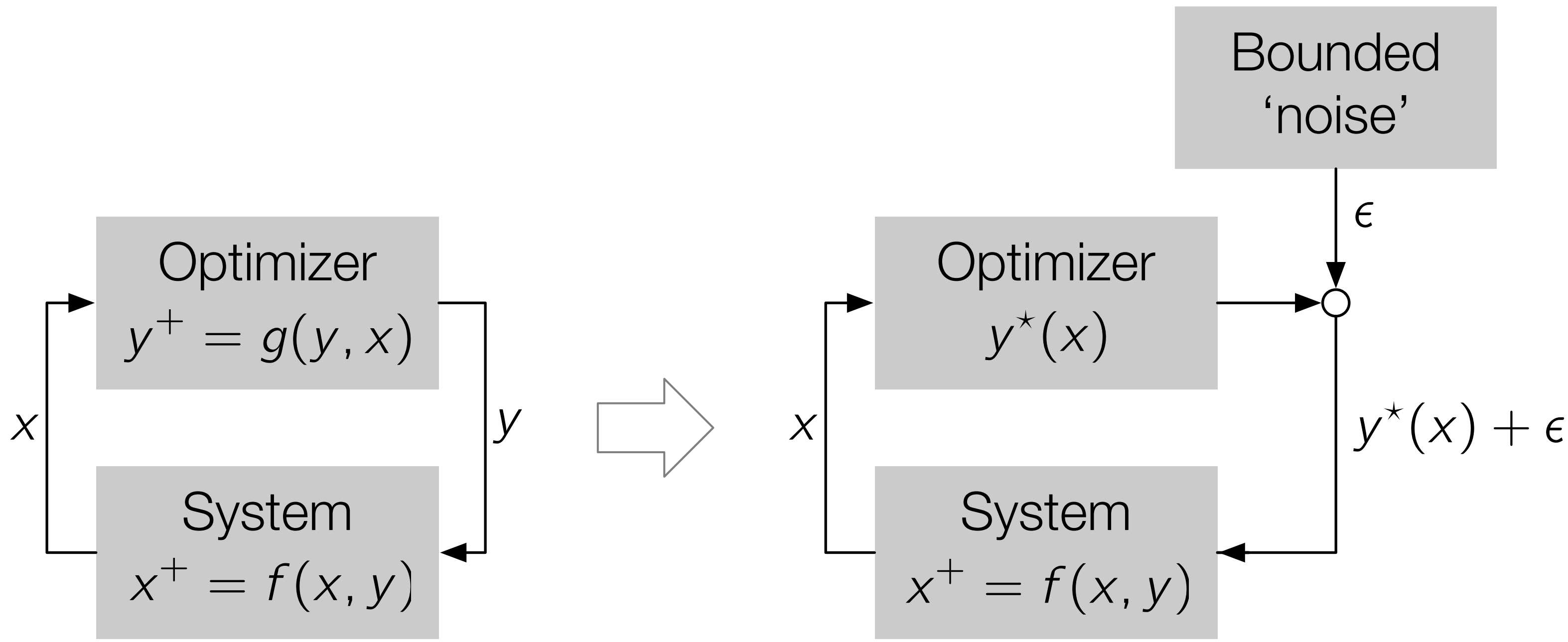


How do we analyse / design "fast" MPC?

Background : Two Roads to Certification

❶ Control perspective

- Bound sub-optimality
- Design robust controller



Challenge: Bounding error is very difficult

Worst-Case Iteration Bounds for 1st Order Methods

How many iterations guarantee convergence to ε-optimality for all states?

$$f(z_k; x) - f^\star(x) \leq r(k) \cdot C \cdot \max_{x \in \mathbb{X}_0} \|z_0(x) - z^\star(x)\| \leq \epsilon$$

① ② ③

① Convergence rate – depends on algorithm

- $1/k$ No assumptions required
- $1/k^2$ Strong convexity of f (possible for many control problems)
- w^k Strong convexity of f and constraints (rarely possible)

② Constant

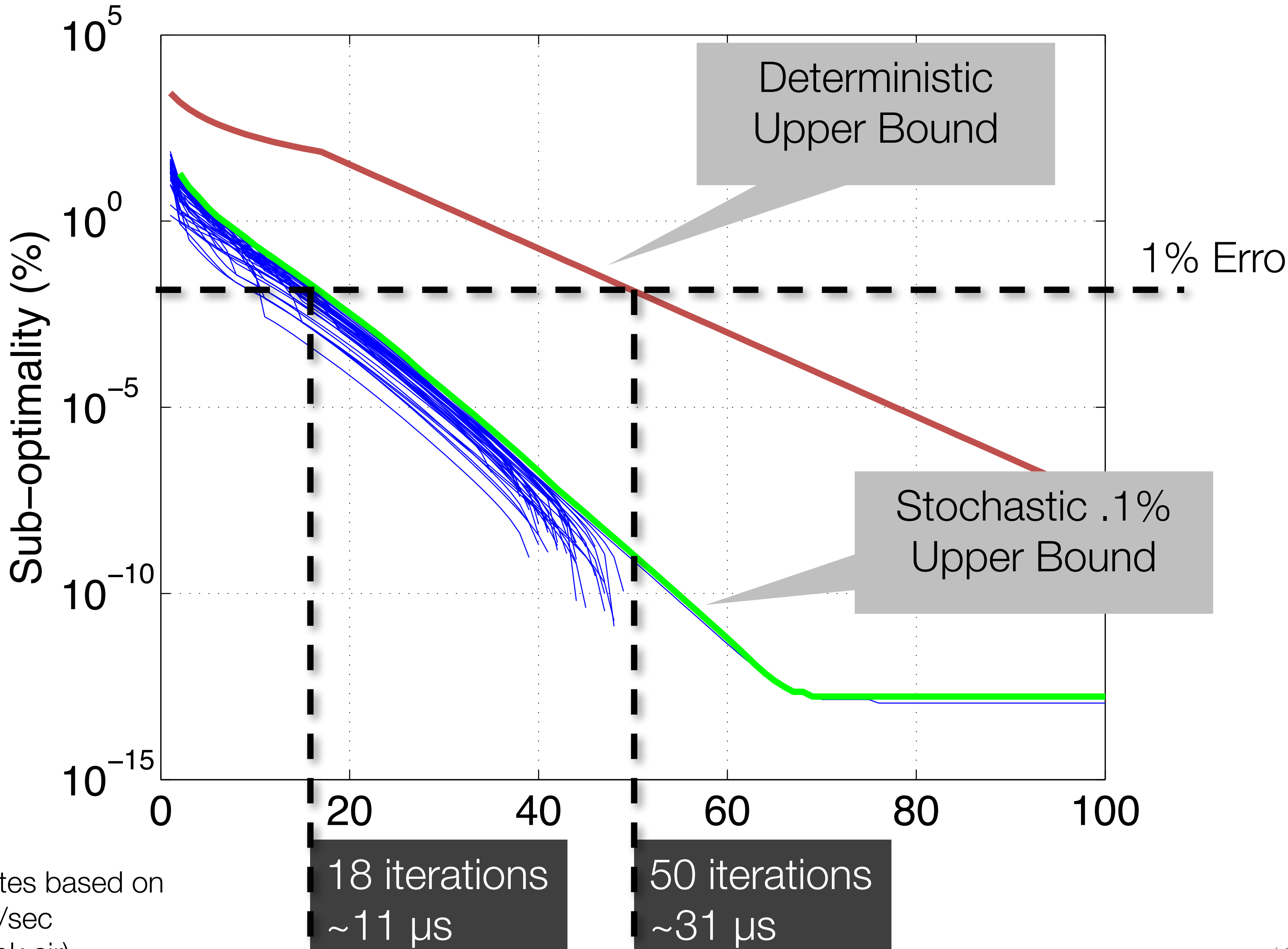
- Function of Lipschitz constants / convexity parameters

③ Initial residual

- Non-convex optimization
- Solvable for some restricted cases

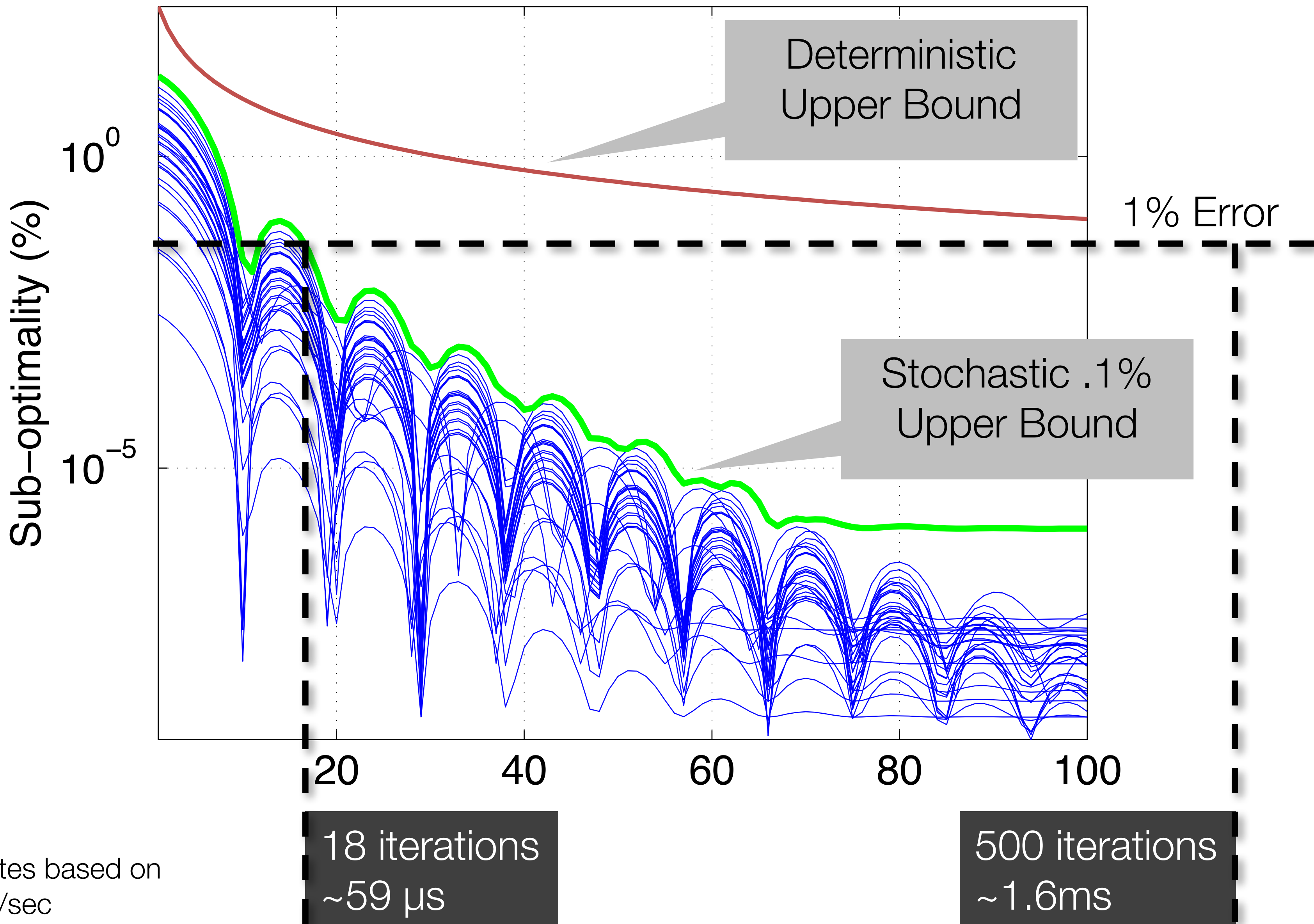
Problem: Computing any of these numbers is extremely difficult!

Ball and Plate / Fast Gradient Method



Time estimates based on
30 GFLOPS/sec
(i.e., macbook air)

Quad-Copter / Fast AMA



Time estimates based on
30 GFLOPS/sec
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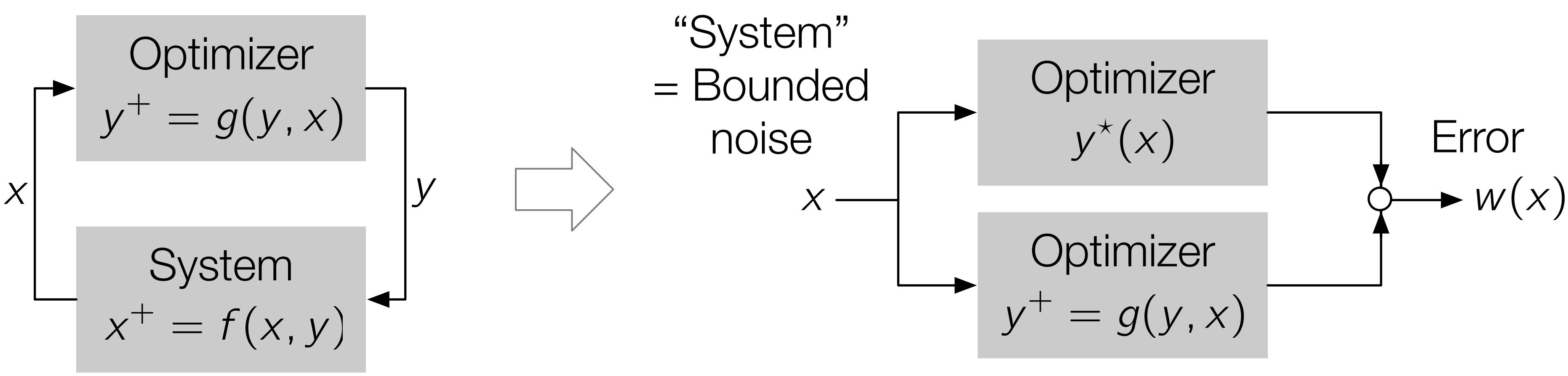
Background : Two Roads to Certification

❷ Optimization perspective

- Bound rate of change of the system (function of sampling rate)
- Show stability of the optimizer



Challenge: No concrete computation – indicates trends

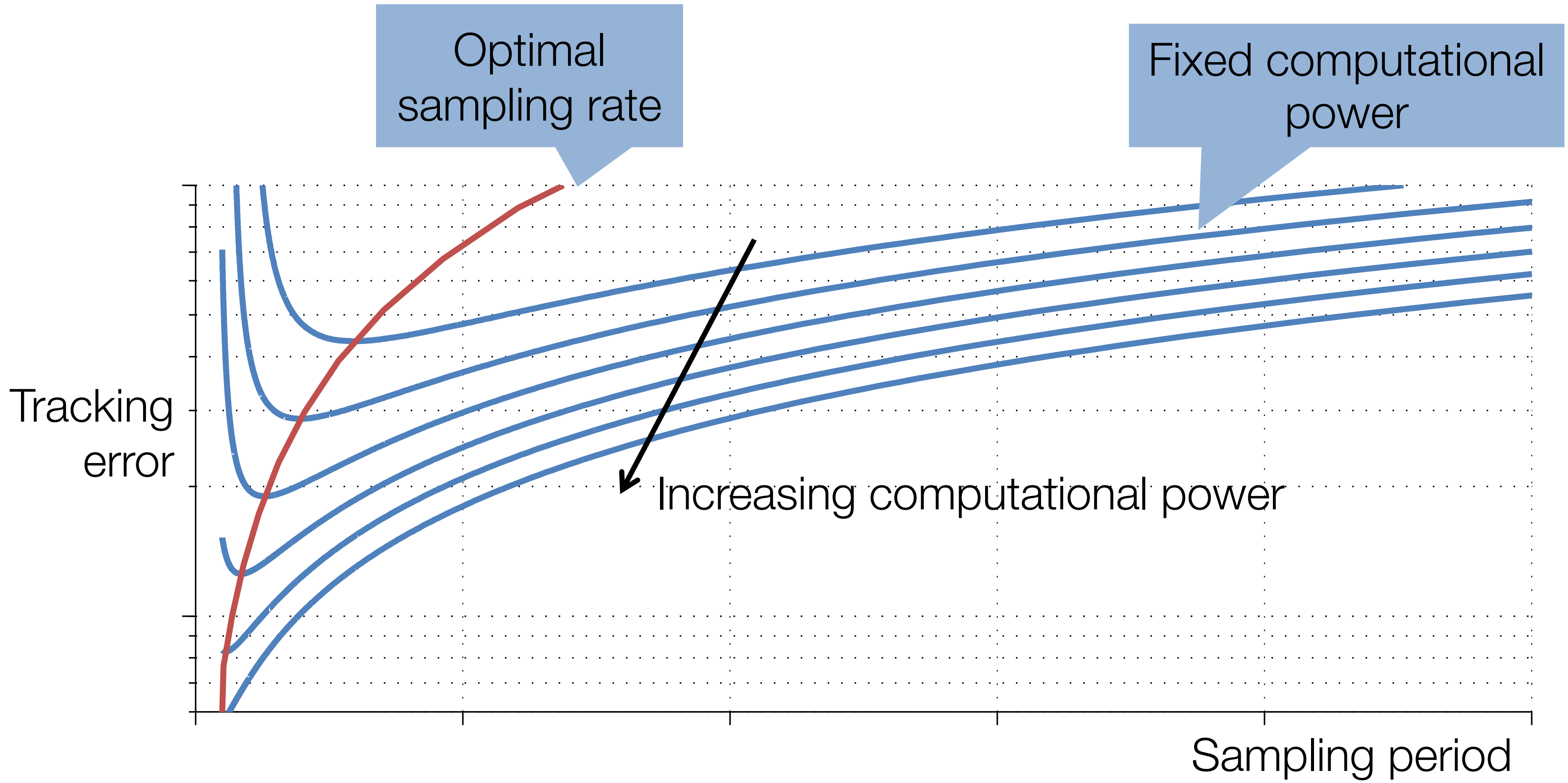
Stability of Parametric Tracking

Define the primal-dual point w

$$\underbrace{\|w_k - w^\star(x_k)\|}_{\substack{\text{Sub-optimality} \\ \text{at time } k}} \le \beta_w \underbrace{\|w_{k-1} - w^\star(x_{k-1})\|}_{\substack{\text{Sub-optimality} \\ \text{at time } k-1}} + \beta_x \underbrace{\|x_k - x_{k-1}\|}_{\substack{\text{Change} \\ \text{in state}}}$$

	Optimizer convergence (β_w)	Sensitivity to system state (β_x)	Recommendation
Interior-point	Small	Large	Sample as quickly as possible
1st order methods & distributed optimization	Large	Small	Sample slowly

Optimal Sample Frequency

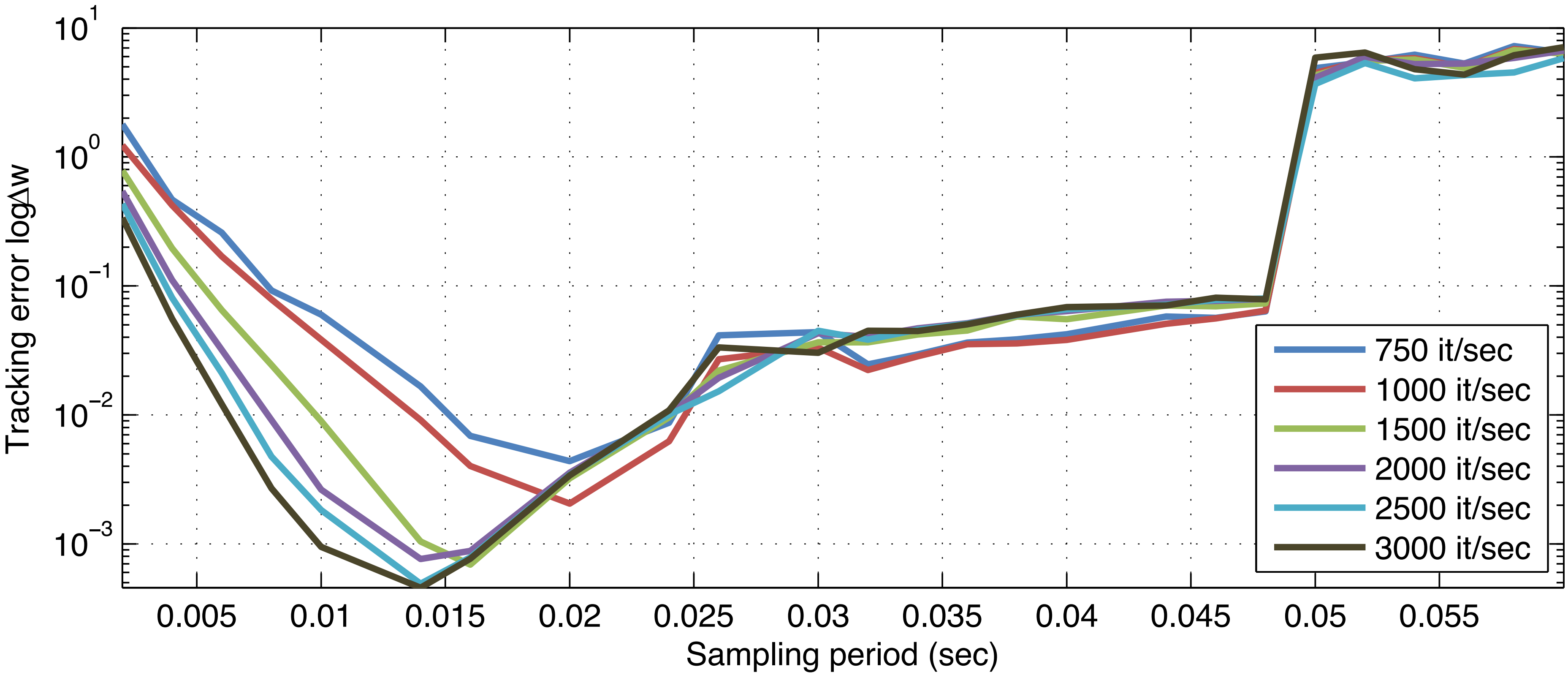


Suggests:

- $\Delta w = C^{-\alpha} \rightarrow$ significantly decreasing gains for increased computing
- Contrary to standard result for second order methods

Experiments Match Predicted Behaviour

1.1 kW Separately Excited DC Motor



Background : Two Roads to Certification

❶ **Control perspective**

- Bound sub-optimality
- Design robust controller

Challenge: Bounding error is very difficult
Result: Effective for some *very* simple systems

❷ **Optimization perspective**

- Bound rate of change of the system (function of sampling rate)
- Show stability of the optimizer

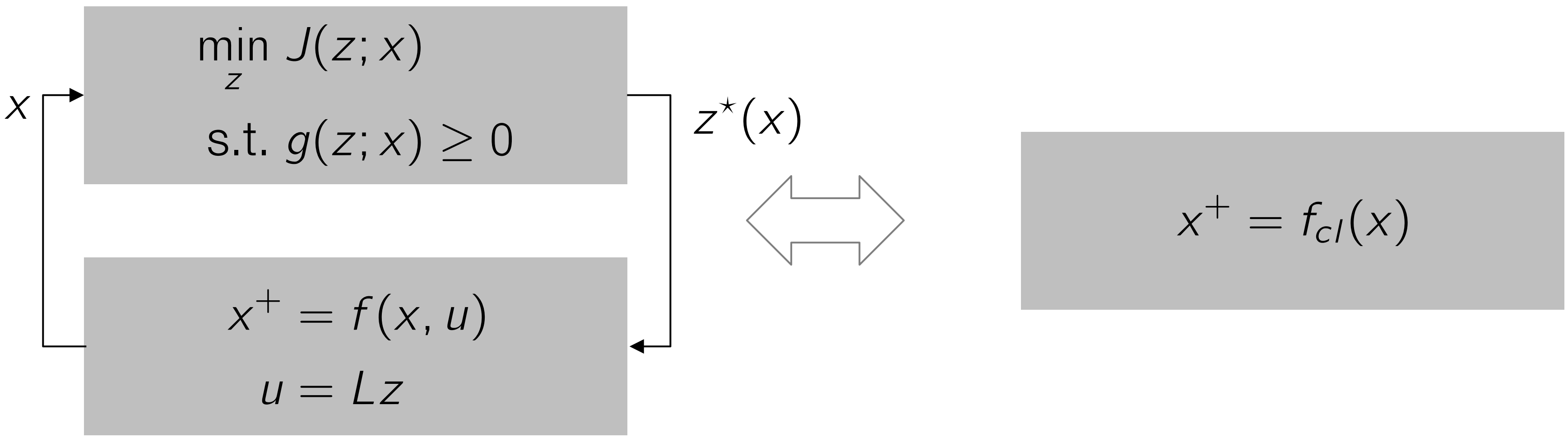
Challenge: No concrete computation
Result: Indicates basic principles

- Interior-point : Sample as fast as possible!
- First-order / distributed : Slower sampling is better

Today: Non-conservative direct verification

Direct Stability Verification

Classic approach : Model closed-loop system and search for Lyapunov function



Stability

$x_t \to 0$ as $t \to \infty$

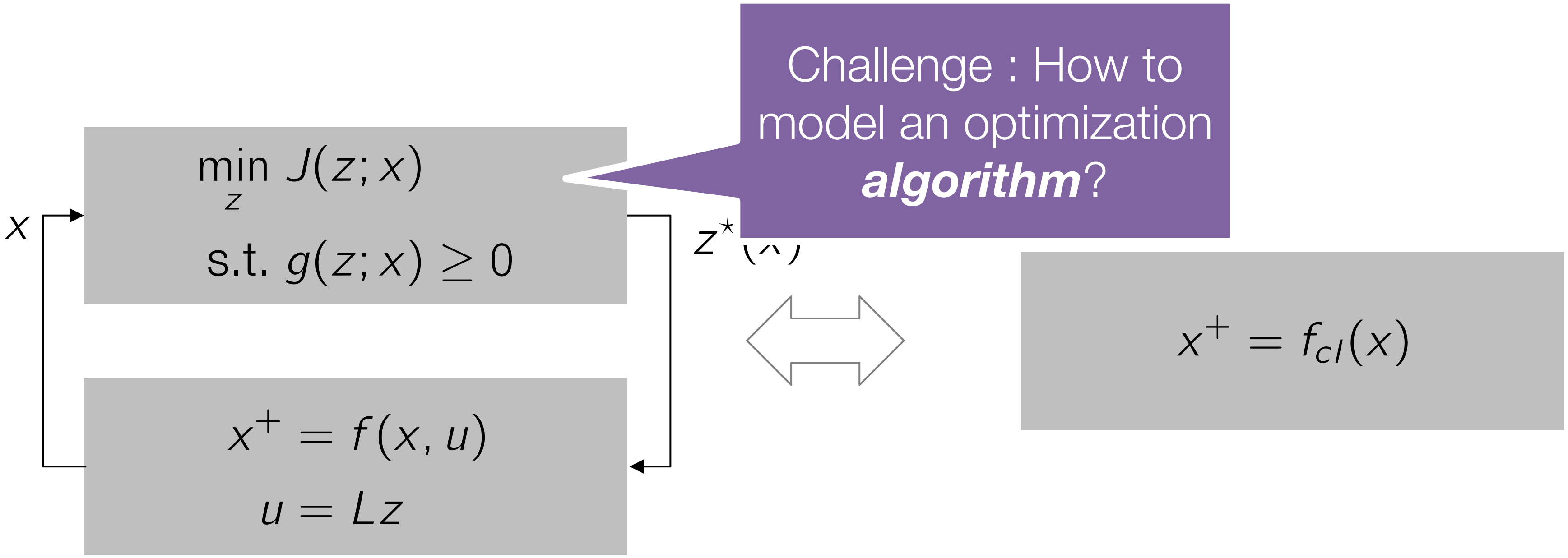
$\Leftarrow$

Lyapunov Function

$$V(f_{\text{cl}}(x)) - V(x) \leq -\|x\|^2$$
$$V(x) \geq \|x\|^2$$

Direct Stability Verification

Classic approach : Model closed-loop system and search for Lyapunov function



Stability

$$x_t \to 0 \text{ as } t \to \infty$$

$\Longleftarrow$

Lyapunov Function

$$V(f_{\text{cl}}(x)) - V(x) \leq -\|x\|^2$$

$$V(x) \geq \|x\|^2$$

Parametric Optimization ➛ Polynomial Constraints

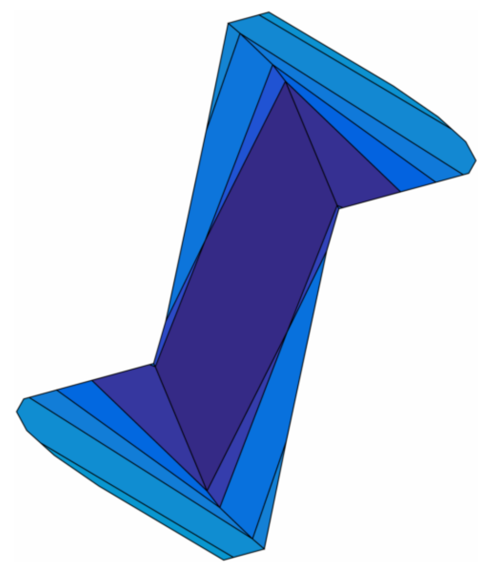
Set of ***all*** optimal solutions

Parametric optimization		KKT conditions		Basic semi-algebraic set
$\min_z J(z;x)$ s.t. $g(z;x) \geq 0$	⇨	$\nabla J(z;x) + \lambda^T \nabla g(z;x) = 0$, $g(z;x),\ \lambda \geq 0$, $\lambda^T g(z;x) = 0$	⇨	$(z, \lambda, x) \in K$

The control law is a projection of this set

$$(x, u) \in \mathrm{proj}_{x, Lz}\ K$$

Piecewise polynomial function



Example: Quadratic program
Piecewise affine function

Parametric Optimization ➛ Polynomial Constraints

Set of ***all*** optimal solutions

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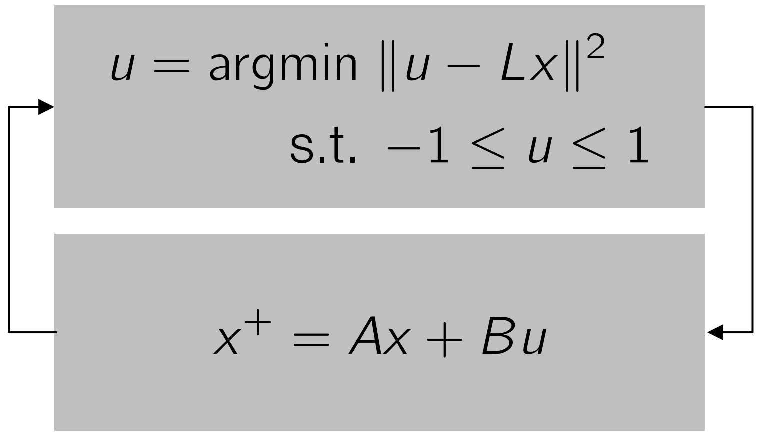
Leave the control law in implicit polynomial form

Closed-loop system

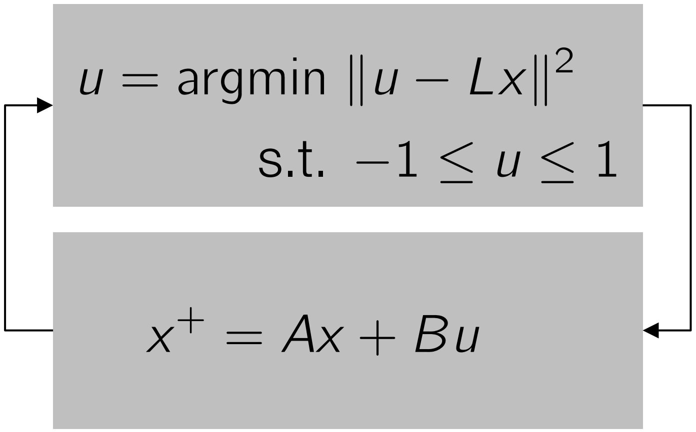
$$x^+ = f(x, Lz) \text{ subject to } (z, \lambda, x) \in K$$

Result : Polynomial representation of an optimization-based control law

Simple Example : Constrained Linear Control



Simple Example : Constrained Linear Control



$$K = \left\{ (x, u, \lambda) \;\middle|\; \begin{aligned} u - Lx - \lambda_l + \lambda_u = 0 \\ -1 \leq u \leq 1 \\ \lambda >= 0 \\ \lambda_l^T(1+u) = \lambda_u^T(1-u) = 0 \end{aligned} \right\}$$

Demonstrate the existence of a function V such that

$$V(x, u, \lambda) \geq 0 \text{ for all } (x, u, \lambda) \in K$$

$$V(0, u, \lambda) = 0 \text{ for all } (0, u, \lambda) \in K$$

$$V(x^+, u^+, \lambda^+) - V(x, u, \lambda) \leq -\|x\| \text{ for all } (x, u, \lambda) \in K,\ (x^+, u^+, \lambda^+) \in K$$

Note

- V can be a function of the primal and dual variables → piecewise polynomial
- Expression is linear in V

Example : Linear Quadratic Predictive Control

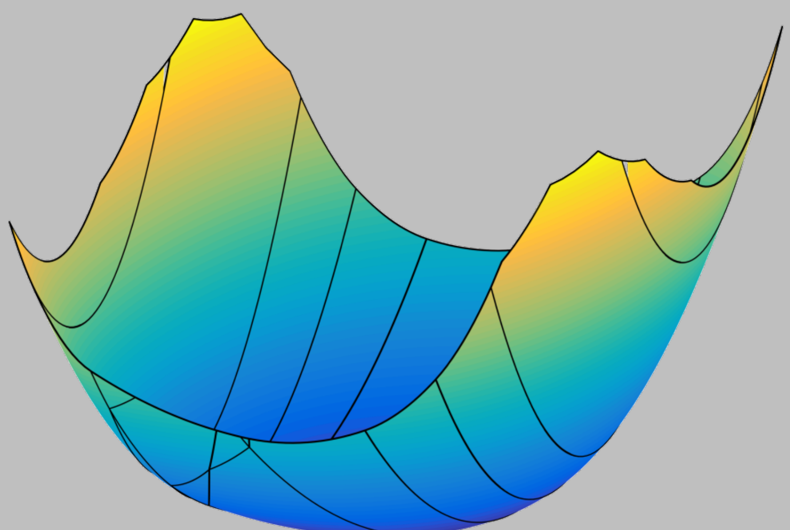
Quadratic function

$$V^\star(x) = \min \overbrace{\sum x_i^\top Q x_i + u_i^\top R u_i}^{V(x,\vec{x},\vec{u})}$$

$$\text{s.t.} \quad x_{i+1} = Ax_i + Bu_i$$

$$(x_i, u_i) \in X \times U$$

Piecewise quadratic function



Quadratic

$$V^\star(x) = \text{proj}_x\{V(x,\vec{x},\vec{u}) \mid (x,\vec{x},\vec{u}) \in K\}$$

Exponentially complex piecewise quadratic

Simple Example : Constrained Linear Control

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$$V(x^+,u^+,\lambda^+) - V(x,u,\lambda) \leq -\|x\| \text{ for all } (x,u,\lambda) \in K, \; (x^+,u^+,\lambda^+) \in K$$

Convex in the coefficients of the red polynomials

$$K = \left\{ \begin{array}{r} u - Lx - \lambda_l + \lambda_u = 0 \\ -1 \leq u \leq 1, \lambda >= 0 \\ \lambda_l^T(1+u) = \lambda_u^T(1-u) = 0 \end{array} \right\}$$

$$\begin{aligned} V(x) - V(Ax+Bu) - \|x\|^2 = {} & \sigma_0 \\ & + {\sigma_l}^T(-1-u) + {\sigma_u}^T(u-1) \\ & - {\sigma_\lambda}^T \lambda \\ & + {\delta_1}^T(u - Lx - \lambda_l + \lambda_u) + \delta_2 \lambda_l^T(1+u) \\ & + \delta_3 \lambda_u^T(1-u) \end{aligned}$$

$$V, \sigma_0, \sigma_l, \sigma_\lambda \in SOS$$

Convex Analysis of Polynomial Systems

Stability certification

$$V(x) \geq 0$$
$$V(f(x,u)) \leq V(x)$$
$$\text{for all } (x,u,\lambda) \in K$$

Discounted cost

If
$$V(x) \geq 0$$
$$\alpha V(f(x,u)) \leq V(x) - l(x,u)$$
$$\text{for all } (x,u,\lambda) \in K$$

then
$$V(x_0) \geq \sum_{t=0}^{\infty} \alpha^t l(x_t, u_t)$$

Sum-of-squares

↓

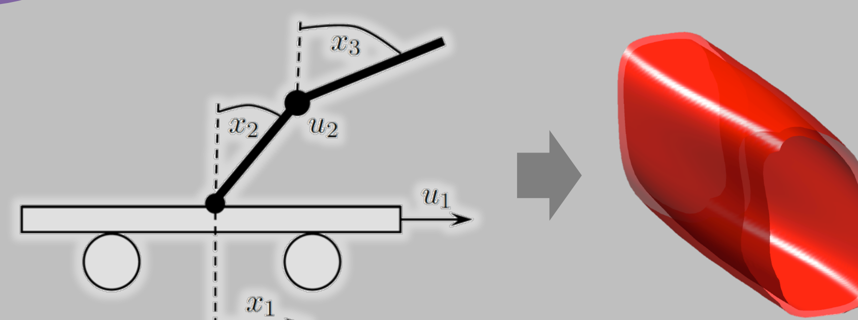
Convex SDP!

ISS-gain

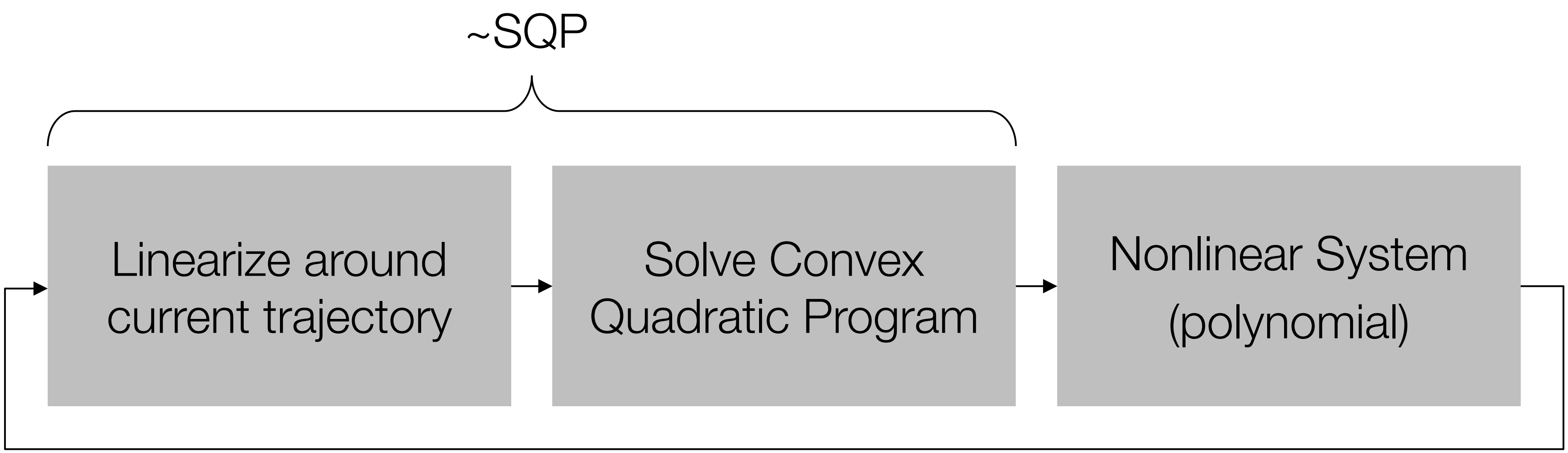
If
$$V(x) \geq 0$$
$$V(f(x,u,w)) \leq V(x) + \alpha_w \|w\|^2$$
$$\text{for all } (x,u,\lambda) \in K$$

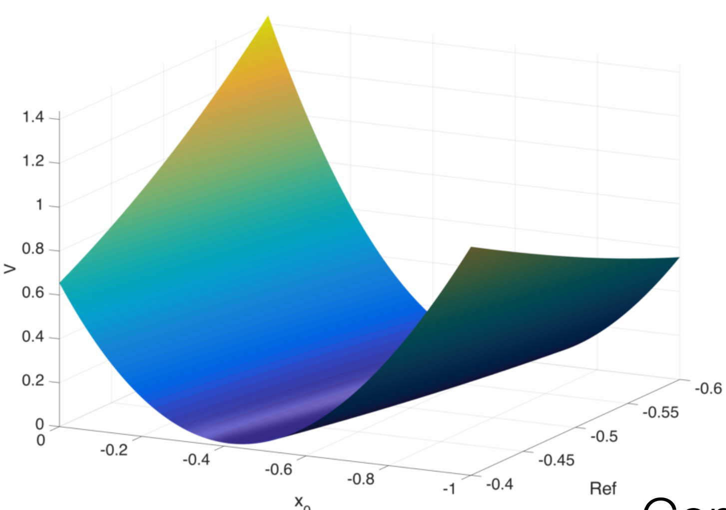
then
$$\sum_{t=0}^{\infty} \|y(x_t)\|_2^2 \leq V(x_0) + \alpha_w \sum_{t=0}^{\infty} \|w_t\|_2^2$$

Region of attraction



Example – Stability of Real-Time Iterations





Lyapunov function for real-time iteration of bi-linear motor

Field current

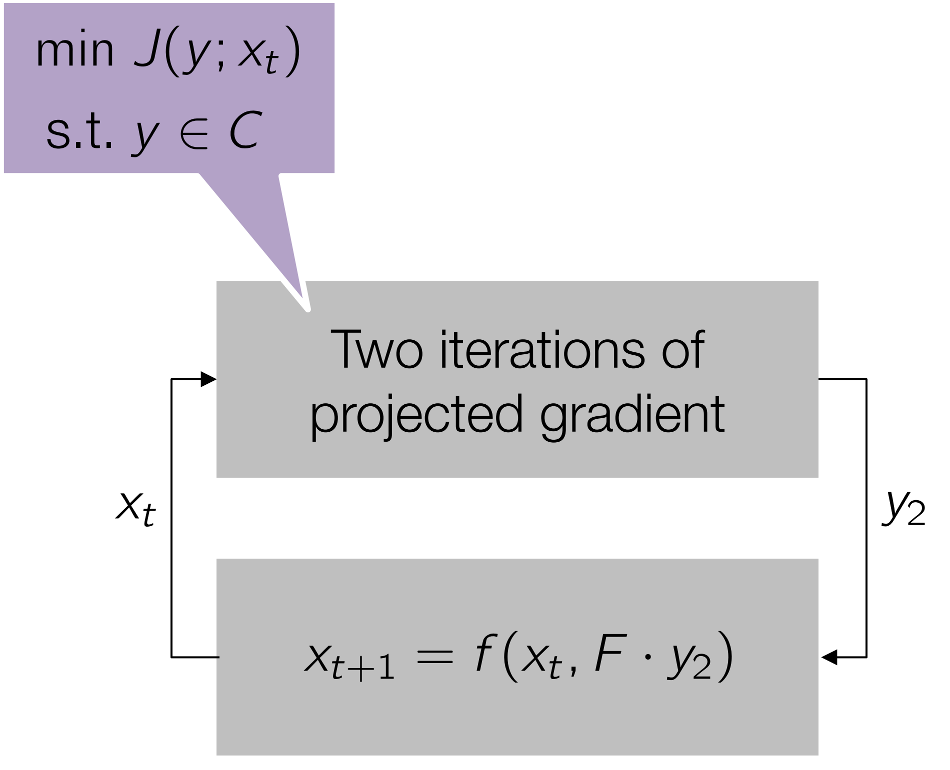
$$x^+ = Ax + Bxu + c$$

Angular speed
Armature current

Computation time: 59sec*

*YALMIP + Mosek

Example – Fixed-time Projected Gradient Method



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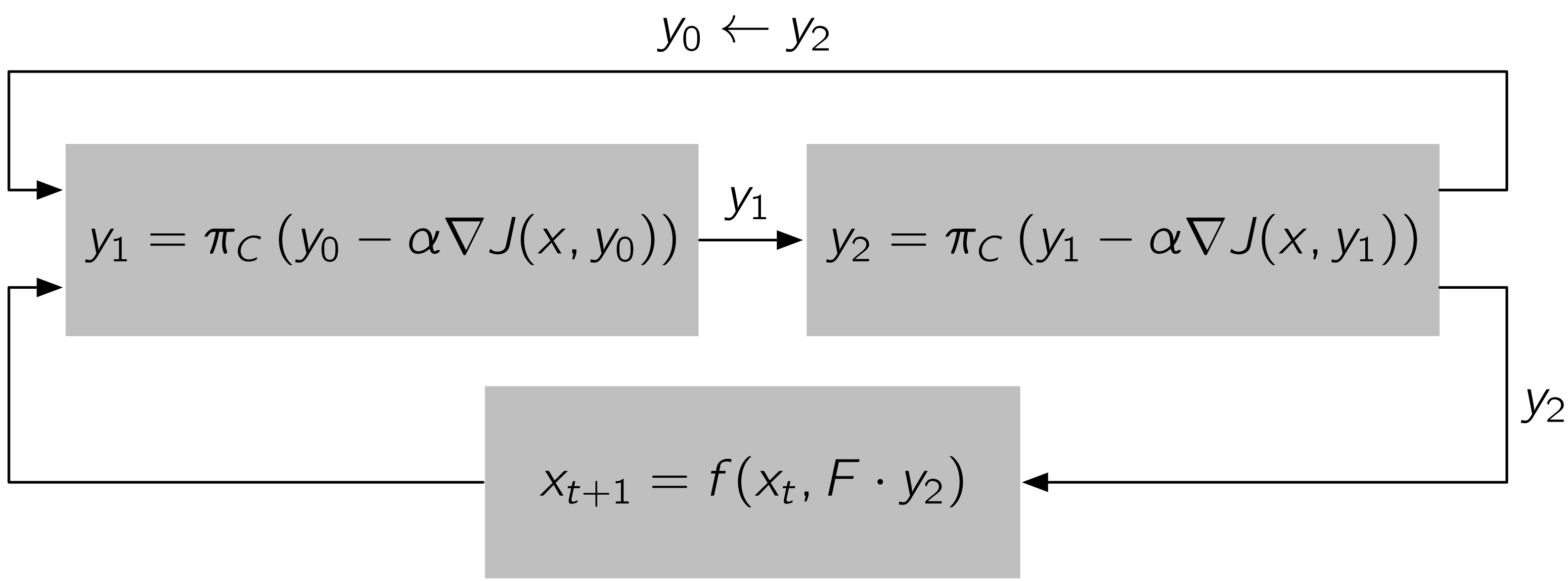

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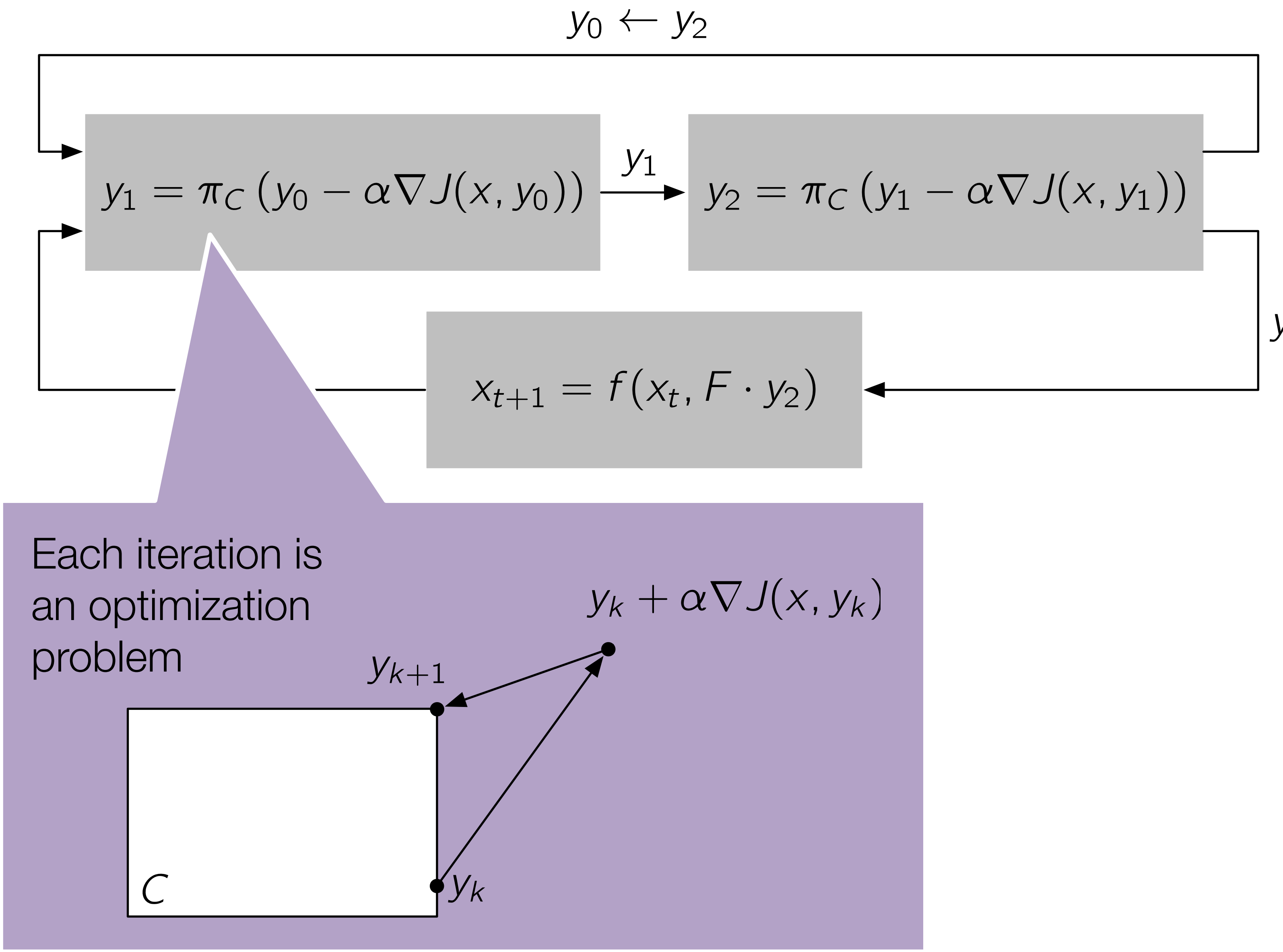

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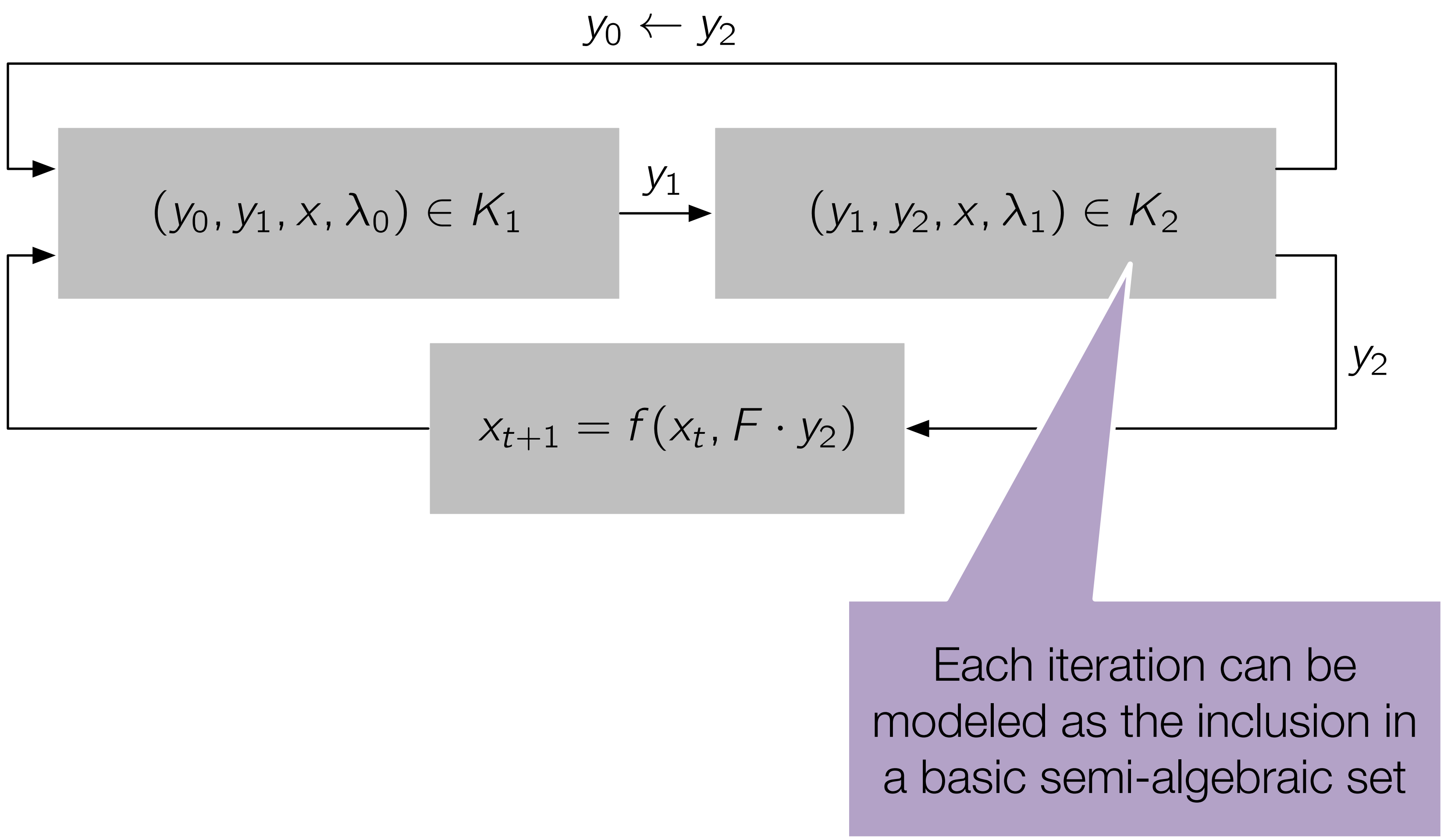
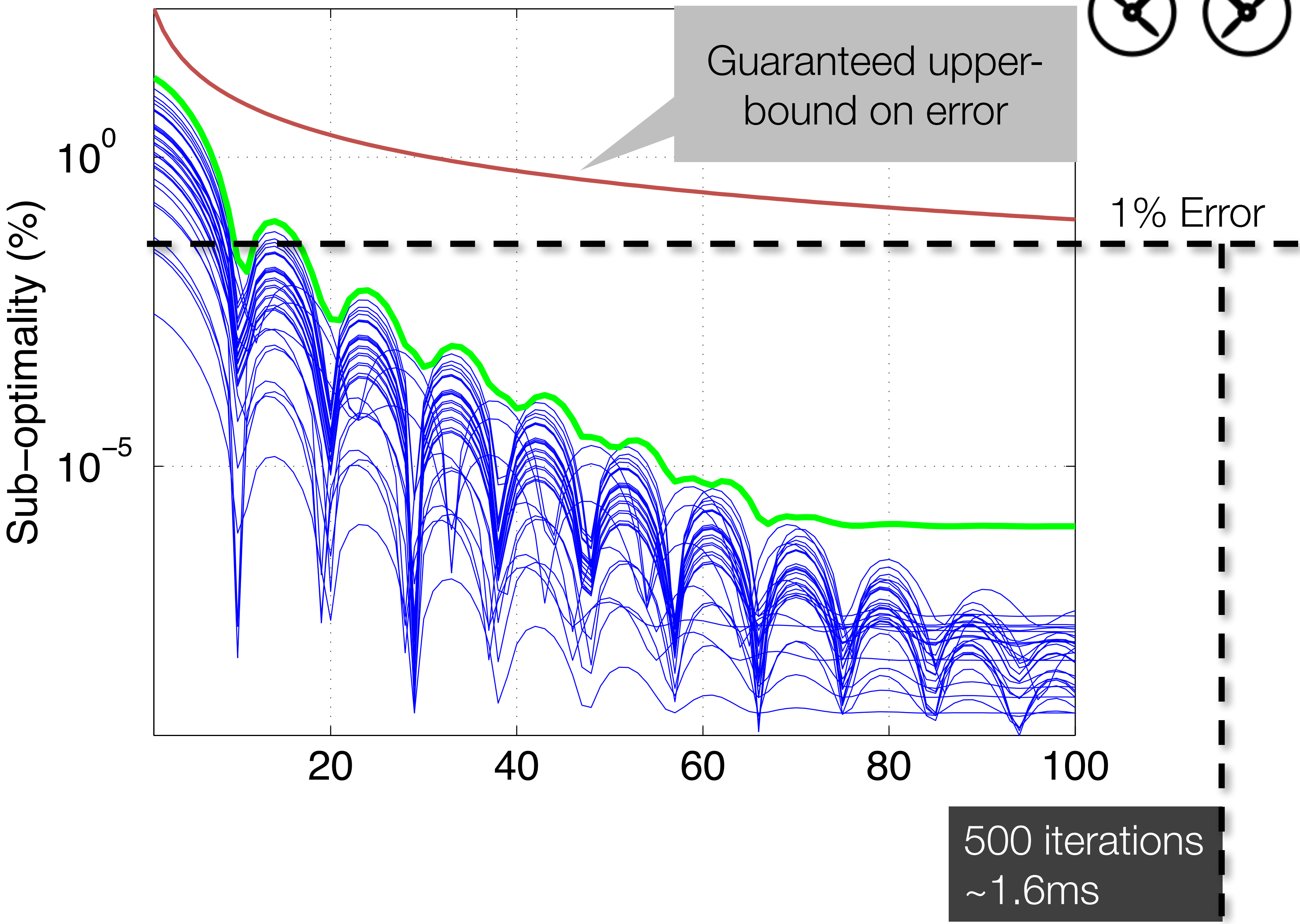


Result : Fixed-time MPC can be modeled as a set of polynomial constraints

Example – FGM for Quad-copter



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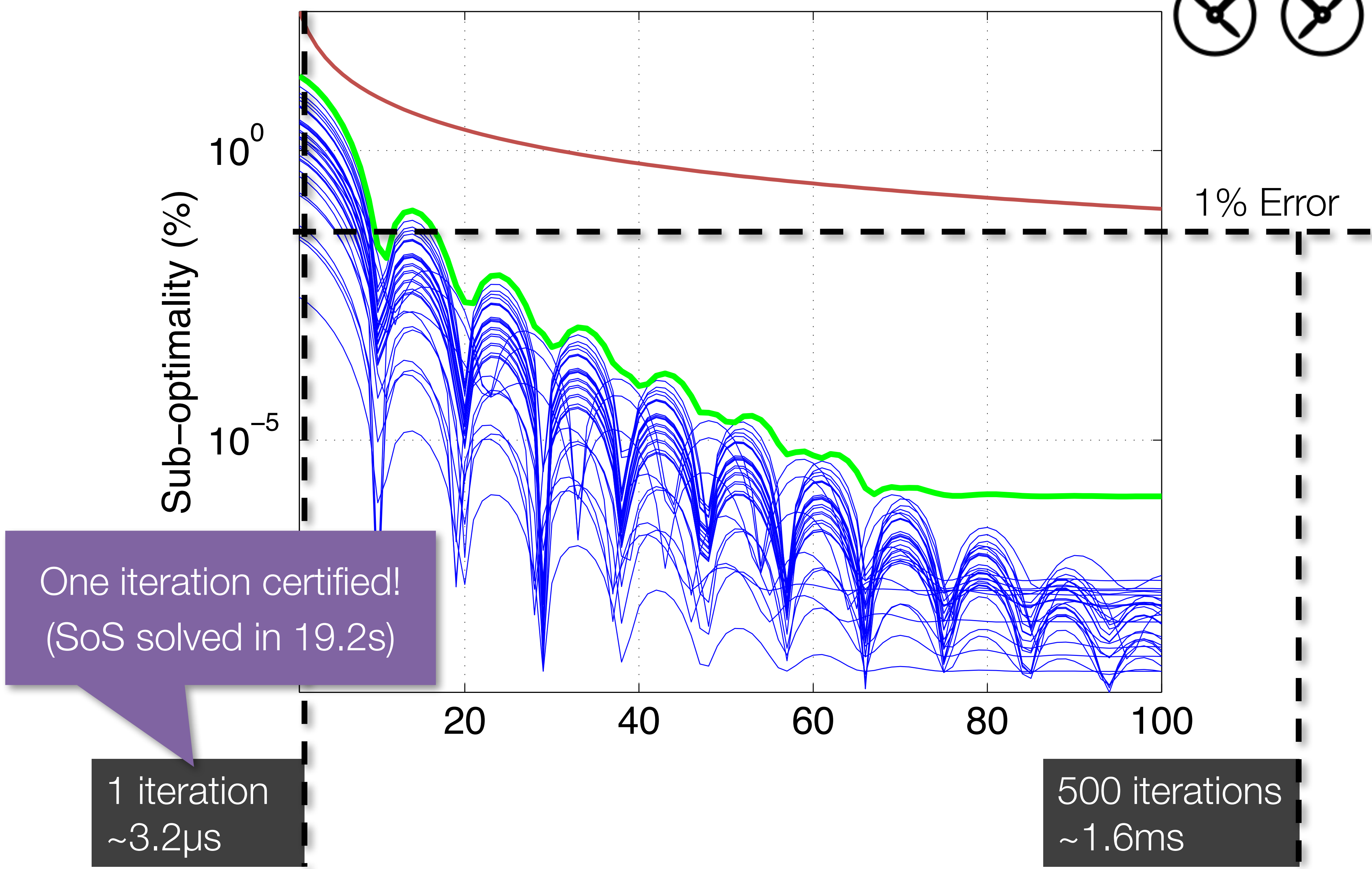
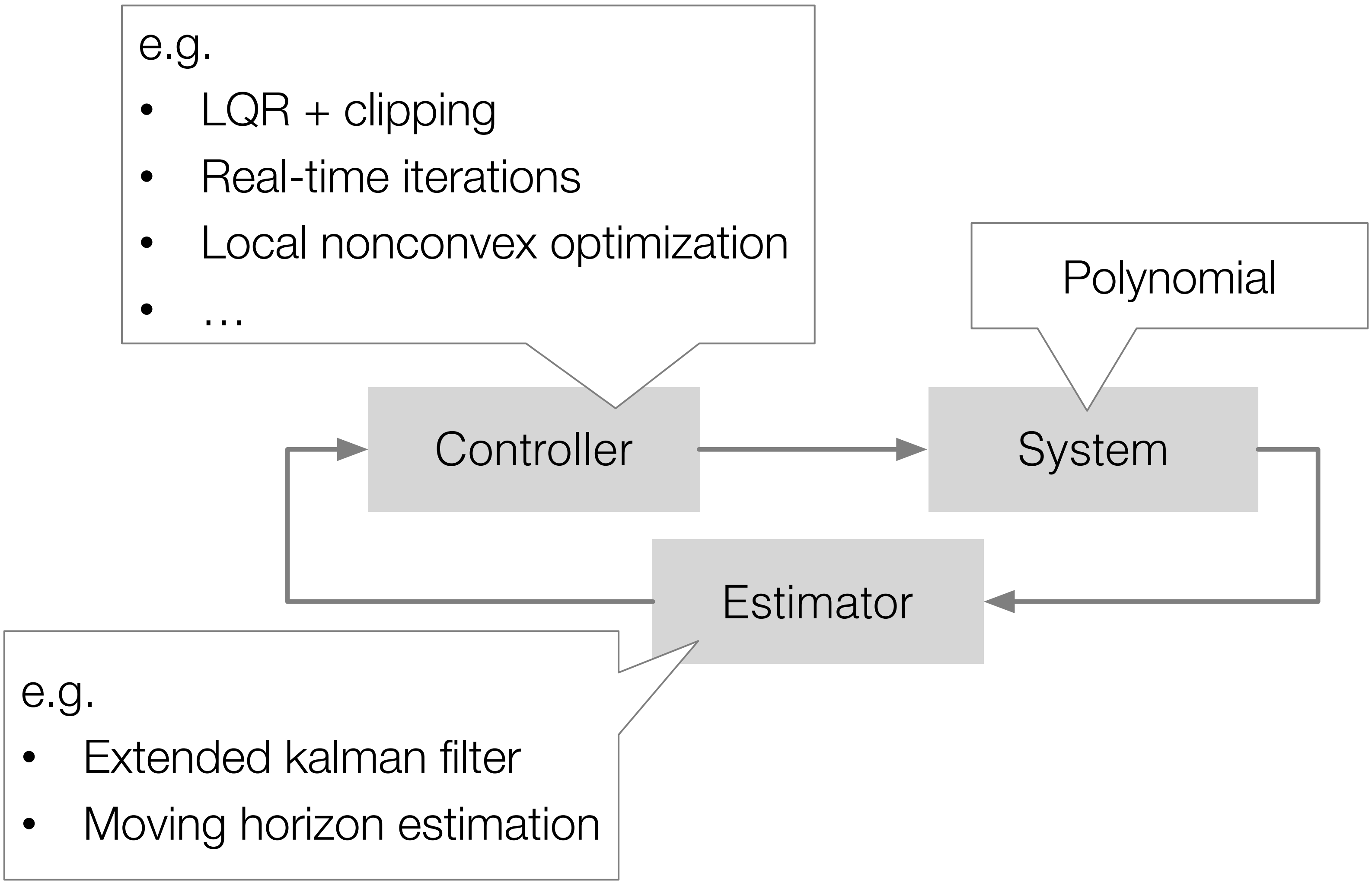


Framework is Very General



Key: Optimizer does not have to be derived from classic MPC theory

From Verification to Synthesis

Goal : 'Optimal' parameters

$$\min_{\theta} \sum_{i=0}^{\infty} l(x_i, u_i)$$

Minimize upper bound

$$V(x) \geq \sum_{i=0}^{\infty} l(x_i, u_i)$$

$$\min_{u} J(u; x; \theta)$$

$$\text{s.t. } g(u; x; \theta) \geq 0$$

$$x^+ = f(x, u_o)$$

$$J(\theta) = \min \int V(x)\rho(x)dx$$

$$\text{s.t.} \quad V(x) \geq 0$$

$$V(f(x,u)) \leq V(x) - l(x,u)$$

$$\text{for all } (x, u, \lambda) \in K(\theta)$$

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$$\text{for all } (x, u, \lambda) \in K(\theta)$$

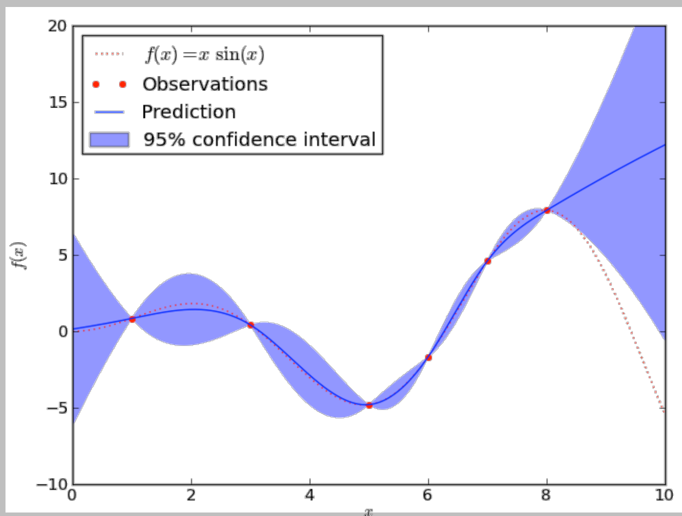
Solve the non-convex problem in θ

$$\min_{\theta} J(\theta)$$

Convex for fixed θ

Global Bayesian Optimization

Fit function $\hat{J}(\theta)$ & confidence estimate (Gaussian process)



Choose sample point most likely to result in improvement

$$\theta_i = \operatorname{argmin}_\theta \ \mathbb{E}[\hat{J}(\theta) \geq J_{\max}]$$

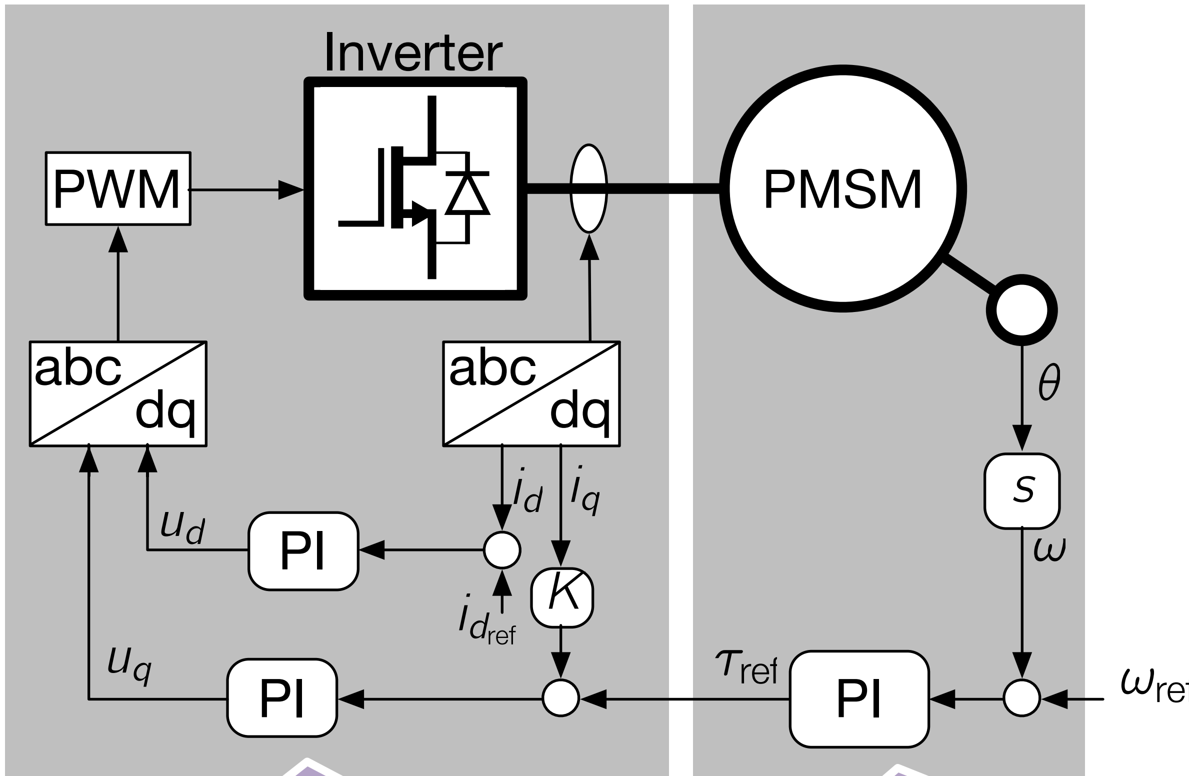
Exploration / exploitation tradeoff

Solve convex verification problem

$$\begin{aligned} J(\theta) = \min \quad & \int V(x)\rho(x)dx \\ \text{s.t.} \quad & V(x) \geq 0 \\ & V(f(x,u)) \leq V(x) - l(x,u) \\ & \text{for all } (x,u,\lambda) \in K(\theta) \end{aligned}$$

Permanent Magnet Synchronous Motor

Standard Control Approach



Current control loop

Track torque commands

100µs

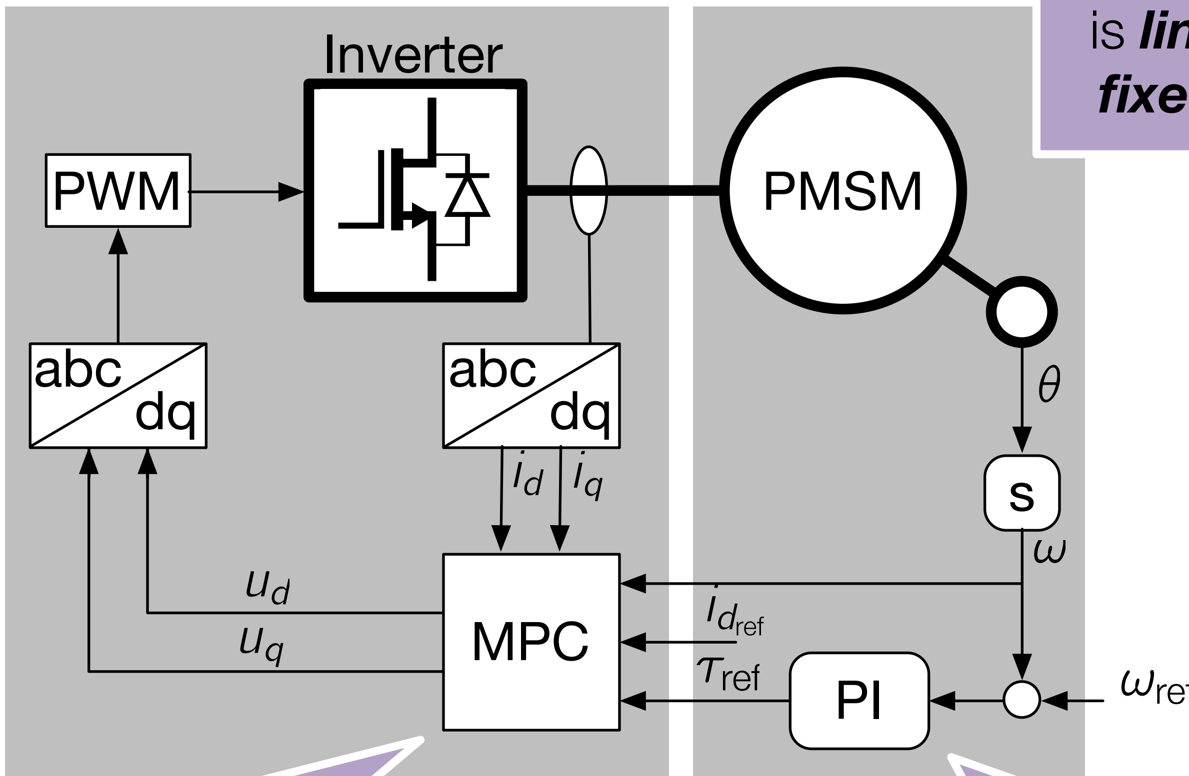
Speed control loop

Outer loop tracks speed commands

1ms

Recent Fast-MPC Approach

Key: Inner loop is ***linear*** for a ***fixed speed***



Linear MPC

Linearize at nominal speed

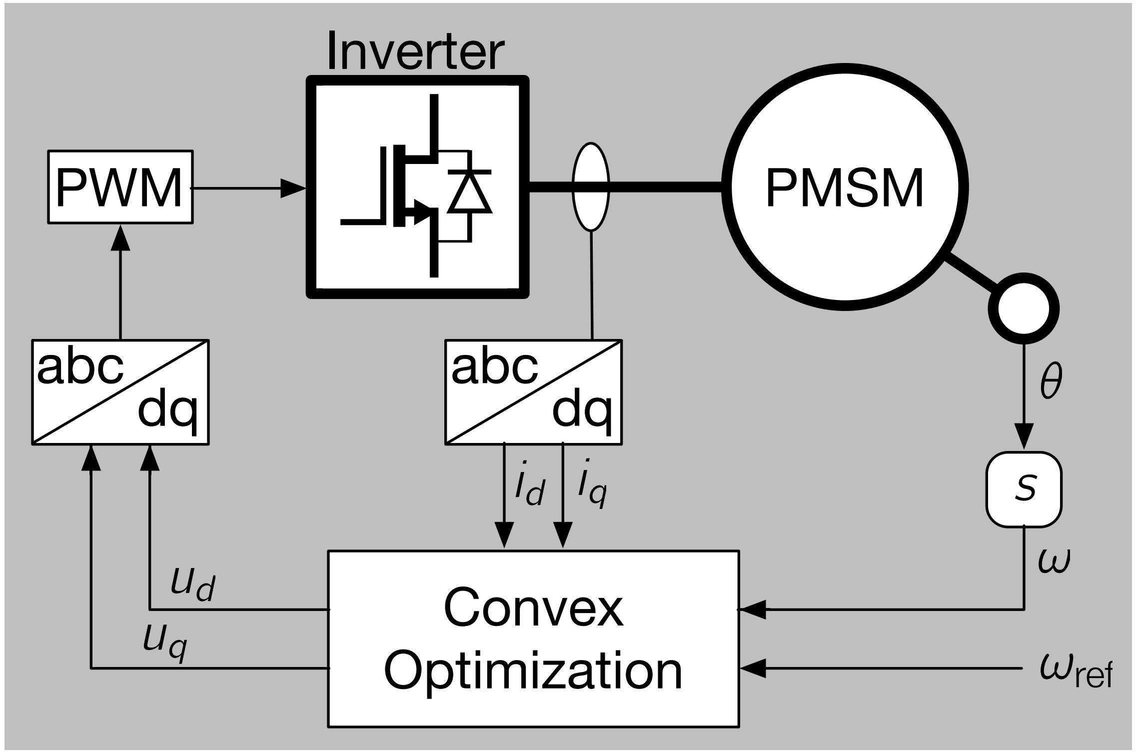
Speed control loop

Low inertia motors: Slower electrical & faster mechanical dynamics

- Faster response and smaller overshoots
- 2.3% and 4.2% improvement in speed and torque tracking

[Cimini, Bernardini, Bemporad, Levijoki, ICIT, 2015]

Convex Controller for Bilinear System



Idea	Replace both loops with a single fast loop
Benefit	Much faster speed response (good for smaller motors)
Challenge	Dynamics are bilinear, resistance changes significantly

Convex Controller for Bilinear System

Varying stator resistance 5x

Bi-linear dynamics

$$\frac{di_d}{dt} = -\frac{r_s}{l_s} i_d + \omega_r i_q + \frac{1}{l_s} u_d$$

$$\frac{di_q}{dt} = -\frac{r_s}{l_s} i_q - \left(i_d + \frac{\phi_0}{l_s} \right) \omega_r + \frac{1}{l_s} u_q$$

$$\frac{d\omega_r}{dt} = K_t i_q - \frac{T_b}{J\omega_b^2} \gamma_L$$

Voltage vector

Speed, current

$$\begin{aligned} \min \quad & \tfrac{1}{2} z^T H z + \Delta x^T F z + m \cdot \delta^2 \\ \text{s.t.} \quad & \|u_s + z\|^2 \leq 1 \\ & \|i_s^+\| \leq 1 + \delta \end{aligned}$$

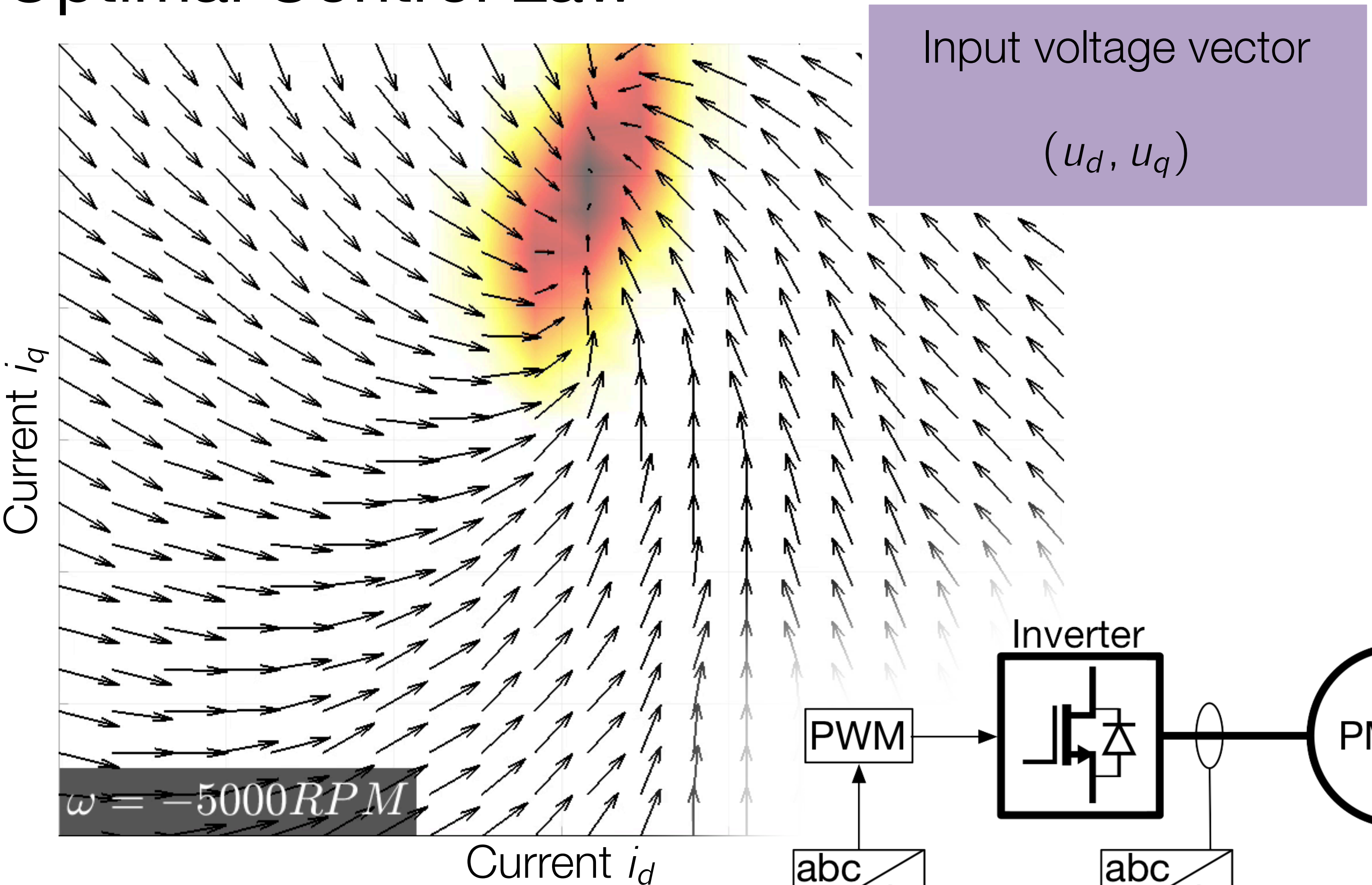
Current magnitude – limits heat (soft in transients)

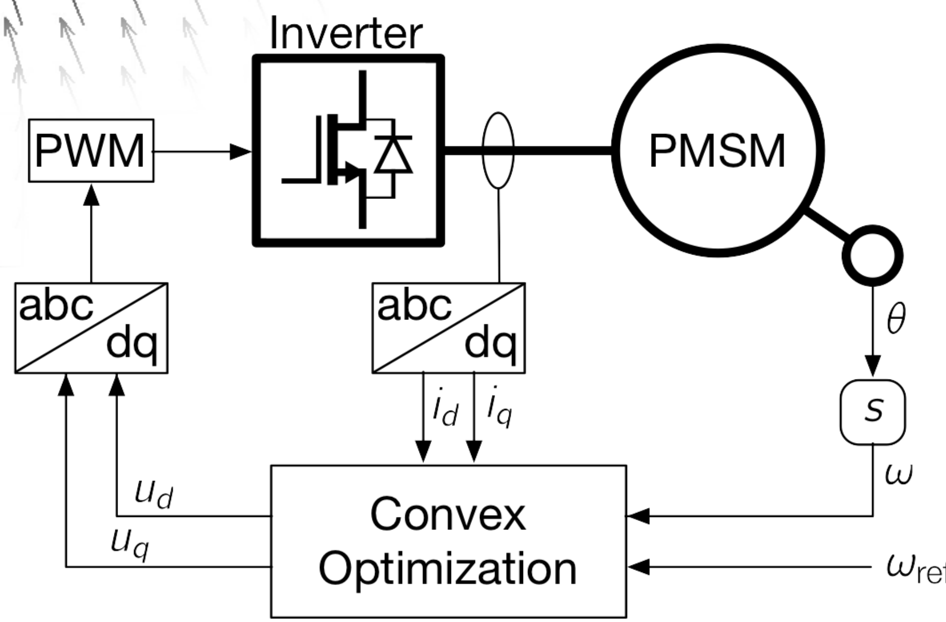
Magnitude of input limited

Optimization: Choose value function for stability and optimal performance

Constraints : Robust stability

Optimal Control Law

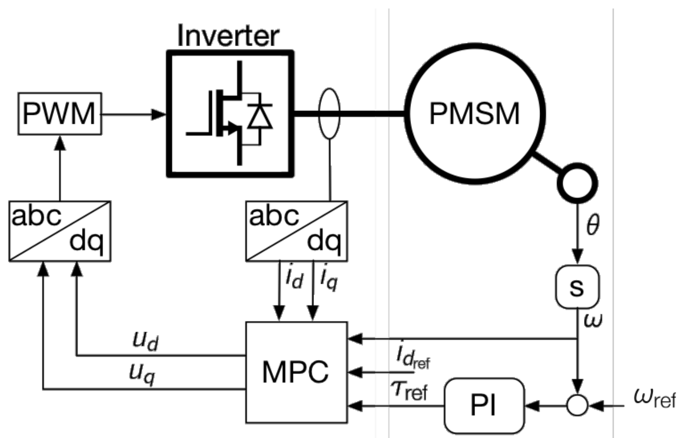


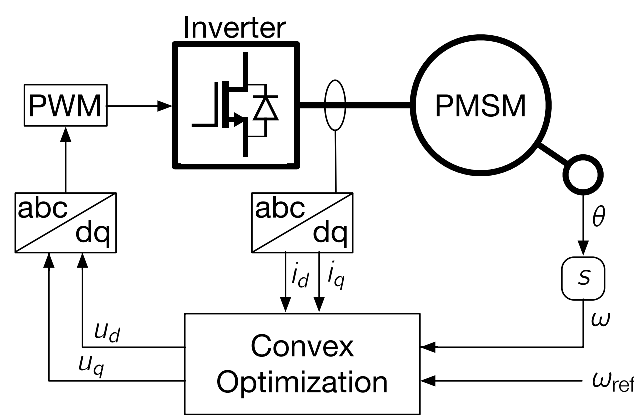


Time to calculate: 19.6 hours

Control Performance

	Linear MPC	**'Optimal' tuning for convex optimizer**
Time from 0-3000 RPM	~60ms	~30ms
Computation time	QP Variables 14 Constraints 122	SOCP / QP Variables 4 / 4 Constraints 2 / 17
Sampling time	Speed : 1 ms Current : 300 μs	100 μs





Note: Problem size estimated from description in paper

Conclusion

Framework for performance and stability analysis of optimization-based controllers

Captures wide array of heuristics used in fast MPC

Enables synthesis of fixed-structure optimization-based controllers

Limit: Sum-of-squares is computationally very intense

Limit: Non-convex global optimization