Moving Horizon Estimation and Minimum Energy Estimation

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x^+ &= f(t, x, u) \\
y &= h(t, x, u) \\
x(0) &= x^0
\end{aligned}
$$

where the state $x(t) \in \mathbb{R}^{n\times 1}$, the control $u(t) \in \mathbb{R}^{m\times 1}$, the measurement $y(t) \in \mathbb{R}^{p\times 1}$ and $x^+(t) = x(t+1)$.

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where the state $x(t) \in I\!R^{n\times 1}$, the control $u(t) \in I\!R^{m\times 1}$, the measurement $y(t) \in I\!R^{p\times 1}$ and $x^+(t) = x(t+1)$.

The filtering problem is to estimate $x(t)$ from $u(s), 0 \le s \le t-1$, from $y(s), 1 \le s \le t-1$ and from some inexact knowledge about x^0 . We denote this estimate by $\hat{x}(t|t-1)$. We also consider estimating $x(t)$ using the additional measurement $y(t)$, we denote this estimate $\hat{x}(t|t)$.

Three Noises

The standard approach is add three noises to the model, a driving noise $v(t) \in \mathbb{R}^{k\times 1}$, an observation noise $w(t) \in \mathbb{R}^{p\times 1}$ and an initial condition noise $\tilde{x}^0 \in \mathbb{R}^{n\times 1}$ to obtain

$$\begin{aligned} x^+ &= f(t,x,u) + g(t,x,u)v \\ y &= h(t,x,u) + w \\ x(0) &= \hat{x}^0 + \tilde{x}^0 \end{aligned}$$

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About the only time it can be solved is when the system is linear for then the conditional density is Gaussian. Then its mean and covariance can be computed by a Kalman filter.

Deterministic Filtering

The minimum energy version of this approach is to assume the noises are deteministic but unknown. One finds the noise triple that minimizes a so-called "energy" like

$$\frac{1}{2}\left(\|\tilde{x}^0\|^2 + \sum_{s=0}^{t-1}\|v(s)\|^2 + \sum_{s=1}^{t}\|w(s)\|^2\right)$$

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The MME is the endpoint of the minimizing state trajectory

$$\hat{x}(t|t) = z(t)$$

Minimum Energy Estimation

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If the system is linear then MME filtering is identical to Kalman filtering provided that the weights of the norms in the deterministic version are the inverses of the covariances of the noises in the stochastic version.

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One chooses a past window of length T . At time t minimize

$$\min_{z^{t-T},v,w} \left\{ \|z^{t-T} - \hat{x}(t-T|t-T)\|^2_{P(t-T)} + \sum_{s=t-T}^{t-1} \|v(s)\|^2_{Q(s)} + \sum_{s=t-T+1}^{t} \|w(s)\|^2_{R(s)} \right\}$$

subject to

$$\begin{aligned} z^+ &= f(s,z,u) + g(s,z,u)v \\ y &= h(s,z,u) + w \end{aligned}$$

where $u(s)$ is the actual input and $y(s)$ is the actual output.

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We believe that the power series techniques described below will allow us to take T as short as 1 time step. Instead of taking a fixed past window of length T we use the whole past but with a forgetting factor $0 < \alpha \leq 1$.

MEE

Since we know $u(t)$ we can simplify notation by redefining

$$\begin{aligned} f(t,x) &= f(t,x,u(t)) \\ g(t,x) &= g(t,x,u(t)) \\ h(t,x) &= h(t,x,u(t)) \end{aligned}$$

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Then the noisy model is

$$\begin{aligned} z(t+1) &= f(t,z(t)) + g(t,z(t))v(t) \\ y &= h(t,z(t)) + w(t) \\ z(0) &= \hat{x}^0 + \tilde{z}^0 \end{aligned}$$

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The simplest definition of "energy' to time t is

$$\frac{\alpha^t}{2}\|z^0\|^2 + \frac{1}{2}\sum_{s=0}^{t-1}\alpha^{t-s}\|v(s)\|^2 + \frac{1}{2}\sum_{s=1}^{t}\alpha^{t-s}\|w(s)\|^2$$

with a discount factor $0 < \alpha \leq 1$.

MEE

We can extend MME by considering a more general form of "energy"

$$\alpha^t \pi^0(z^0) + \frac{1}{2}\sum_{s=0}^{t-1} \alpha^{t-s}\|v(s)\|^2_{Q(s)} + \frac{1}{2}\sum_{s=1}^{\tau} \alpha^{t-s}\|w(s)\|^2_{R(s)}$$

where $Q(s) > 0,\ R(s) > 0$ **, an initial energy** $\pi^0(x) \geq 0$ **and**

$$\|v(s)\|^2_{Q(s)} = v'(s)Q(s)v(s), \qquad \|w(s)\|^2_{R(s)} = w'(s)R(s)w(s)$$

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Or even more generally

$$\alpha^t \pi^0(z^0) + \frac{1}{2} \sum_{s=0}^{t-1} \alpha^{t-s} l(s, z(s), v(s), w(s))$$

subject to constraints on the Lagrangian $l(s, z, v, w)$. For simplicity of exposition we shall stick with quadratic Lagrangians.

MME Function

The MEE function $\pi(x, t|\tau)$ is the minimum of the generalized energy subject to

$$\begin{aligned} z^{+} &= f(s, z(s)) + g(s, z(s))v(s) \\ y(s) &= h(s, z) + w(s) \\ z(0) &= \hat{x}^{0} + \tilde{x}^{0} \\ z(t) &= x \end{aligned}$$

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We are primarily interested in the cases where $\tau = t - 1$ or $\tau = t$.

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Finally $\pi(x, t|\tau)$ might have multiple local minima. As these change with t the global minimum might jump from one to another.

Except for the residual, this additional information is not available with MHE.

Prediction Step

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In MEE the prediction step starts with $\hat{x}(t|t)$ and $\pi(x,t|t)$ and computes $\hat{x}(t+1|t)$ and $\pi(x,t+1|t)$ by a solving functional equation for π of dynamic programming type,

$$\pi(x,t+1|t) \;\; = \;\; \min_{z,v} \left\{ \alpha\pi(z,t|t) + \frac{1}{2}\|v(t)\|^2_{Q(t)} \right\}$$

subject to the constraint

$$x \;\; = \;\; f(t,z) + g(t,z)v$$

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subject to the constraint

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The predicted estimate $\hat{x}(t+1|t)$ is given by

$$\hat{x}(t+1|t) \;=\; \text{argmin}_x \pi(x,t+1|t)$$

Assimilation Step

When $y(t)$ becomes known the assimilation step is given by the functional equation,

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The assimilated estimate $\hat{x}(t|t)$ is

$$\hat{x}(t|t) \;\; = \;\; \text{argmin}_x \pi(x,t|t)$$

Constrained Optimiztion

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To get unconstrained problems we add the constraint to the criterion with a Lagrange multiplier $\lambda(t,x) \in \mathbb{R}^{n\times 1}$ to get a family of unconstrained minimization problems indexed by t, x ,

$$\min_{z,v,\lambda} \left\{ \alpha\pi(z,t|t) + \frac{1}{2}\|v\|^2_{Q(t)} + \lambda'(t,x)\left(x - f(t,z) - g(t,z)v\right) \right\}$$

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To simplify the presentation we shall assume $g(t,x) = G(t)x$. The general case follows in a similar fashion.

Necessary Conditions

Setting the partial of the augmented criterion with respect to $\boldsymbol{\lambda}$ to zero yields the constraint

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Then the constraint becomes

$$x = f(t, z) + \bar{Q}(t)\lambda(t, x)$$

where

$$\bar{Q}(t) = G(t)Q^{-1}(t)G'(t)$$

Necessary Conditions

Setting the partial of of the augmented criterion with respect to z to zero yields

$$0 = \alpha \frac{\partial \pi}{\partial z}(z(t,x), t|t) - \lambda'(t,x) \frac{\partial f}{\partial z}(t, z(t,x))$$

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So at each t, x we must solve this equation and the constraint equation

$$x = f(t, z(t,x)) + \bar{Q}(t)\lambda(t,x)$$

in the two unknowns $z(t,x)$, $\lambda(t,x)$.

Necessary Conditions

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Solving these two nonlinear equations is a daunting task so we turn to a series approach.

Series Approach

Expand f and π in series in $\tilde{z} = z - \hat{x}(t|t)$,

$$\begin{aligned} f(t,z) &= f^{[0]}(t) + F(t)\tilde{z} + f^{[2]}(t,\tilde{z}) + \dots \\ \pi(z,t|t) &= \pi^{[0]}(t|t) + \frac{1}{2}\tilde{z}' P(t|t)\tilde{z} + \pi^{[3]}(\tilde{z},t|t) + \dots \end{aligned}$$

where $^{[d]}$ denotes a homogeneous polynomial term of degree d in $\tilde{z}$ with time dependent coefficients.

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It is not hard to see that

$$\begin{aligned} \pi^{[0]}(t|t) &= \min_x \pi(x,t|t) \\ \hat{x}(t|t) &= \operatorname{argmin}_x \pi(x,t|t) \\ & \min_x \pi(x,t+1|t) = \alpha \pi^{[0]}(t|t) \\ \hat{x}(t+1|t) &= \operatorname{argmin}_x \pi(x,t+1|t) = f^{[0]}(t) \end{aligned}$$

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Expand $\tilde{z}(t,x)$ and $\lambda(t,x)$ in series in $\tilde{x} = x - \hat{x}(t+1|t)$,

$$\begin{aligned}
\tilde{z}(t,x) &= Z(t)\tilde{x} + z^{[2]}(t,\tilde{x}) + z^{[3]}(t,\tilde{x}) + \dots \\
\lambda(t,x) &= \Lambda(t)\tilde{x} + \lambda^{[2]}(t,\tilde{x}) + \lambda^{[3]}(t,\tilde{x}) + \dots
\end{aligned}$$

The two nonlinear equations become

$$0 = \alpha \frac{\partial \pi}{\partial z}(z(t,x), t|t) - \lambda'(t,x)\frac{\partial f}{\partial z}(t, z(t,x))$$

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$$\tilde{x} = f(t, z(t,x)) - f^{[0]}(t) + \bar{Q}(t)\lambda(t,x)$$

We collect from these equations the terms linear in $\tilde{x}$ to obtain

$$\begin{aligned} \tilde{x} &= F(t)Z(t)\tilde{x} + \bar{Q}(t)\Lambda(t)\tilde{x} \\ 0 &= \alpha P(t|t)Z(t)\tilde{x} - F'(t)\Lambda(t)\tilde{x} \end{aligned}$$

First Degree Terms

These equations must hold for any $\tilde{x}$ so $Z(t), \Lambda(t)$ must satisfy

$$\begin{bmatrix} I \\ 0 \end{bmatrix} = \mathcal{H}(t) \begin{bmatrix} Z(t) \\ \Lambda(t) \end{bmatrix}$$

where

$$\mathcal{H}(t) = \begin{bmatrix} F(t) & \bar{Q}(t) \\ \alpha P(t|t) & -F'(t) \end{bmatrix}$$

The solvability of these equations depends on the invertibility of $\mathcal{H}(t)$ which we will discuss in a moment.

Second Degree Terms

The terms of second degree in $\tilde{x}$ are

$$\mathcal{H}(t) \begin{bmatrix} z^{[2]}(t,\tilde{x}) \\ \lambda^{[2]}(t,\tilde{x}) \end{bmatrix} = -\begin{bmatrix} k^{[2]}(t,\tilde{x})) \\ l^{[2]}(t,\tilde{x}) \end{bmatrix}$$

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where

$$\begin{aligned} k^{[2]}(t,\tilde{x}) &= f^{[2]}(t,Z(t)\tilde{x}) \\ l^{[2]}(t,\tilde{x}) &= \alpha\left(\frac{\partial \pi^{[3]}}{\partial z}(t,Z(t)\tilde{x})\right)' \\ &\quad -\left(\frac{\partial f^{[2]}}{\partial z}(t,Z(t)\tilde{x})\right)'\Lambda(t)\tilde{x} \end{aligned}$$

Again the solvability of these equations depends on the invertibility of $\mathcal{H}(t)$.

Third Degree Terms

The degree three terms are

$$\mathcal{H}(t)\begin{bmatrix} z^{[3]}(t,\tilde{x}) \\ \lambda^{[3]}(t,\tilde{x}) \end{bmatrix} = -\begin{bmatrix} k^{[3]}(t,\tilde{x}) \\ l^{[3]}(t,\tilde{x}) \end{bmatrix}$$

Third Degree Terms

where

$$k^{[3]}(t,\tilde{x}) = f^{[3]}(t, Z(t)\tilde{x}) + \left(f^{[2]}(t, Z(t)\tilde{x}) + z^{[2]}(t,\tilde{x})\right)^{[2]}$$

$$l^{[3]}(t,\tilde{x}) = \alpha\left(\left(\frac{\partial \pi^{[3]}}{\partial z}(t, Z(t)\tilde{x} + z^{[2]}(t,\tilde{x})\right)^{[3]}\right)'$$

$$+ \quad \alpha\left(\frac{\partial \pi^{[4]}}{\partial z}(t, Z(t)\tilde{x})\right)' - \left(\frac{\partial f^{[2]}}{\partial z}(t, Z(t)\tilde{x})\right)' \lambda^{[2]}(t,\tilde{x})$$

$$- \quad \left(\frac{\partial f^{[2]}}{\partial z}(t, z^{[2]}(t,\tilde{x})\right)' \Lambda(t)\tilde{x} - \left(\frac{\partial f^{[3]}}{\partial z}(t, Z(t)\tilde{x})\right)' \Lambda(t)\tilde{x}$$

and $(\cdot)^{[d]}$ indicates the degree d terms of the enclosed expression.

Third Degree Terms

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k^{[3]}(t,\tilde{x}) &= f^{[3]}(t,Z(t)\tilde{x}) + \left(f^{[2]}(t,Z(t)\tilde{x}) + z^{[2]}(t,\tilde{x})\right)^{[2]} \\
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+ \quad & \alpha\left(\frac{\partial \pi^{[4]}}{\partial z}(t,Z(t)\tilde{x})\right)' - \left(\frac{\partial f^{[2]}}{\partial z}(t,Z(t)\tilde{x})\right)' \lambda^{[2]}(t,\tilde{x}) \\
- \quad & \left(\frac{\partial f^{[2]}}{\partial z}(t,z^{[2]}(t,\tilde{x})\right)' \Lambda(t)\tilde{x} - \left(\frac{\partial f^{[3]}}{\partial z}(t,Z(t)\tilde{x})\right)' \Lambda(t)\tilde{x}
\end{aligned}$$

and $(\cdot)^{[d]}$ indicates the degree d terms of the enclosed expression.

The higher degree terms are found in a similar fashion.

Prediction Step

Suppose we have the expansions

$$\begin{aligned}\tilde{z}(t,x) &\approx Z(t)\tilde{x} + z^{[2]}(t,\tilde{x}) + z^{[3]}(t,\tilde{x}) \\ \lambda(t,x) &\approx \Lambda(t)\tilde{x} + \lambda^{[2]}(t,\tilde{x}) + \lambda^{[3]}(t,\tilde{x})\end{aligned}$$

then

$$\begin{aligned}v(t,x) &= Q^{-1}(t)G'(t)\lambda(t,x) \\ &\approx Q^{-1}(t)G'(t)\left(\Lambda(t)\tilde{x} + \lambda^{[2]}(t,\tilde{x}) + \lambda^{[3]}(t,\tilde{x})\right)\end{aligned}$$

and

$$\pi(x,t+1|t) \approx \alpha\pi(z(t,x),t|t) + \frac{1}{2}\|v(t,z(t,x),x)\|^2_{Q(t)}$$

$$\hat{x}(t+1|t) \approx \textsf{argmin}_x \pi(x,t+1|t)$$

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This completes the prediction step.

Relation to EKF

If $d = 1$ the prediction step from $\hat{x}(t|t)$ to $\hat{x}(t+1|t)$ is the same as that of the Extended Kalman Filter (EKF)

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In the EKF, the matrices $P(t|\tau)$ are covariances and the EKF prediction step is

$$P(t+1|t) = F(t)P(t|t)F'(t) + G(t)G'(t)$$

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In the MEE, the matrices $P(t|\tau)$ are Hessians and loosely speaking the inverses of the covariances. The MEE prediction step is

$$P(t+1|t) \;=\; \alpha Z'(t)P(t|t)Z(t) + \Lambda'(t)\bar{Q}\Lambda(t)$$

Assimilation Step

The assimilation step starts with $y(t)$, $\hat{x}(t|t-1)$ and a degree $d+1$ polynomial approximation to $\pi(x,t|t-1)$ expanded in $\tilde{x} = x - \hat{x}(t|t-1)$ and computes $\hat{x}(t|t)$ and a degree $d+1$ polynomial approximation to $\pi(x,t|t)$.

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So we can use Newton's method to minimize

$$\pi(x,t|t) \;\; = \;\; \pi(x,t|t-1) + \frac{1}{2}\|y(t) - h(t,x)\|^2_{R(t)}$$

with initial guess $\hat{x}(t|t-1)$. The argminimum is $\hat{x}(t|t)$.

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This completes the assimilation step.

Open Question

But suppose $\pi(x,t|t)$ has several local minima that are close to $\hat{x}(t|t-1)$, what can happen?

Theorem

Suppose that $P(0|0)$ is positive definite and the pair $F(t), G(t)$ is stabilizable at each $t \geq 0$ then $\mathcal{H}(t)$ is invertible for each $t \geq 0$.

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Stabilizable is in the continuous time sense. Why?

Example

Here is the Euler discretion of the Van der Pol Oscillator with time step 0.1 second,

$$
\begin{aligned}
x_1^+ &= x_1 + 0.1x_2 \\
x_2^+ &= -0.1x_1 + x_2 + 0.1(1 - x_1^2)x_2
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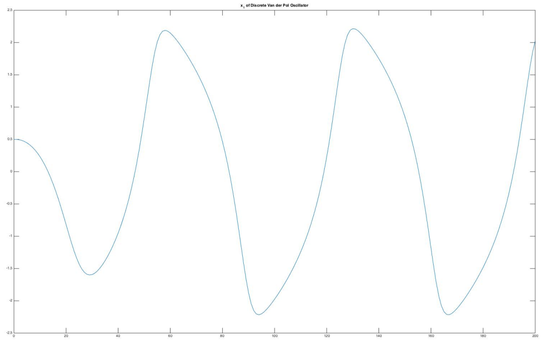
A typical trajectory of x_1 is below Notice that the period of the limit cycle is approximately seventy time steps.



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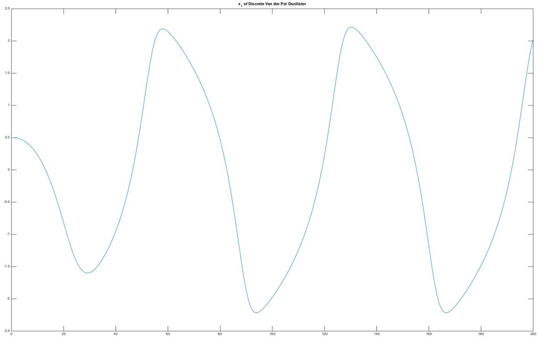

Example

The corresponding phase portrait is below.

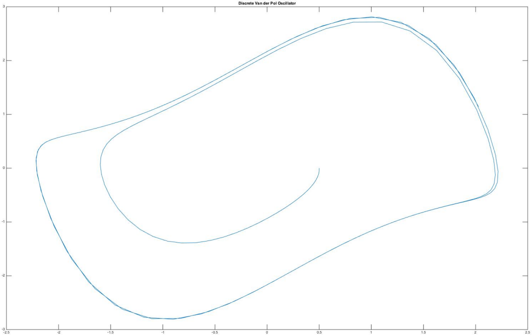

Figure: Typical Phase Portrait

Example

We constructed a MEE filter with

$$
\begin{aligned}
G &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
Q &= 0.1 \\
R &= 0.01 \\
\alpha &= 0.5 \\
x(0) &= \begin{bmatrix} 0.5 \\ 0 \end{bmatrix} \\
\hat{x}(0|0) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
\pi(x, 0|-1)) &= \|x - \hat{x}(0|0)\|^2
\end{aligned}
$$

Example

The error $\tilde{x}_1(t|t)$ is shown below and is essentially zero after about twenty time steps.

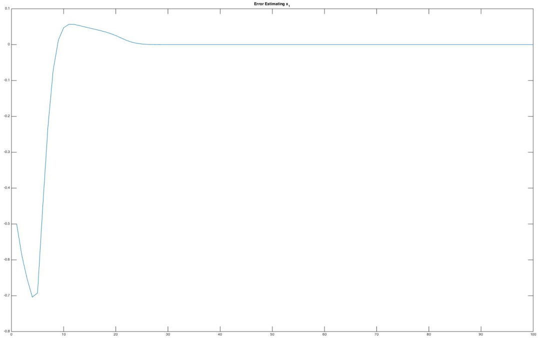

Figure: $\tilde{x}_1(t|t)$

Example

The phase portrait of $x(t)$ is in blue and that of $\hat{x}(t|t)$ is in red. Notice that $\hat{x}(t|t)$ converges to $x(t)$ faster than $x(t)$ converges to the limit cycle.

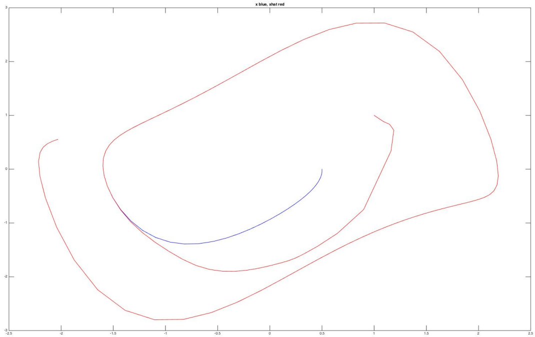

Figure: $\boldsymbol{x(t)}$ blue, $\boldsymbol{\hat{x}(t|t)}$ red

Conclusion

We have extended Minimum Energy Estimation of Mortenson and Hijab to discrete time systems.

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Thank You.

Questions?