NAVAL POSTGRADUATE SCHOOL

Some Examples of Sparse Grid Characteristics Method for Optimal Control and HJB Equations

Wei Kang, Lucas Wilcox and Oleg Yakimenko

U.S. Naval Postgraduate School

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Outline

- Introduction - the curse of dimensionality
- The sparse grid characteristics (SGC) method
- Optimal control and the HJB equation - rigid body
- Optimal control and the HJB equation - UAVs
- Conclusions and remarks

The Curse of Dimensionality

The curse of dimensionality: the dimension of the discretized problem increases exponentially with the number of variables in the HJB equation.

Some approaches

- Open-loop optimal control: *pros* - numerical methods exist to solve complicated problems; *cons* - computational load and convergence issues may limit its real-time applications.
- Dynamic Programming based on the HJB equation: *pros* - real-time feedback law is simple; *cons* - the curse of dimensionality and ......

Sparse Grid Characteristics Method: Mitigating the curse of dimensionality for systems of moderate dimensions ($N = 6$ in this example).

A hierarchy of grids

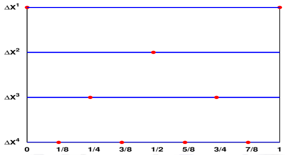
$$X^1 = \{0, 1\}, X^2 = \{0, \tfrac{1}{2}, 1\}, X^3 = \{0, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, 1\}, \cdots$$
$$X^i = \{\tfrac{k-1}{2^{i-1}};\ \ k = 1, 2, \cdots m_i\},\ \ m_i = 2^{i-1} + 1$$

The sequence has a telescopic structure

$$X^1 \subset X^2 \subset X^3 \subset X^4 \subset \cdots$$

Define

$$\Delta X^1 = X^1,\ \ \Delta X^i = X^i \setminus X^{i-1}, i \geq 2$$
$$\Delta m_i = |\Delta X^i|$$



ΔX^i for $i = 1, 2, 3, 4$

A hierarchy of grids in $\Re^d$

Vector index

$$\begin{aligned} &\mathbf{i} = \left[\begin{array}{cccc} i_1 & i_2 & \cdots & i_d \end{array}\right], \qquad |\mathbf{i}| = i_1 + i_2 + \cdots + i_d \\ &\mathbf{j} = \left[\begin{array}{cccc} j_1 & j_2 & \cdots & j_d \end{array}\right] \\ &\Delta X^{\mathbf{i}} = \Delta X^{i_1} \times \cdots \times \Delta X^{i_d}, \quad \Delta m^{\mathbf{i}} = \left[\begin{array}{ccc} \Delta m^{i_1} & \cdots & \Delta m^{i_d} \end{array}\right] \\ &x^{\mathbf{i}}_{\mathbf{j}} = (x^{i_1}_{j_1}, \cdots, x^{i_d}_{j_d}) \in \Delta X^{\mathbf{i}} \end{aligned}$$

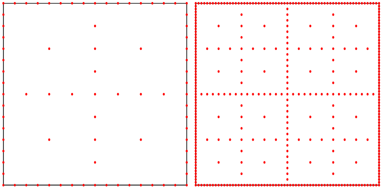
The dense grid is

$$X^q \times \cdots \times X^q = \bigcup_{1 \le \mathbf{i} \le q} \Delta X^{\mathbf{i}}$$

Following Smolyak's approximation algorithm, the sparse grid is

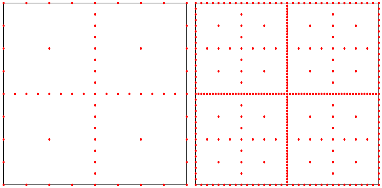
$$G^q_{\mathrm{sparse}} = \bigcup_{|\mathbf{i}| \le q} \Delta X^{\mathbf{i}}$$

Examples of sparse grids



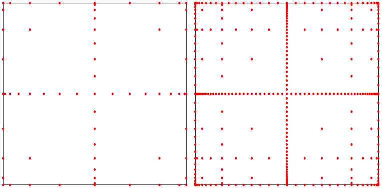
G^{q}_{sparse} in $[0,1]^2$, $q = 6$ and $q = 8$

A modified sparse grid



$G^{q}_{\text{sparse}}(\mathit{Modified})$ $(q = 6,\ q = 8)$

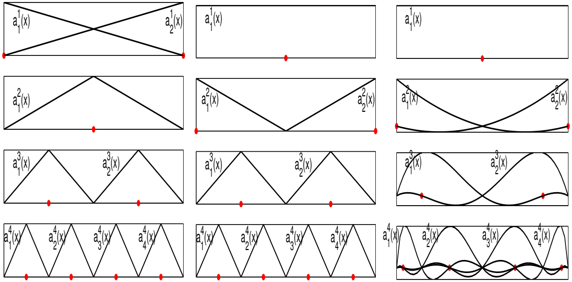
CGL (Chebyshev-Gauss-Lobatto) type



$G^{q}_{\text{sparse}}(\mathit{CGL})$ $(q = 6,\ q = 8)$

Interpolation

Basis functions



Hierarchical surpluses

$$I^q(f) = I^{q-1}(f) + \Delta I^q(f), \quad q \geq d$$

$$\Delta I^q(f) = \sum_{|\mathbf{i}|=q} \sum_{1 \leq \mathbf{j} \leq \Delta m^{\mathbf{i}}} w_{\mathbf{j}}^{\mathbf{i}} a_{\mathbf{j}}^{\mathbf{i}}$$

$$w_{\mathbf{j}}^{\mathbf{i}} = f(x_{\mathbf{j}}^{\mathbf{i}}) - I^{q-1}(f)(x_{\mathbf{j}}^{\mathbf{i}})$$

Sparse vs. dense

Grid	Grid size	Interpolation error
Dense	N^d	$O(\frac{1}{N^2})$
Sparse	$O(N(\log N)^{d-1})$	$O(\frac{(\log N)^{d-1}}{N^2})$

Comparison based upon linear interpolation on standard sparse grids for functions with bounded second order derivatives.

Remark: Essentially, we pay the price of $(\log N)^{d-1}$ in accuracy in exchange for the reduction of gird size: $O(N(\log N)^{d-1})$.

Causality Free Computation

Causality free algorithm

A computational algorithm is spatially causality free if the value of $V(x, t)$ at x is computed independently from the value at nearby grid points

- Hopf-Lax formula for HJ equations.
- Lax-Oleinik formula for conservation laws.
- Characteristic methods for PDEs.
- Pontryagin's maximum principle (TPBVP).

The Bolza Problem

$$\begin{aligned} \min_{u(\cdot)} \quad & \int_0^{t_f} L(x(t), u(t))dt + h(x(t_f)), \\ \dot{x} \quad = \quad & f(x, u), \qquad x \in \Re^d, u \in \Re^m \\ x(0) \quad = \quad & x_0 \end{aligned}$$

Define the Hamiltonian

$$H(x, \lambda, u) = L(x, u) + \lambda^T f(x, u), \quad \lambda \in \Re^d$$

SGC Methods

The PMP or **the characteristic equations** are causality free.

Given any x_0, consider the TPBVP

$$\begin{aligned} &\dot{x} = \frac{\partial H}{\partial \lambda}(x, \lambda, u^*) \\ &\dot{\lambda} = -\frac{\partial H}{\partial x}(x, \lambda, u^*) \\ &\dot{z} = L(x, u^*), \qquad u^* = \arg\min_u H(x, \lambda, u) \\ &x(0) = x_0,\ \lambda(t_f) = h_x^T(x(t_f)),\ z(0) = 0 \end{aligned}$$

Then

$$\boxed{\begin{aligned} &V(0, x_0) = z(t_f) + h(x(t_f)) \\ &V_x(0, x_0) = \lambda(0), \\ &\text{or } u_{\text{optimal}}(0, x_0) = u^*(0, x_0, \lambda(0)) \end{aligned}}$$

We adopted a 5th order Lobatto IIIa method to solve the TPBVP (Kierzenka - Shampine 2008).

SGC Method

Some remarks

1. The SGC method is perfectly parallel in computation.
2. The optimal feedback law

$$u_{\text{optimal}}(t,x) = u^*\left(x, \lambda(t,x)\right)$$

The value of costate, λ, is available on all grid points.

3. The value function solves the HJB equation

$$\frac{\partial V(t,x)}{\partial t} + H^*\left(x, \frac{\partial V(t,x)}{\partial x}\right) = 0$$
$$V(t_f, x) = h(x)$$

SGC Methods

Error analysis

$$\bar{V}(t,x) = V(t,x) + e_{interp} + e_{BVP}$$

where $\bar{V}(t,x)$ is the approximate value function. Sparse grid with piecewise linear interpolation,

$$\frac{||e_{BVP}||_{L^\infty}}{\epsilon} = O\left((\log N)^{d-1}\right).$$

CGL sparse grid with polynomial interpolation,

$$\frac{||e_{BVP}||_{L^\infty}}{\epsilon} = O\left((\log N)^{2d-1}\right).$$

SGC Methods

Error analysis If the TPBVP solution, $\tilde{V}(t,x)$, has high accuracy, error can be approximated by

$$|e_{interp} + e_{BVP}| \approx |\bar{V}(t,x) - \tilde{V}(t,x)|$$

on a random set of points.

The computation is perfectly parallel.

Optimal Attitude Control

The system model of attitude control using momentum wheels

$$\dot{v} = E(v)\omega$$
$$J\dot{\omega} = S(\omega)R(v)H + Bu$$

where

$v = \begin{bmatrix} \phi & \theta & \psi \end{bmatrix}^T$	Euler angles
$\omega = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \end{bmatrix}^T$	angular velocity
$H \in \Re^3$	constant angular momentum
$B \in \Re^{3\times m}$	$m = \#$ of control momentum wheels
$J \in \Re^{3\times 3}$	inertia matrix

Optimal Attitude Control

$$E(v) = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi/\cos\theta & \cos\phi/\cos\theta \end{bmatrix}$$

$$S(\omega) = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

$$R(v) = \begin{bmatrix} \cos\theta\cos\psi & \cos\theta\sin\psi & -\sin\theta \\ \sin\phi\sin\theta\cos\psi - \cos\phi\sin\psi & \sin\phi\sin\theta\sin\psi + \cos\phi\cos\psi & \cos\theta\sin\phi \\ \cos\phi\sin\theta\cos\psi + \sin\phi\sin\psi & \cos\phi\sin\theta\sin\psi - \sin\phi\cos\psi & \cos\theta\cos\phi \end{bmatrix}$$

Optimal Attitude Control

The cost functional:

$$\int_0^{t_f} L(v,\omega,u)dt + h(v(t_f),\omega(t_f))$$
$$L(v,\omega,u) = \frac{W_1}{2}||v||^2 + \frac{W_2}{2}||\omega||^2 + \frac{W_3}{2}||u||^2$$
$$h(v(t_f),\omega(t_f)) = \frac{W_4}{2}||v(t_f)||^2 + \frac{W_5}{2}||\omega(t_f)||^2$$

W_i, $i = 1,2,3,4,5,$ are constant weights.

The parameters

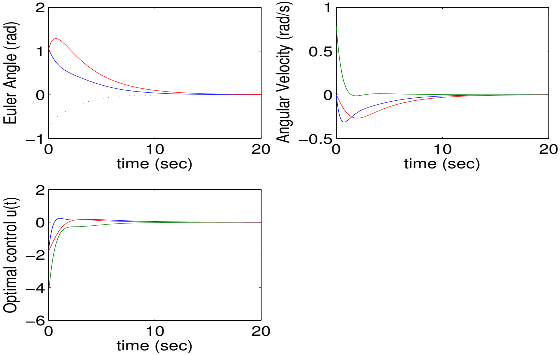
$$J = \text{diag}(2,3,4), \qquad H = [1\ 1\ 1]^T$$
$$B = \begin{bmatrix} 1 & \frac{1}{20} & \frac{1}{10} \\ \frac{1}{15} & 1 & \frac{1}{10} \\ \frac{1}{10} & \frac{1}{15} & 1 \end{bmatrix} \qquad \begin{matrix} W_1 = 1, & W_2 = 10 \\ W_3 = \frac{1}{2}, & W_4 = 1 \\ W_5 = 1 & \end{matrix}$$
$$-\frac{\pi}{3} \le \phi,\theta,\psi \le \frac{\pi}{3}, \qquad -\frac{\pi}{4} \le \omega_i \le \frac{\pi}{4}$$

Numerical results

q	$\lvert G^q_{\text{sparse}} \rvert$ CGL	Dense grid size	# of Processors	MAE $N = 1280$ samples
$q = 13$	$44,698$	$> 10^{12}$	512	7.3 e-4

The error at 1280 points are computed in parallel using 128 CPU cores. The error tolerance of $\tilde{V}(t,x)$ is 10^{-9}.

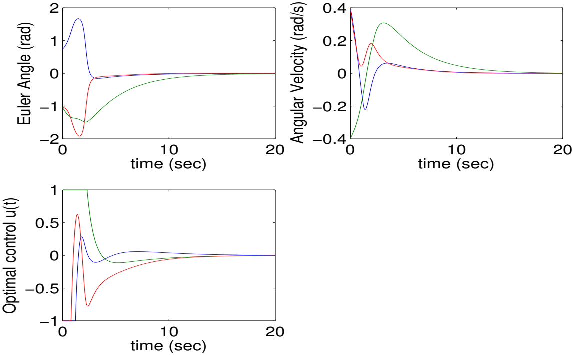
An example of optimal trajectory



Control with saturation $u \leq 1$

q	$\lvert G^q_{\text{sparse}} \rvert$ CGL	Dense grid size	# of Processors	MAE $N = 1280$ samples
$q = 13$	$44,698$	$> 10^{12}$	512	2.2 e-2

An example of optimal trajectory



Inner-loop error tolerance = 1e-4; final loop tolerance = 1e-9, MAE is computed in parallel using 128 CPU cores.

Optimal Attitude Control

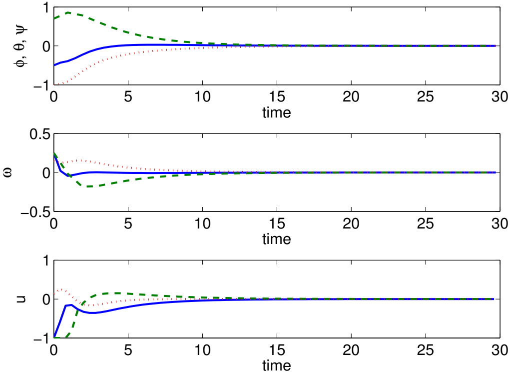
Closed-loop control with saturation

- A zero-order hold MPC controller is adopted.
- The sampling rate is 10 Hz.
- $u_{\text{optimal}}(x)$ is computed using interpolation of costates on the sparse grid.

An attitude trajectory

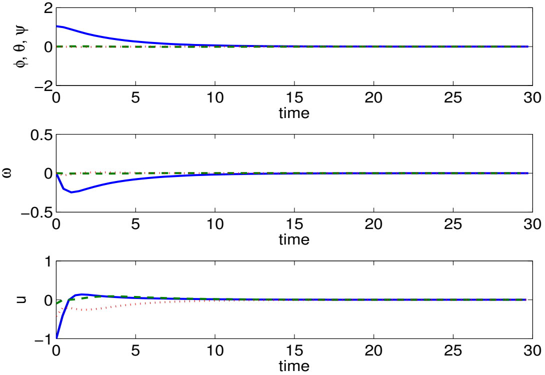
$$\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.7 \\ -1.0 \end{bmatrix}$$

$$\begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix} = \begin{bmatrix} 0.25 \\ 0.25 \\ 0.15 \end{bmatrix}$$



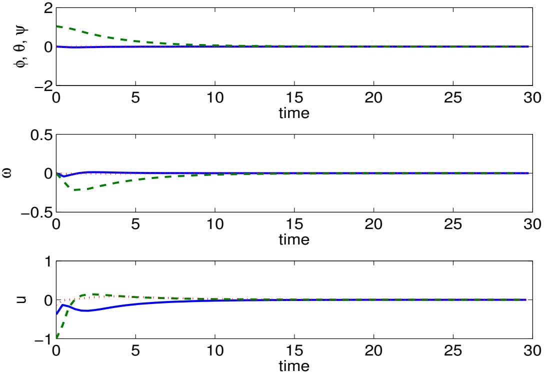
Closed-loop control with saturation

A $\frac{\pi}{3}$-attitude-slew maneuver



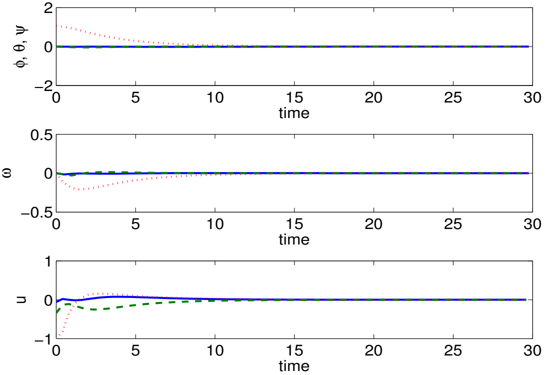
Closed-loop control with saturation

A $\frac{\pi}{3}$-attitude-slew maneuver



Closed-loop control with saturation

A $\frac{\pi}{3}$-attitude-slew maneuver



An uncontrollable system using two momentum wheels

$$\min_{u} \int_0^{t_f} \frac{W_1}{2}||v - v_e(v,\omega)||^2 + \frac{W_2}{2}||\omega||^2 + \frac{W_3}{2}||u||^2 dt$$

subject to

$$\dot{v} = E(v)\omega$$

$$J\dot{\omega} = S(\omega)R(v)H + Bu$$

Parameters

$$B = \begin{bmatrix} 1 & \frac{1}{10} \\ 0 & 1 \\ \frac{1}{12} & 0 \end{bmatrix}, \qquad H = [12\ 12\ 6]^T$$

$$W_1 = 1, \quad W_2 = 2$$

$$W_3 = \frac{1}{2},$$

$$J = \mathrm{diag}(2,3,4),$$

$$-\frac{\pi}{6} \le \phi, \theta, \psi \le \frac{\pi}{6}, \qquad -\frac{\pi}{8} \le \omega_i \le \frac{\pi}{8}$$

Equilibrium: $v = v_e(v, \omega), \;\; \omega = 0$

$$\min_{v_e} ||R(v_e) - I||_{\max}$$

subject to

$$C^T R(v_e) H = C^T (R(v)H - J\omega)$$

where $C \in \Re^3$ is a constant vector satisfying

$$C^T B = 0$$

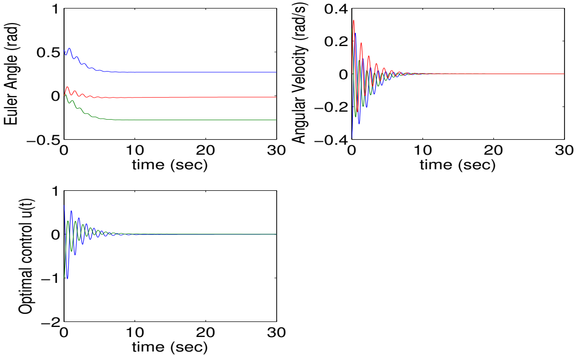
The equilibrium, v_e, is computed numerically. The process is equivalent to maximizing trace$(R(v_e))$.

Optimal Attitude Control

Numerical results ($m = 2$)

q	$\lvert G^q_{\text{sparse}} \rvert$	Dense grid size	# of Processors	MAE $N = 1280$ samples
$q = 13$	$44,698$	$> 10^{12}$	512	8.5 e-3

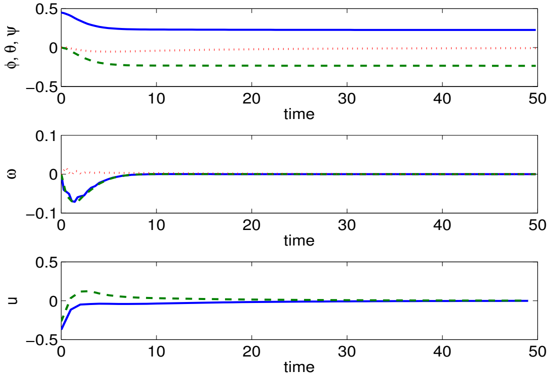
An example trajectory



Inner-loop error tolerance = 1e-4; final loop tolerance = 1e-9

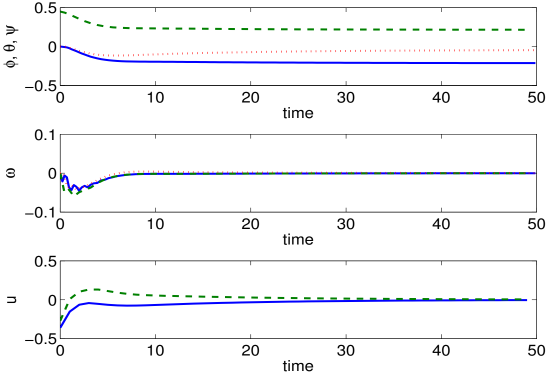
Closed-loop control $(m = 2)$

A $\frac{\pi}{6}$-attitude-slew maneuver



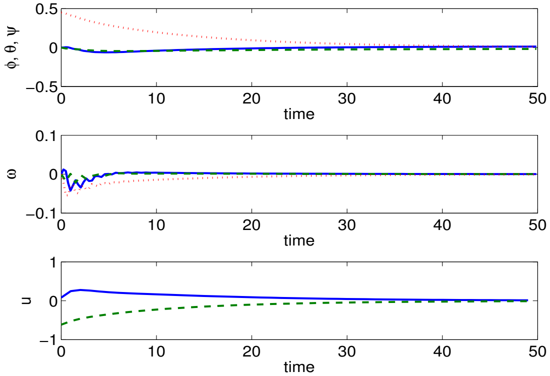
Closed-loop control $(m = 2)$

A $\frac{\pi}{6}$-attitude-slew maneuver



Closed-loop control ($m = 2$)

A $\frac{\pi}{6}$-attitude-slew maneuver



The system model

Prof. Oleg Yakimenko

$$\dot{x}_1 = V\cos\gamma\cos\Psi, \quad \dot{x}_2 = V\cos\gamma\sin\Psi, \quad \dot{x}_3 = -V\sin\gamma$$

$$\dot{V} = \frac{1}{m}T - \frac{C_{D0}\rho S}{2m}V^2 - gA_1 n_z - \frac{2mg^2A_2}{\rho S}\frac{{n_z}^2}{V^2} - g\sin\gamma$$

$$\dot{\gamma} = \frac{g}{V}(n_z\cos\phi - \cos\gamma)$$

$$\dot{\Psi} = \frac{g}{V\cos\gamma}n_z\sin\phi$$

$$\dot{\phi} = u_\phi$$

Parameters adopted from foam Unicorn wing

(x_1, x_2, x_3) - location in NED frame
V - speed
γ - path angle
ψ - heading
ϕ - bank angle

n_z - vertical lift
T - throttle
u_ϕ - bank angle rate

ρ - air density
$S = 0.321\ m^2$
$mg = 9.34\ N$
$CD_0 = 0.0213$
$A_1 = -0.056$
$A_2 = 0.22$

The cost functional:

$$
\begin{aligned}
\mathcal{J} &= \int_0^{t_f} L(V,\gamma,\Psi,\phi,u)dt \\
L(V,\gamma,\Psi,\phi,u) &= \frac{W_1}{2}||V-V^d||^2 + \frac{W_2}{2}||\gamma-\gamma^d||^2 + \frac{W_3}{2}||\Psi-\Psi^d|| \\
&+ \frac{W_4}{2}||\phi-\phi^d||^2 \\
&+ \frac{W_5}{2}||T-T^d||^2 + \frac{W_6}{2}||n_z-n_z^d||^2 + \frac{W_7}{2}||u_\phi-u_\phi^d||^2
\end{aligned}
$$

W_i, $i=1,2,3,4,5,6$, are constant weights, $(V^d,\gamma^d,\Psi^d,\phi^d)$ is the desired target state, (T^d,n_z^d,u_ϕ^d) makes final state an equilibrium.

The parameters

$$W_1=\frac{1}{4}, W_2=1, W_3=1, W_4=1, W_5=0.2, W_6=0.2, W_7=0.2$$

The Hamiltonian

$$\begin{aligned} H(V,\gamma,\Psi,\phi,u,\lambda) &= H_1(V,\gamma,\Psi,\phi,\lambda) + A_T T^2 + B_T T \\ &+ A_{n_z} {n_z}^2 + B_{n_z} n_z + A_{u_\phi} {u_\phi}^2 + B_{u_\phi} u_\phi \end{aligned}$$

$$A_T = \frac{W_5}{2}, \quad B_T = \lambda_1 \alpha_1 - W_5 T^d$$

$$A_{n_z} = \frac{W_6}{2} - \frac{\lambda_1 \alpha_4}{V^2}, \quad B_{n_z} = \frac{\lambda_2 g}{V} \cos\phi + \frac{\lambda_3 g}{V \cos\gamma} \sin\phi - \lambda_1 \alpha_3 - n_z^d W_6$$

$$A_{u_\phi} = \frac{W_7}{2}, \quad B_{u_\phi} = \lambda_4 - W_7 u_\phi^d$$

$$H_1(V,\gamma,\Psi,\phi,\lambda) = \text{all other terms of states and co-states}$$

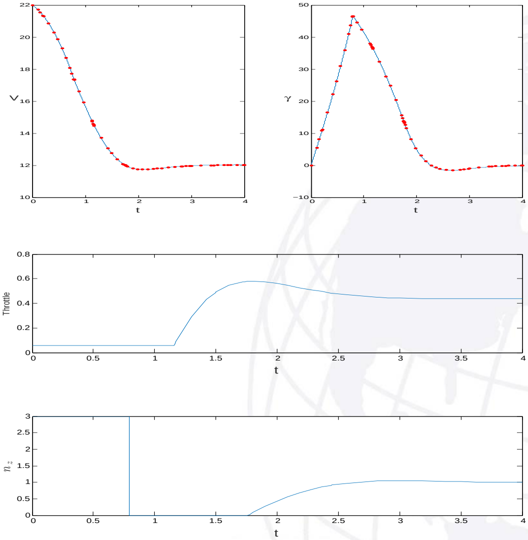
$A > 0$

$$u^* = \begin{cases} -\frac{B}{2A}, & u_{\min} < -\frac{B}{2A} < u_{\max} \\ u_{\min}, & -\frac{B}{2A} \leq u_{\min} \\ u_{\max}, & -\frac{B}{2A} \geq u_{\max} \end{cases}$$

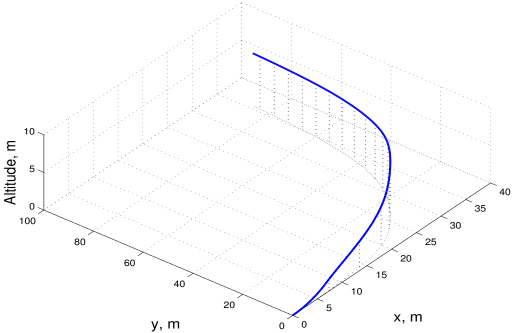
$A < 0$

$$u^* = \begin{cases} u_{\min}, & Au_{\min}^2 + Bu_{\min} \\ & \leq Au_{\max}^2 + Bu_{\max} \\ u_{\max}, & \text{otherwise} \end{cases}$$

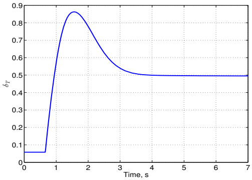
Optimal Trajectories

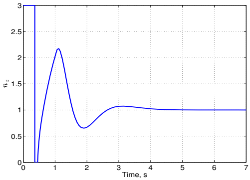


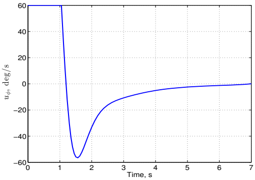
Nominal Trajectory



Optimal control





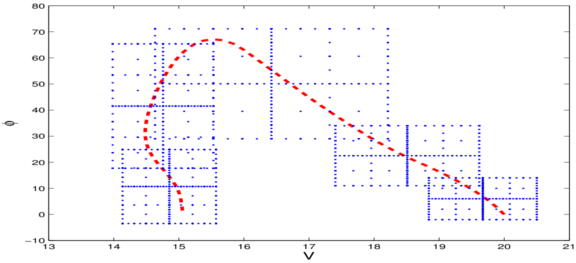


Numerical results - patchy sparse grids

q	$\lvert G^q_{\text{sparse}} \rvert$ Linear Interpolation	Dense grid size	# of windows
$q = 9$	$1,105$	$> 10^6$	5

Window 1	Window 2	Window 3	Window 4	Window 5
5.8e-5	1.2e-4	5.9e-4	2.0e-4	6.1 e-5

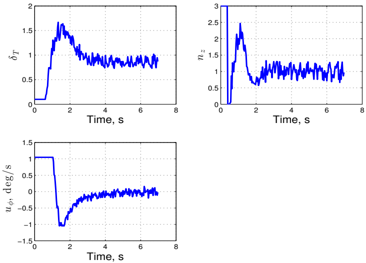
MAE is computed at 1100 random points in each window.



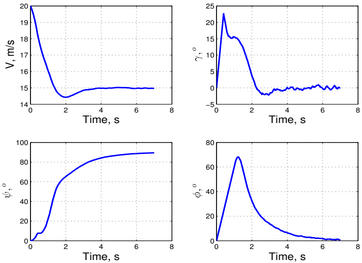
Closed-loop control with saturation

- Controller: zero-order hold MPC at 30 Hz..
- Sensor error: uniform distribution ($e_V : \pm 0.2 m/s; e_\gamma, e_\psi, e_\phi : \pm 2^\circ$)
- Feedback: interpolation of costates in $u^*(x, \lambda)$.

Control input



Trajectory



Conclusions

Some remarks

- The sparse grid characteristics method is causality free.
- The method has perfect parallelism.
- The algorithm has no spacial error propagation.
- Patchy sparse grids further reduces the grid size
- Some causality free methods are based on TPBVP, an effective solver is the key for success.
- For discontinuous control, sparse grid interpolation has low accuracy. Interpolation of costates is recommended.

THANK YOU