Taylor expansions for the HJB equation associated with a bilinear control problem

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Motivation

- dragged Brownian particle following Langevin equation

  $dX_s = Y(s)ds,\ dY_s = -\beta Y_s ds + F(X, s)ds + \sqrt{2\beta kT/m}dB(s)$

  particle confined by potential V: force $F(X, s) = -\nabla V(X, s)$
  $\beta >>,\ t = \frac{s}{\beta},\ \nu = \frac{kT}{m},$

- Smoluchowski equation

$$dX_t = -\nabla V(X, t) + \sqrt{2\nu}dB_t$$

- atomic force microscopy, single molecule pulling, optical trapping

- collinate light from laser into aperature of microscope objetive

- control by optical tweezer $V(X_t, t) = W(X_t) + u(X_t, t)$.

The Fokker-Planck equation

Consider probability distribution function

$$\rho(x,t)\mathrm{d}x = \mathbb{P}[X_t \in [x, x+\mathrm{d}x)]$$

Fokker-Planck equation

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nu \Delta \rho + \nabla \cdot (\rho \nabla V) && \text{in } \Omega \times (0,\infty), \\ 0 &= (\nu \nabla \rho + \rho \nabla V) \cdot \vec{n} && \text{on } \Gamma \times (0,\infty), \\ \rho(x,0) &= \rho_0(x) && \text{in } \Omega, \end{aligned}$$

- $\Omega \subset \mathbb{R}^n$ bounded open set with boundary $\Gamma = \partial\Omega$,
- ρ_0 initial probability distribution with $\int_\Omega \rho_0(x)\mathrm{d}x = 1$.

[e.g. Borzi et al., Hartmann et al.

Control of the Fokker-Planck equation

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nu \Delta \rho + \nabla \cdot (\rho \nabla V) && \text{in } \Omega \times (0, \infty), \\ 0 &= (\nu \nabla \rho + \rho \nabla V) \cdot \vec{n} && \text{on } \Gamma \times (0, \infty), \\ \rho(x, 0) &= \rho_0(x) && \text{in } \Omega, \end{aligned}$$

- system converges to stationary distribution $\rho_\infty(x)$, (Boltzmann distribution)
- particles have to cross energy barrier between potential wells, $\rightsquigarrow$ may be inadequately slow, $\sim \exp(\Delta W / \nu)$
- can we control the potential $V(x, t) = W(x) + \alpha(x) u(t)$,
- if we can, what are "good" choices for α?

Assumptions:

- $W \in W^{2,\max(2,n+\varepsilon)}(\Omega), \varepsilon > 0$
- $\alpha \in W^{2,\max(2,n+\varepsilon)}(\Omega), \varepsilon > 0$ with $\nabla \alpha \cdot \vec{n} = 0$ on Γ

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Solutions to the Fokker-Planck equation

For $T > 0$ we call ρ a (variational) solution on $(0, T)$ if

$$\rho \in W(0,T) = L^2(0,T; H^1(\Omega)) \cap H^1(0,T; (H^1(\Omega))^*)$$

and

$$\langle \rho_t(t), v\rangle + \langle \nu \nabla \rho(t) + \rho(t)\nabla W, \nabla v\rangle + u(t)\langle \rho(t)\nabla\alpha, \nabla v\rangle = 0, \ \forall v \in H^1(\Omega)$$

$$\rho(0) = \rho_0.$$

Proposition

(i) $u \in L^2(0,T), \rho_0 \in L^2(\Omega) \Rightarrow \exists!$ solution $\rho \in W(0,T)$

(ii) If moreover $\Delta\alpha \in L^\infty(\Omega), \rho_0 \in H^1(\Omega)$

$\Rightarrow \rho_t \in L^2(0,T; L^2(\Omega)), \ \rho \in C([0,T]; H^1(\Omega))$

Proposition

Let $u \in L^2(0,T)$ and $\rho_0 \in L^2(\Omega)$.

(i) For every $t \in [0,T]$ we have $\int_\Omega \rho(t)\,\mathrm{d}x = \int_\Omega \rho_0\,\mathrm{d}x$.

(ii) If $\rho_0 \geq 0$ a.e. on Ω, then $\rho(x,t) \geq 0 \ \forall t > 0$ and a.e. on Ω.

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A bilinear control system

Consider the bilinear control system

$$\dot{\rho}(t) = \mathcal{A}\rho(t) + u(t)\mathcal{N}\rho(t),$$
$$\rho(0) = \rho_0,$$

where the operators $\mathcal{A}$ and $\mathcal{N}$ are defined as follows

$$\mathcal{A}\colon \mathcal{D}(\mathcal{A}) \subset L^2(\Omega) \to L^2(\Omega),$$
$$\mathcal{D}(\mathcal{A}) = \left\{\rho \in H^2(\Omega) \,|\, (\nu\nabla\rho + \rho\nabla W)\cdot\vec{n} = 0 \text{ on } \Gamma\right\},$$
$$\mathcal{A}\rho = \nu\Delta\rho + \nabla\cdot(\rho\nabla W),$$
$$\mathcal{N}\colon H^1(\Omega) \to L^2(\Omega), \quad \mathcal{N}\rho = \nabla\cdot(\rho\nabla\alpha).$$

$\sigma(\mathcal{A}) \subset \mathbb{R}_-$, $\qquad \Phi(x) = \log(\nu) + \frac{W(x)}{\nu}$

$\rho_\infty = e^{-\Phi}$ is an eigenfunction of $\mathcal{A}$ with $\mathcal{A}\rho_\infty = 0$

A shifted problem

Instead of ρ, consider the shifted state $y := \rho - \rho_\infty$

$$\dot{y}(t) = \mathcal{A}y(t) + u(t)\mathcal{N}y(t) + \mathcal{B}u(t),$$
$$y(0) = \rho_0 - \rho_\infty,$$

with $\mathcal{B} = \mathcal{N}\rho_\infty$, thus $\mathcal{B}u(t) = u(t)\mathcal{N}\rho_\infty = u(t)\nabla \cdot (\rho_\infty \nabla \alpha)$.

We decompose our state space

$$L^2(\Omega) =: \mathcal{Y}_\mathcal{P} \oplus \mathcal{Y}_\mathcal{Q}$$
$$\mathcal{Y}_\mathcal{P} = \{v \in L^2(\Omega) \colon \int_\Omega v \,\mathrm{d}x = 0\}, \quad \mathcal{Y}_\mathcal{Q} = \operatorname{span}\{\rho_\infty\}$$
$$y = y_\mathcal{P} + y_\mathcal{Q} = \mathcal{P}y + \mathcal{Q}y.$$

Decoupling the problem

$\rightsquigarrow$ applying $\mathcal{P}$ and $\mathcal{Q}$ yields

$$\begin{pmatrix} \dot{y}_{\mathcal{P}} \\ \dot{y}_{\mathcal{Q}} \end{pmatrix} = \begin{pmatrix} \mathcal{P}\mathcal{A} & \mathcal{P}\mathcal{A} \\ \mathcal{Q}\mathcal{A} & \mathcal{Q}\mathcal{A} \end{pmatrix} \begin{pmatrix} y_{\mathcal{P}} \\ y_{\mathcal{Q}} \end{pmatrix} + u \begin{pmatrix} \mathcal{P}\mathcal{N} & \mathcal{P}\mathcal{N} \\ \mathcal{Q}\mathcal{N} & \mathcal{Q}\mathcal{N} \end{pmatrix} \begin{pmatrix} y_{\mathcal{P}} \\ y_{\mathcal{Q}} \end{pmatrix} + u \begin{pmatrix} \mathcal{P}\mathcal{B} \\ \mathcal{Q}\mathcal{B} \end{pmatrix}.$$

$\rightsquigarrow$ simplifies to

$$\begin{pmatrix} \dot{y}_{\mathcal{P}} \\ \dot{y}_{\mathcal{Q}} \end{pmatrix} = \begin{pmatrix} \mathcal{P}\mathcal{A} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_{\mathcal{P}} \\ y_{\mathcal{Q}} \end{pmatrix} + u \begin{pmatrix} \mathcal{P}\mathcal{N} & \mathcal{P}\mathcal{N} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_{\mathcal{P}} \\ y_{\mathcal{Q}} \end{pmatrix} + u \begin{pmatrix} \mathcal{P}\mathcal{B} \\ 0 \end{pmatrix}.$$

$\rightsquigarrow$ simplifies to

$$\dot{y}_{\mathcal{P}} = \widehat{\mathcal{A}} y_{\mathcal{P}} + u \widehat{\mathcal{N}} y_{\mathcal{P}} + \widehat{\mathcal{B}} u, \quad y_{\mathcal{P}}(0) = \mathcal{P}\rho_0,$$
$$y_{\mathcal{Q}}(t) = \mathcal{Q}\rho_0 - \rho_\infty = 0, \; t \geq 0.$$

An LQR problem for the linearized system

For $\delta > 0$ let us focus on the linearized system

$$\dot{y}_{\mathcal{P}} = (\widehat{\mathcal{A}} + \delta I)y_{\mathcal{P}}(t) + \widehat{\mathcal{B}}u, \quad y_{\mathcal{P}}(0) = \mathcal{P}\rho_0,$$

together with the quadratic cost functional

$$J(y_{\mathcal{P}}, u) = \frac{1}{2}\int_0^\infty \langle y_{\mathcal{P}}(t), \mathcal{M}y_{\mathcal{P}}(t)\rangle_{L^2(\Omega)}\,\mathrm{d}t + \frac{1}{2}\int_0^\infty |u(t)|^2\,\mathrm{d}t,$$

where $\mathcal{M} \in \mathcal{L}(\mathcal{Y}_{\mathcal{P}})$ is a self-adjoint nonnegative operator on $\mathcal{Y}_{\mathcal{P}}$.

Riccati-based feedback law: $u = -\widehat{\mathcal{B}}^*\widehat{\Pi}y_{\mathcal{P}}$

$$(\widehat{\mathcal{A}}^* + \delta I)\widehat{\Pi} + \widehat{\Pi}(\widehat{\mathcal{A}} + \delta I) - \widehat{\Pi}\widehat{\mathcal{B}}\widehat{\mathcal{B}}^*\widehat{\Pi} + \mathcal{M} = 0, \quad \widehat{\Pi} \in \mathcal{L}(\mathcal{Y}_{\mathcal{P}}).$$

The ∞-dimensional Hautus test: The pair $(\widehat{\mathcal{A}}, \widehat{\mathcal{B}})$ is δ-stabilizable if

$$\ker(\lambda I - \widehat{\mathcal{A}}^*) \cap \ker(\widehat{\mathcal{B}}^*) = \{0\} \quad \text{for } \lambda \in \overline{\mathbb{C}}_{-\delta} \cap \sigma(\widehat{\mathcal{A}}^*).$$

$\rightsquigarrow$ into nonlinear system: compare e.g.: M. Badra, I. Lasiecka, J.-P. Raymond, R. Triggiani,...

Fokker-Planck equation

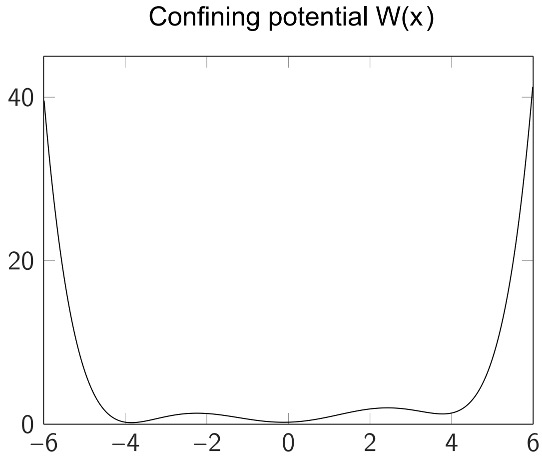
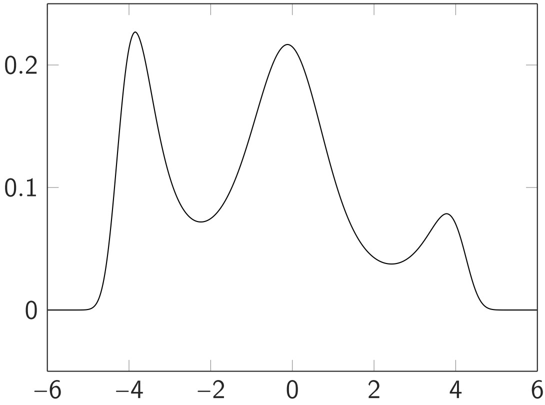


Figure: 1D Fokker-Planck equation, $n = 1024$.

Fokker-Planck equation

Stationary distribution $\rho_\infty(x)$



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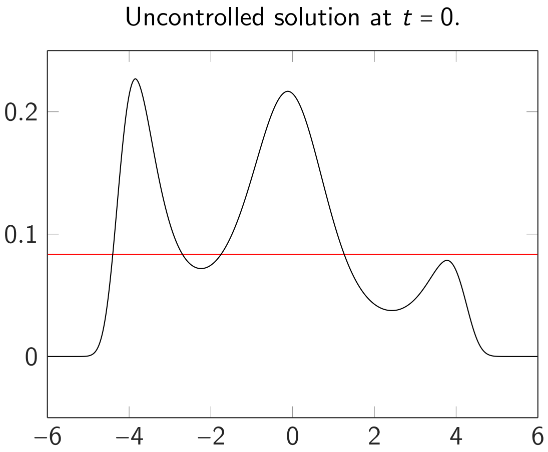

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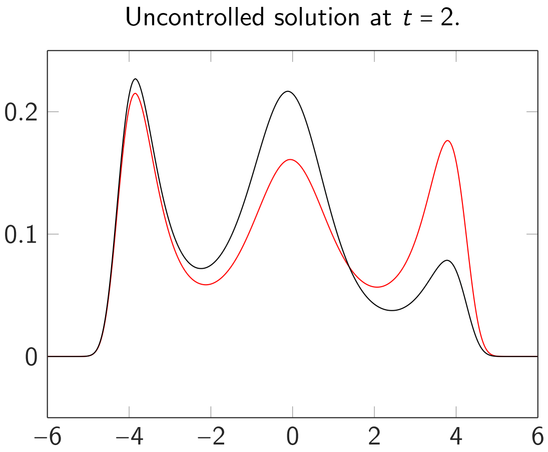

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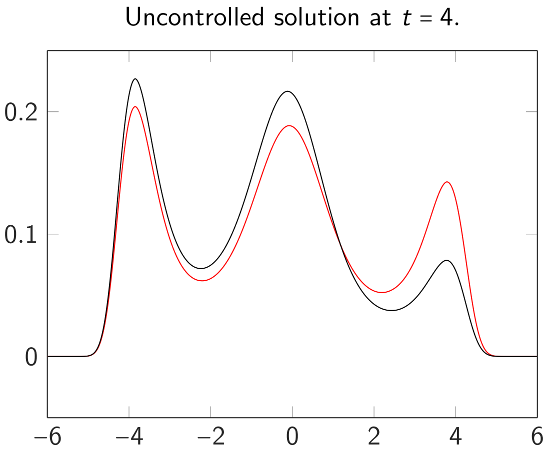

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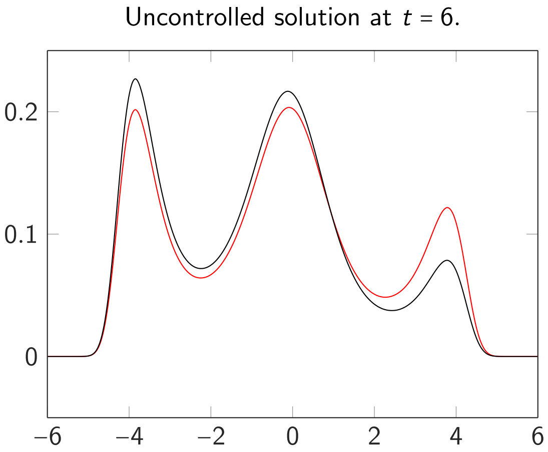

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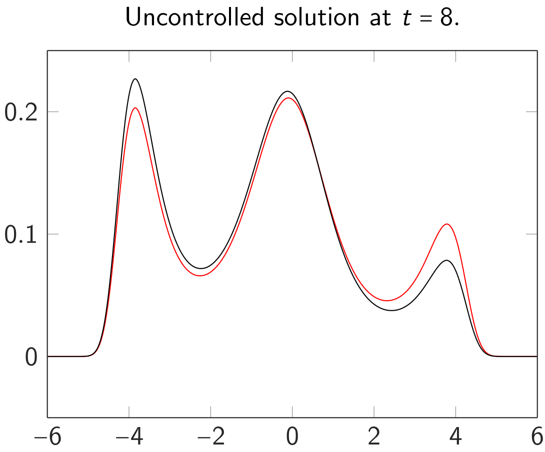

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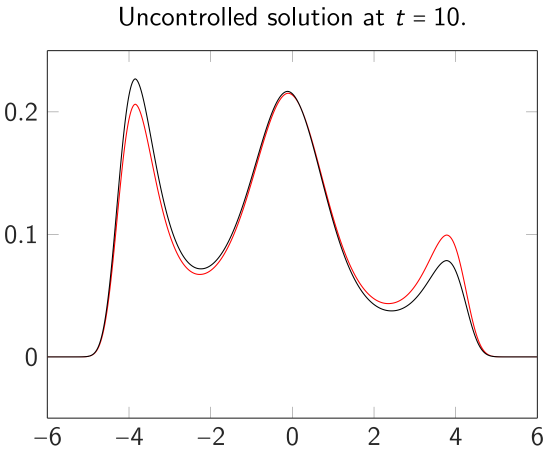

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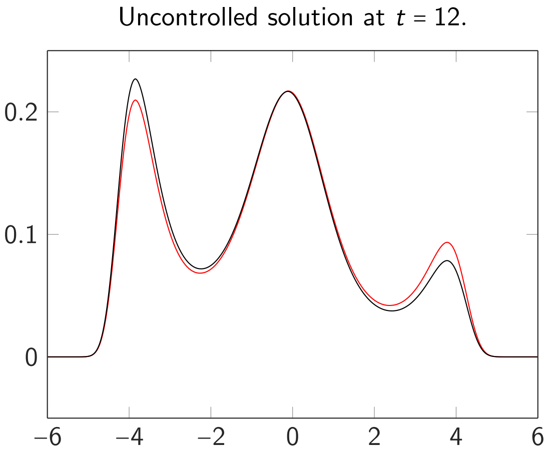

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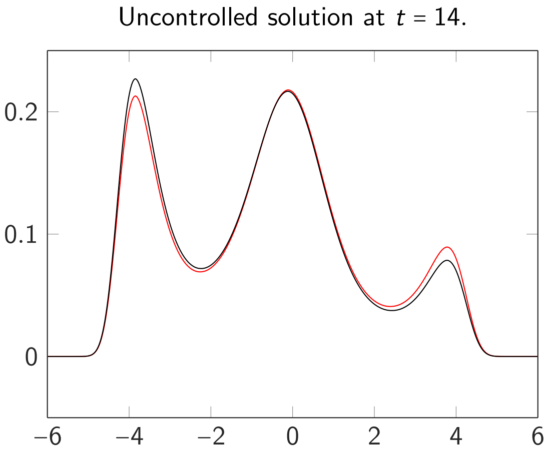

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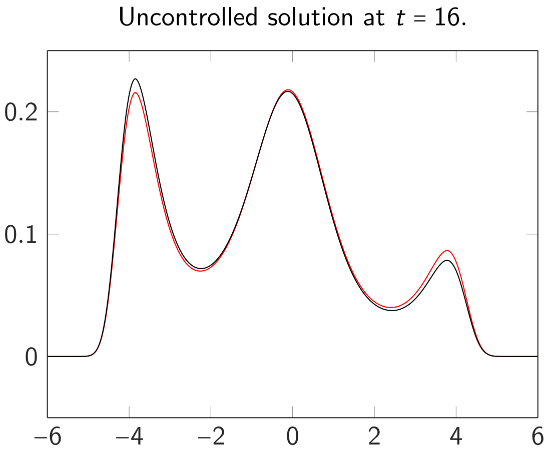

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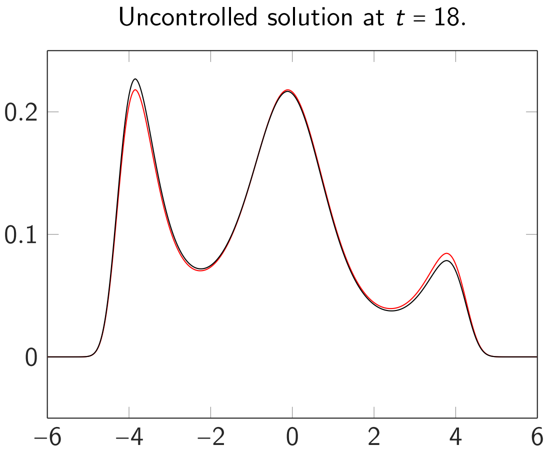

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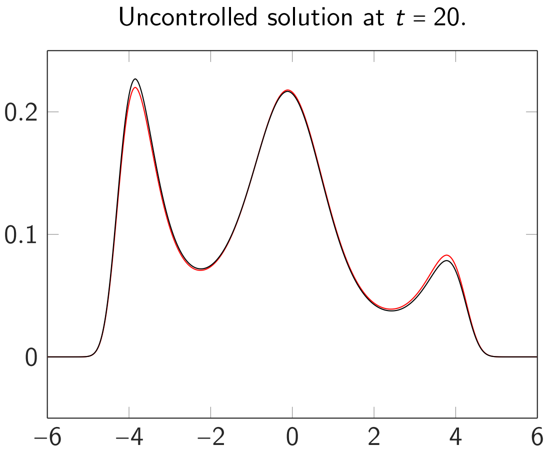

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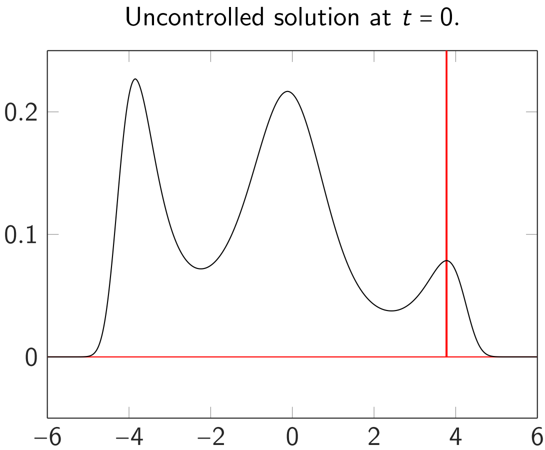

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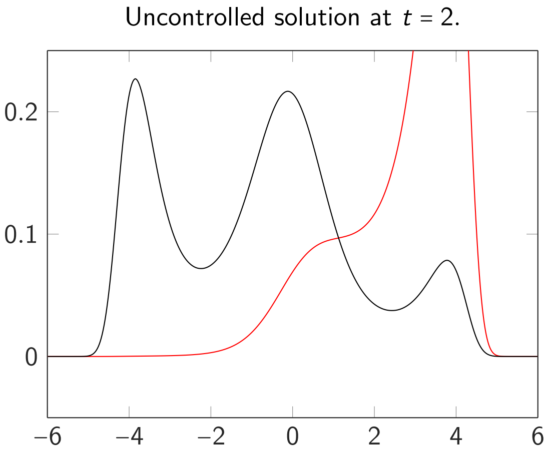

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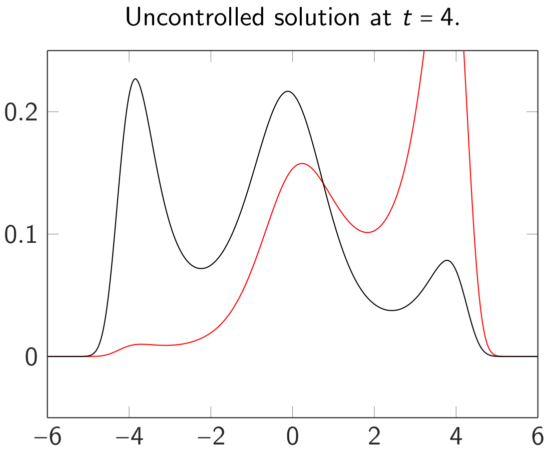

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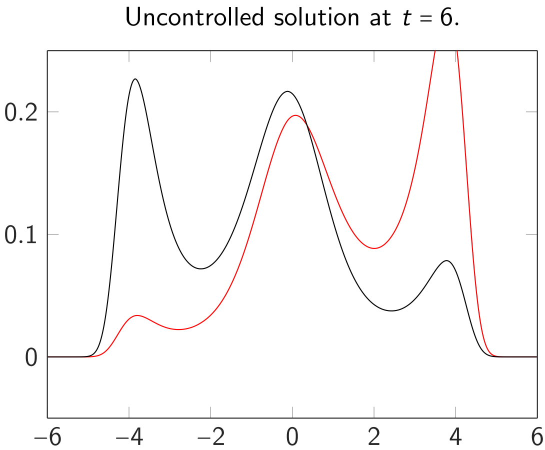

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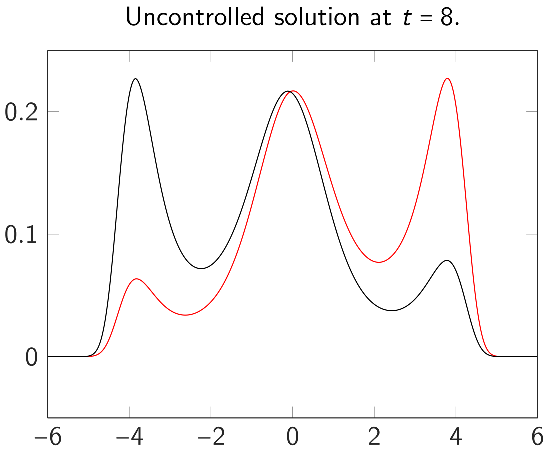

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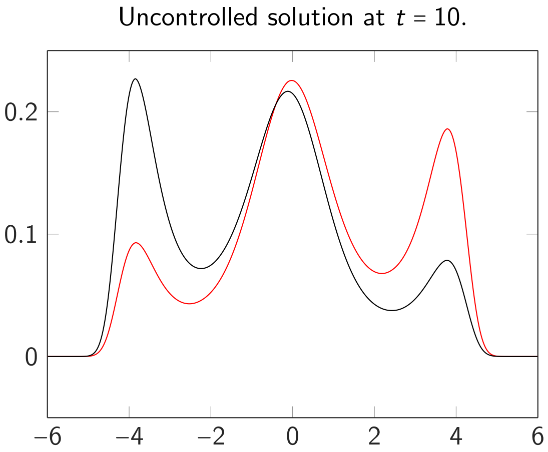

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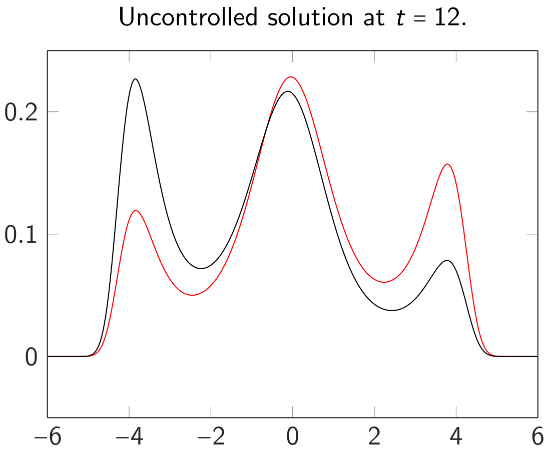

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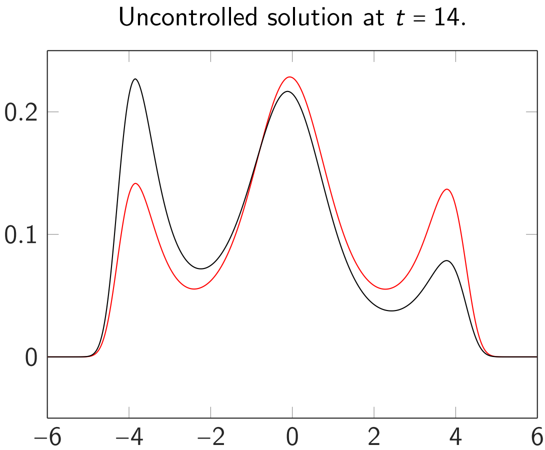

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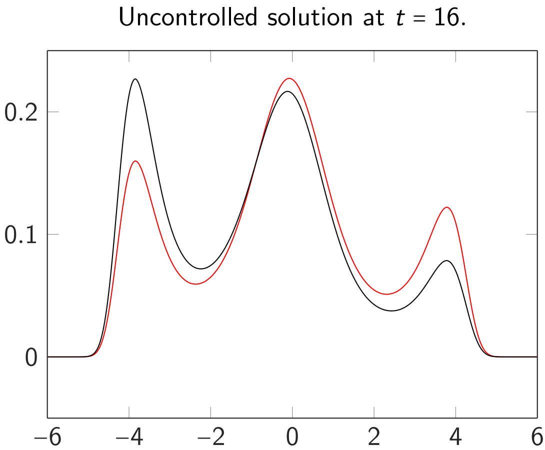

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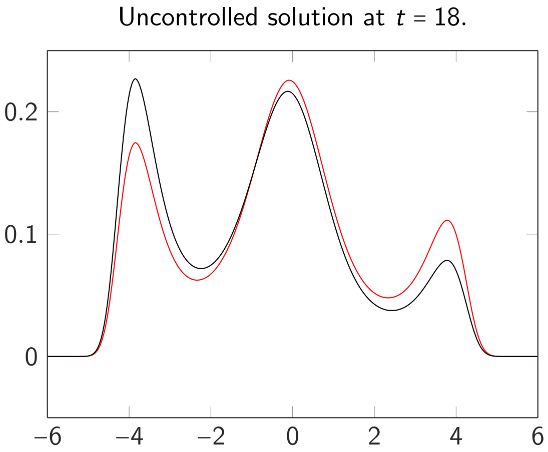

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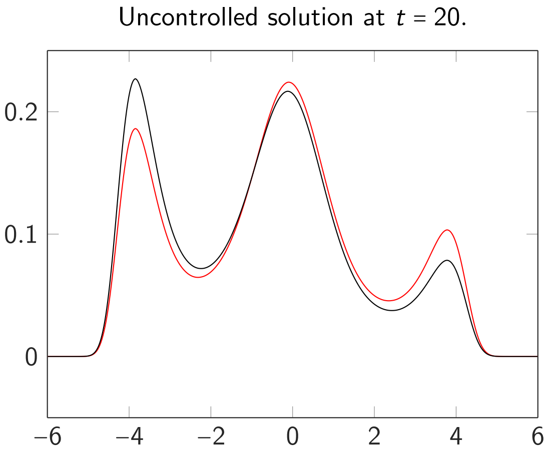

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Back to the 1D Fokker-Planck equation

(Un)controlled solution at $t = 0$.

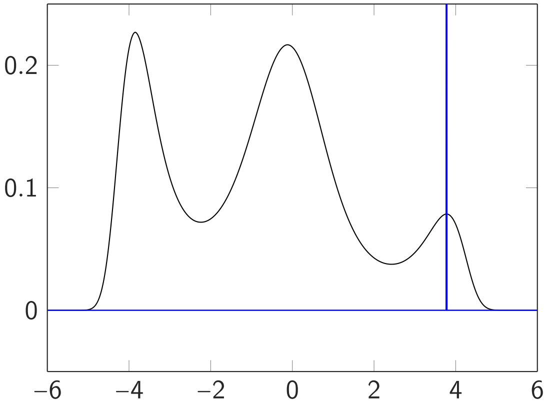

Figure: Fokker-Planck, $n = 1024, r = 10, \beta = 10^{-5}$.

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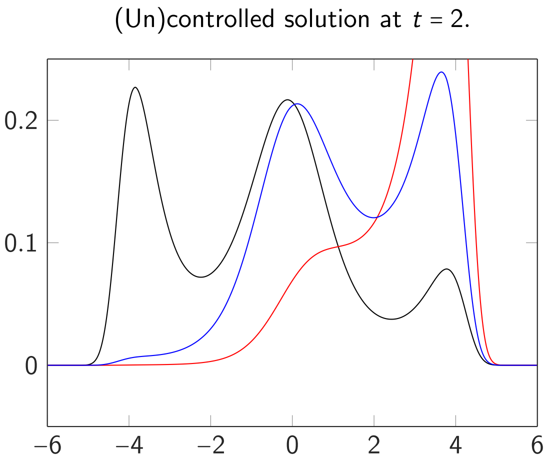

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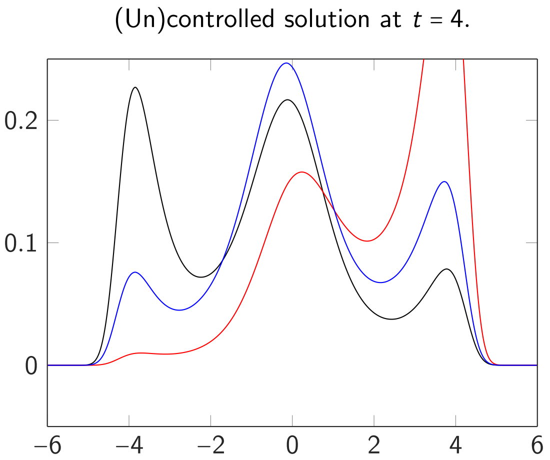

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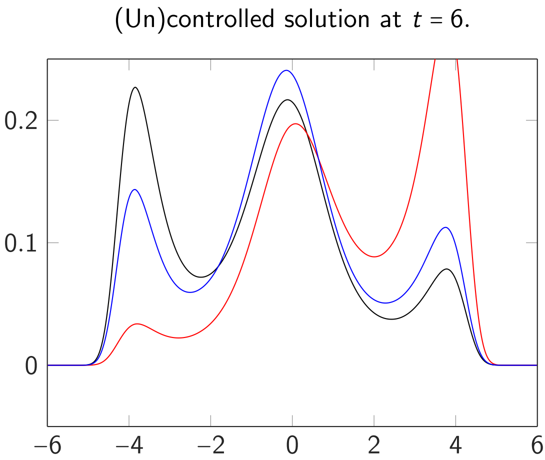

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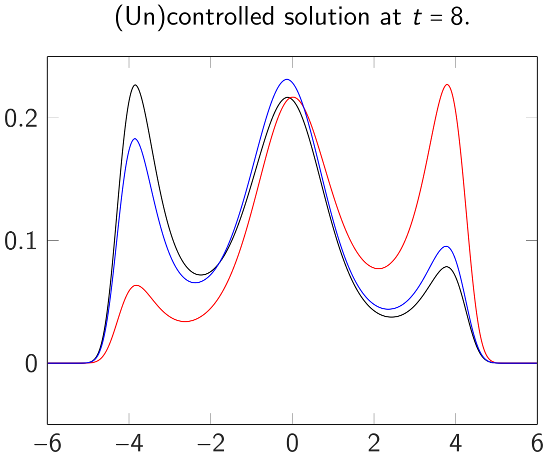

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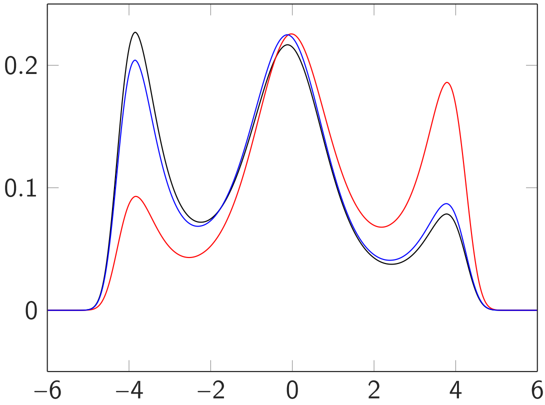
(Un)controlled solution at $t = 10$.



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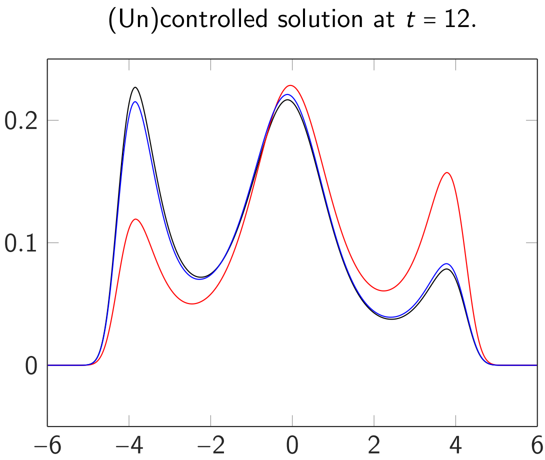

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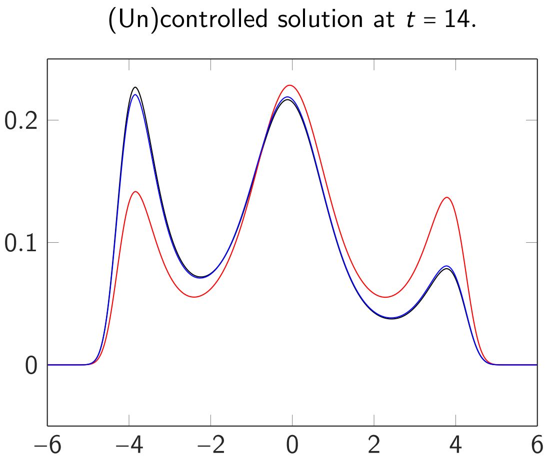

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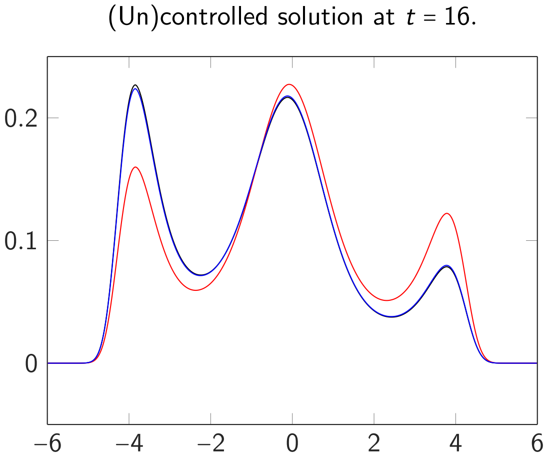

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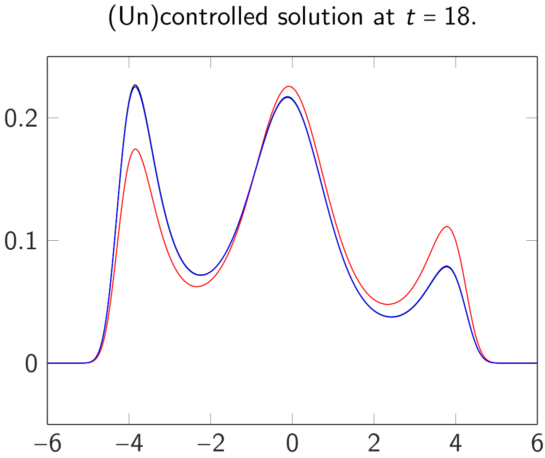

Figure: Fokker-Planck, $n = 1024, r = 10, \beta = 10^{-5}$.

Back to the 1D Fokker-Planck equation

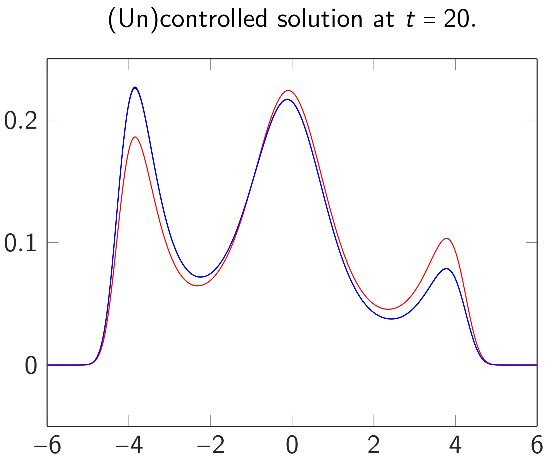

Figure: Fokker-Planck, $n = 1024, r = 10, \beta = 10^{-5}$.

2-D and movies removed

Comparison of control laws $u(t)$

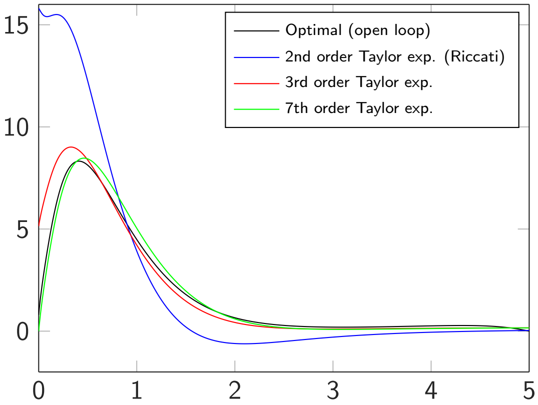

Figure: 1D Fokker-Planck equation, $n = 1024$.

Bilinear quadratic optimization

Consider a bilinear control system

$$\dot{x}(t) = Ax(t) + Nx(t)u(t) + Bu(t), \quad x(0) = x_0,$$
$$y(t) = Cx(t),$$

- $A, N \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}$,
- control $u\colon [0,\infty) \to \mathbb{R}$ and
- output $y\colon [0,\infty) \to \mathbb{R}^p$ of the system,
- (A, B) stabilizable.

For this system, we introduce the minimal value functional

$$\mathcal{V}(x_0) = \inf_{u\in L^2(0,\infty)} \frac{1}{2}\int_0^\infty \|y(t)\|^2 \mathrm{d}t + \frac{\alpha}{2}\int_0^\infty u(t)^2 \mathrm{d}t.$$

Dynamic programming

By the dynamic programming principle, for any x_0 and $\tau > 0$:

$$\mathcal{V}(x_0) = \inf_{u \in L^2(0,\tau)} \int_0^\tau \ell(y(u,x_0;t),u(t))\mathrm{d}t + \mathcal{V}(x(u,x_0;\tau)),$$

henceforth $\ell(y,u) = \frac{1}{2}\|y\|^2 + \frac{\alpha}{2}u^2$.

Under smoothness assumptions on $\mathcal{V}$, we obtain

$$\min_{u \in \mathbb{R}} \left[(Ax + (Nx + B)u)^T \nabla \mathcal{V}(x) + \frac{1}{2}\|Cx\|^2 + \frac{\alpha}{2}u^2 \right] = 0, \quad \mathcal{V}(0) = 0.$$

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The Hamilton-Jacobi-Bellman equation

Consider again

$$\min_{u\in\mathbb{R}}\left[(Ax+(Nx+B)u)^T\nabla\mathcal{V}(x)+\frac{1}{2}\|y\|^2+\frac{\alpha}{2}u^2\right]=0,\quad \mathcal{V}(0)=0.$$

Minimization yields Hamilton-Jacobi-Bellman (HJB) equation

$$x^TA^T\nabla\mathcal{V}(x)+\frac{1}{2}\|Cx\|^2-\frac{1}{2\alpha}((Nx+B)^T\nabla\mathcal{V}(x))^2=0,\quad \mathcal{V}(0)=0.$$

Optimal feedback law via solving HJB equation

$$u_{\mathrm{opt}}(x)=-\frac{1}{\alpha}(Nx+B)^T\nabla\mathcal{V}(x).$$

Problem: The HJB equation is a nonlinear n-dimensional PDE...

The Hamilton-Jacobi-Bellman equation

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Problem: The HJB equation is a nonlinear *n*-dimensional PDE...

HJB equation: the linear case

For the linear case, we obtain

$$x^T A^T \nabla \mathcal{V}(x) + \frac{1}{2}\|Cx\|^2 - \frac{1}{2\alpha}(B^T \nabla \mathcal{V}(x))^2 = 0, \quad \mathcal{V}(0) = 0.$$

The ansatz $\mathcal{V}(x) = \frac{1}{2}x^T \Pi x$, with $\Pi = \Pi^T \in \mathbb{R}^{n \times n}$ yields

$$x^T A^T \Pi x + \frac{1}{2}\|Cx\|^2 - \frac{1}{2\alpha}(B^T \Pi x)^2 = 0.$$

The more familiar expression is

$$\frac{1}{2}x^T \left(A^T \Pi + \Pi A + C^T C - \frac{1}{\alpha} \Pi B B^T \Pi \right) x = 0.$$

$\Rightarrow$ algebraic Riccati equation

Taylor expansions – basic idea

Assume that $\mathcal{V}$ can be expanded around 0 as follows

$$\mathcal{V}(x) = \underbrace{\mathcal{V}(0)}_{\in\mathbb{R}} + \underbrace{D\mathcal{V}(0)}_{\in\mathbb{R}^n}(x) + \frac{1}{2!}\underbrace{D^2\mathcal{V}(0)}_{\in\mathbb{R}^{n\times n}}(x,x) + \frac{1}{3!}\underbrace{D^3\mathcal{V}(0)}_{\in\mathbb{R}^{n\times n\times n}}(x,x,x) + \dots$$

Feedback law can be determined via

$$u = -\frac{1}{\alpha}\sum_{k=2}^{\infty}\frac{1}{(k-1)!}D^k\mathcal{V}(0)(Nx+B,x,\dots,x)$$

Finite-dimensional case: [Lukes, Cebuhar/Costanza, Krener]

Infinite-dimensional case: [Thevenet/Buchot/Raymond]

Question: Precise structure of $D^k\mathcal{V}(0)$?

Taylor expansions – basic idea

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Question: Precise structure of $D^k\mathcal{V}(0)$?

Taylor expansions – Differentiating HJB

Let us come back to

$$x^T A^T \nabla \mathcal{V}(x) + \frac{1}{2}\|Cx\|^2 - \frac{1}{2\alpha}((Nx + B)^T \nabla \mathcal{V}(x))^2 = 0,$$

Taylor expansions – Differentiating HJB

Let us come back to

$$x^T A^T \nabla \mathcal{V}(x) + \frac{1}{2}\|Cx\|^2 - \frac{1}{2\alpha}((Nx + B)^T \nabla \mathcal{V}(x))^2 = 0,$$

$\Rightarrow$ **one differentiation** in direction $z_1 \in \mathbb{R}^n$ yields

$$D^2\mathcal{V}(x)(Ax, z_1) + D\mathcal{V}(x)Az_1 + \langle Cx, Cz_1 \rangle$$
$$- \frac{1}{\alpha}\big(D^2\mathcal{V}(x)(Nx + B, z_1) + D\mathcal{V}(x)Nz_1\big)\big(D\mathcal{V}(x)(Nx + B)\big)] = 0.$$

Taylor expansions – Differentiating HJB

Let us come back to

$$x^T A^T \nabla \mathcal{V}(x) + \frac{1}{2}\|Cx\|^2 - \frac{1}{2\alpha}((Nx+B)^T \nabla \mathcal{V}(x))^2 = 0,$$

$\Rightarrow$ **two differentiations** in directions $z_1, z_2 \in \mathbb{R}^n$ yield

$$\begin{aligned}
&D^3\mathcal{V}(x)(Ax, z_1, z_2) + D^2\mathcal{V}(x)(Az_2, z_1) + D^2\mathcal{V}(x)(Az_1, z_2) + \langle Cz_1, Cz_2 \rangle \\
&\quad - \frac{1}{\alpha}\Big(D^2\mathcal{V}(x)(Nx+B, z_1) + D\mathcal{V}(x)Nz_1\Big) \cdot \\
&\qquad\qquad \Big(D^2\mathcal{V}(x)(Nx+B, z_2) + D\mathcal{V}(x)Nz_2\Big) \\
&\quad - \frac{1}{\alpha}\Big(D^3\mathcal{V}(x)(Nx+B, z_1, z_2) + D^2\mathcal{V}(x)(Nz_2, z_1) + D^2\mathcal{V}(x)(Nz_1, z_2)\Big) \\
&\qquad\qquad \Big(D\mathcal{V}(x)(Nx+B)\Big) = 0.
\end{aligned}$$

Taylor expansions – Differentiating HJB

Let us come back to

$$x^T A^T \nabla \mathcal{V}(x) + \frac{1}{2}\|Cx\|^2 - \frac{1}{2\alpha}\left((Nx+B)^T \nabla \mathcal{V}(x)\right)^2 = 0,$$

$\Rightarrow$ **two differentiations** in directions $z_1, z_2 \in \mathbb{R}^n$ yield

$$\begin{aligned}
&\cancel{D^3\mathcal{V}(0)(A0, z_1, z_2)} + D^2\mathcal{V}(0)(Az_2, z_1) + D^2\mathcal{V}(0)(Az_1, z_2) + \langle Cz_1, Cz_2\rangle \\
&\quad - \frac{1}{\alpha}\left(D^2\mathcal{V}(0)(\cancel{N0} + B, z_1) + \cancel{D\mathcal{V}(0)Nz_1}\right) \cdot \\
&\qquad \left(D^2\mathcal{V}(0)(\cancel{N0} + B, z_2) + \cancel{D\mathcal{V}(0)Nz_2}\right) \\
&\quad - \frac{1}{\alpha}\left(D^3\mathcal{V}(0)(\cancel{N0} + B, z_1, z_2) + D^2\mathcal{V}(0)(Nz_2, z_1) + D^2\mathcal{V}(0)(Nz_1, z_2)\right) \\
&\qquad \left(\cancel{D\mathcal{V}(0)(N0 + B)}\right) = 0.
\end{aligned}$$

This is the Riccati equation...

Taylor expansions – Differentiating HJB

Let us come back to

$$x^T A^T \nabla \mathcal{V}(x) + \frac{1}{2}\|Cx\|^2 - \frac{1}{2\alpha}((Nx+B)^T \nabla \mathcal{V}(x))^2 = 0,$$

$\Rightarrow$ **three differentiations** in directions $z_1, z_2, z_3 \in \mathbb{R}^n$ yield

$$\begin{aligned}
&D^3\mathcal{V}(0)(Az_3, z_1, z_2) + D^3\mathcal{V}(0)(Az_2, z_1, z_3) + D^3\mathcal{V}(0)(Az_1, z_2, z_3)\\
&\quad - \frac{1}{\alpha}\Big(D^3 V(0)(B, z_1, z_3) + D^2\mathcal{V}(0)(Nz_3, z_1) + D^2\mathcal{V}(0)(Nz_1, z_3)\Big)\Big(D^2\mathcal{V}(0)(B, z_2)\Big)\\
&\quad - \frac{1}{\alpha}\Big(D^3\mathcal{V}(0)(B, z_2, z_3) + D^2\mathcal{V}(0)(Nz_3, z_2) + D^2\mathcal{V}(0)(Nz_2, z_3)\Big)\Big(D^2\mathcal{V}(0)(B, z_1)\Big)\\
&\quad - \frac{1}{\alpha}\Big(D^3\mathcal{V}(0)(B, z_1, z_2) + D^2\mathcal{V}(0)(Nz_2, z_1) + D^2\mathcal{V}(0)(Nz_1, z_2)\Big)\Big(D^2\mathcal{V}(0)(B, z_3)\Big)\\
&\quad = 0.
\end{aligned}$$

Taylor expansions – Differentiating HJB

Let us come back to

$$x^T A^T \nabla \mathcal{V}(x) + \frac{1}{2}\|Cx\|^2 - \frac{1}{2\alpha}((Nx+B)^T \nabla \mathcal{V}(x))^2 = 0,$$

$\Rightarrow$ **three differentiations** in directions $z_1, z_2, z_3 \in \mathbb{R}^n$ yield

$$\begin{aligned}
&D^3\mathcal{V}(0)(Az_3, z_1, z_2) + D^3\mathcal{V}(0)(Az_2, z_1, z_3) + D^3\mathcal{V}(0)(Az_1, z_2, z_3) \\
&\quad - \frac{1}{\alpha}\Big(D^3V(0)(B, z_1, z_3) + D^2\mathcal{V}(0)(Nz_3, z_1) + D^2\mathcal{V}(0)(Nz_1, z_3)\Big)\Big(D^2\mathcal{V}(0)(B, z_2)\Big) \\
&\quad - \frac{1}{\alpha}\Big(D^3\mathcal{V}(0)(B, z_2, z_3) + D^2\mathcal{V}(0)(Nz_3, z_2) + D^2\mathcal{V}(0)(Nz_2, z_3)\Big)\Big(D^2\mathcal{V}(0)(B, z_1)\Big) \\
&\quad - \frac{1}{\alpha}\Big(D^3\mathcal{V}(0)(B, z_1, z_2) + D^2\mathcal{V}(0)(Nz_2, z_1) + D^2\mathcal{V}(0)(Nz_1, z_2)\Big)\Big(D^2\mathcal{V}(0)(B, z_3)\Big) \\
&\quad = 0.
\end{aligned}$$

☹ Looks complicated ☺ Linear in $D^3\mathcal{V}(0)$

The general structure

For i, $j \in \mathbb{N}$, consider the following set of permutations:

$$S_{i,j} = \{\sigma \in S_{i+j} \,|\, \sigma(1) < \dots < \sigma(i) \text{ and } \sigma(i+1) < \dots < \sigma(i+j)\},$$

where S_{i+j} is the set of permutations of $\{1, \dots, i+j\}$.

Example

$$\begin{aligned} S_{2,2} &= \{\sigma \in S_4 \,|\, \sigma(1) < \sigma(2) \text{ and } \sigma(3) < \sigma(4)\} \\ &= \{(1,2,3,4),(1,3,2,4),(1,4,2,3), \\ &\qquad (2,3,1,4),(2,4,1,3),(3,4,1,2)\} \end{aligned}$$

For given multilinear form $\mathcal{T}$ (of order $i+j$), we define

$$\mathrm{Sym}_{i,j}(\mathcal{T})(z_1, \dots, z_{i+j}) := \binom{i+j}{i}^{-1} \Big[\sum_{\sigma \in S_{i,j}} \mathcal{T}(z_{\sigma(1)}, \dots, z_{\sigma(i+j)}) \Big].$$

The general structure

Define $A_\Pi = A - \frac{1}{\alpha} BB^* \Pi$. For $k \geq 3$ and $z_1, \ldots, z_k \in \mathbb{R}^n$ consider

$$\sum_{i=1}^{k} D^k \mathcal{V}(0)(z_1, ..., z_{i-1}, A_\Pi z_i, z_{i+1}, ..., z_k) = \frac{1}{2\alpha} \mathcal{R}_k(z_1, \ldots, z_k) \quad (*),$$

where $\mathcal{R}_k(z_1, \ldots, z_k)$ is given by:

$$\mathcal{R}_k(z_1, \ldots, z_k) = 2k(k-1)\mathrm{Sym}_{1,k-1}\left(\mathcal{C}_1(z_1)\mathcal{G}_{k-1}(z_2, \ldots, z_k)\right)$$
$$+ \sum_{i=2}^{k-2} \binom{k}{i} \mathrm{Sym}_{i,k-i}\Big((\mathcal{C}_i(z_1, \ldots, z_i) + i\mathcal{G}_i(z_1, \ldots, z_i))$$
$$\times (\mathcal{C}_{k-i}(z_{i+1}, \ldots, z_k) + (k-i)\mathcal{G}_{k-i}(z_{i+1}, \ldots, z_k)) \Big),$$

where:

$$\mathcal{C}_i(z_1, ..., z_i) = D^{i+1}\mathcal{V}(0)(B, z_1, ..., z_i)$$

$$\mathcal{G}_i(z_1, ..., z_i) = \frac{1}{i}\Big[\sum_{j=1}^{i} D^i \mathcal{V}(0)(z_1, ..., z_{j-1}, Nz_j, z_{j+1}, ..., z_i) \Big].$$

Tensor calculus

Main numerical task:

$$\text{Solve} \quad \underbrace{\left(\sum_{i=1}^{k} I^{k-i} \otimes A_{\Pi}^T \otimes I^{i-1}\right)}_{\mathbf{A}} T_k = \underbrace{R_k(T_2, \ldots, T_{k-1})}_{\text{? low rank ?}}.$$

Since $\mathbf{A}$ is stable: $\mathbf{A}^{-1} = -\int_0^\infty e^{t\mathbf{A}}\,\mathrm{d}t = -\int_0^\infty \bigotimes_{i=1}^{k} e^{tA_{\Pi}^T}\,\mathrm{d}t.$

Approximate by quadrature formula

[Grasedyck,Hackbusch,Stenger]

$$\mathbf{A}^{-1} \approx -\sum_{j=-r}^{r} w_j \bigotimes_{i=1}^{k} e^{t_j A_{\Pi}^T},$$

with suitable quadrature weights w_j and points t_j.

The infinite dimensional setup

Let us focus on the abstract bilinear control system

$$\dot{x}(t) = \mathcal{A}x(t) + \mathcal{N}x(t)u(t) + \mathcal{B}u(t), \quad x(0) = x_0 \in X,$$

- $V \subset X \subset V^*$ Gelfand triple of Hilbert spaces
- $a\colon V \times V \to \mathbb{R}$ a bounded V-X bilinear form, i.e., $\exists \nu > 0$ and $\lambda \in \mathbb{R}$
$$a(v,v) \geq \nu \|v\|_V^2 - \lambda \|v\|_X^2 \ \forall v \in V$$
- $\mathcal{N} \in \mathcal{L}(V,X) \cap \mathcal{L}(\mathcal{D}(\mathcal{A}),V), \ \ \mathcal{N}^* \in \mathcal{L}(V,X), \ \ B \in X$
- For $\beta > 0$ large enough define $\mathcal{A}_0 := -\mathcal{A} + \beta I$
$$\Rightarrow [\mathcal{D}(\mathcal{A}_0), X]_{\frac{1}{2}} = [\mathcal{D}(\mathcal{A}_0^*), X]_{\frac{1}{2}} = V$$
- $(\mathcal{A}, \mathcal{B})$ stabilizable

A multilinear operator equation

Well-posedness of $\mathcal{T}_k \equiv D^k \mathcal{V}(0)$

For $k \geq 3$, and $z_1, \ldots, z_k \in X$ define the multilinear form

$$\mathcal{T}_k \colon X \times \cdots \times X \to \mathbb{R},$$
$$\mathcal{T}_k(z_1, \ldots, z_k) = -\frac{1}{2\alpha} \int_0^\infty \mathcal{R}_k(e^{\mathcal{A}_\Pi t} z_1, \ldots, e^{\mathcal{A}_\Pi t} z_k) \,\mathrm{d}t.$$

Then $\mathcal{T}_k$ is the unique solution of $(*)$. Moreover, it holds that

$$|\mathcal{T}_k(z_1, \ldots, z_k)| \leq C \prod_{i=1}^{k} \|z_i\|_X.$$

Definition:

$$\mathcal{V}_p \colon Y \to \mathbb{R}, \quad \mathcal{V}_p(y) = \sum_{k=2}^{p} \frac{1}{k!} \mathcal{T}_k(y, \ldots, y).$$

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A suboptimal feedback law

Consider now the polynomial feedback law

$$u_p(x) = -\sum_{k=2}^{p} \frac{1}{(k-1)!} \mathcal{T}_k(\mathcal{N}x + \mathcal{B}, x, \dots, x)$$

and the corresponding (nonlinear) closed-loop system

$$\text{(CL)} \qquad \dot{x} = \mathcal{A}x - (\mathcal{N}x + \mathcal{B})u_p(x), \quad x(0) = x_0.$$

Local well-posedness

There exist constants $C_1, C_2 > 0$ such that: if $\|x_0\|_X \leq C_1$, then

- (CL) admits a unique solution $x \in W(0,\infty) = \{\varphi \in L^2(0,\infty; X) | \varphi_t \in L^2(0,\infty; V^*)\}$
- this solution satisfies $\|x\|_{W(0,\infty)} \leq C_2$
- and $\lim_{t\to\infty} \|x(t)\|_X = 0$

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and the corresponding (nonlinear) closed-loop system

$$\text{(CL)} \qquad \dot{x} = \mathcal{A}x - (\mathcal{N}x + \mathcal{B})u_p(x), \quad x(0) = x_0.$$

Local suboptimality

There exists a constant $C_3 > 0$, C_4 such that: if $\|x_0\|_X \leq C_3$,

- $\int_0^\infty \ell(x(u_p, x_0; t), u_p(t))\, \mathrm{d}t \leq \mathcal{V}(x_0) + C_4(\|x_0\|_X^{p+1})$
- $|\mathcal{V}(x_0) - \mathcal{V}_p(x_0)| \leq C\|y_0\|_X^{p+1}$
- $\|\bar{x}(\bar{u}, x_0) - x(u_p, x_0)\|_{W(0,\infty)} \leq C_4\|x_0\|_X^{\frac{p+1}{2}}$
- $\|\bar{u} - u_p\|_{W(0,\infty)} \leq C_4\|x_0\|_X^{\frac{p+1}{2}}$

Comparison of control laws $u(t)$

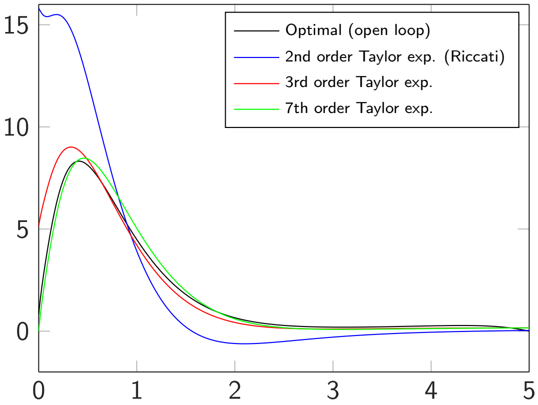

Figure: 1D Fokker-Planck equation, $n = 1024$.

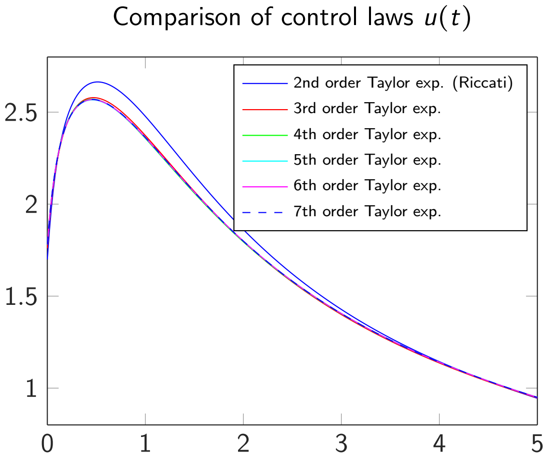

Figure: Fokker-Planck, $n = 1024$, $r = 10$, $\beta = 10^{-4}$.

Miscellanea

- Applicable to Fokker Planck
- Feasible for general infinite dimensional control systems
- Combined with balanced truncation

- When is higher order useful ?
- Efficient tensor numerics

Finite-dimensional properties

Assume spatial discretization yields $A, N, \rho^d, e = \frac{1}{d}\left(1, \ldots, 1\right)^T$.

- $\int_\Omega \rho(t)\,\mathrm{d}x = \int_\Omega \rho_0\,\mathrm{d}x \rightsquigarrow e^T\rho^d(t) = e^T\rho_0^d = 1$
- $\rho_0 \geq 0 \Rightarrow \rho(x,t) \geq 0\ \forall t \rightsquigarrow A$ is a Metzler matrix
- $A_s = DAD^{-1}$ with $D = \mathrm{diag}(e^{\frac{\Phi^d}{2}})$ is a symmetric matrix
- $A^T e = N^T e = 0 = A\rho_\infty^d$
- $\mu = \max(\varepsilon, \varepsilon + \frac{1}{2}\lambda_{\max}(\widehat{A} + \widehat{A}^T))$

A two dimensional double well potential

As a numerical example, let us consider

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \nu \Delta \rho + \nabla \cdot (\rho \nabla W) + u \nabla \cdot (\rho \nabla \alpha) && \text{in } \Omega \times (0, \infty), \\ 0 &= (\nu \nabla \rho + \rho \nabla W) \cdot \vec{n} && \text{on } \Gamma \times (0, \infty), \\ \rho(x, 0) &= \rho_0(x) && \text{in } \Omega. \end{aligned}$$

- $\Omega = (-1.5, 1.5) \times (-1.1)$
- $W(x) = 3(x_1^2 - 1)^2 + 6x_2^2$
- finite differences with $n_x \cdot n_y = 96 \cdot 64 = 6144$ grid points
- upwind scheme for convective terms

Conclusion

- Fokker-Planck equation yields a bilinear control system.
- Decoupled the system by spectral projections.
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Thank you for your attention!