Stationary Action for Fundamental Solution of TPBVPs

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Roma

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Action Functional

- We look at conservative dynamical systems.
- Simplest case: Point-mass in a field.
- Position component of the state at time, t, is denoted by $\xi(t) \in \boldsymbol{R}^n$.
- Potential energy will be induced by some field: $V : \boldsymbol{R}^n \to \boldsymbol{R}$.
- Kinetic energy: $T(\dot{\xi}(t)) \doteq \frac{1}{2}\dot{\xi}^T(t)\mathcal{M}\dot{\xi}(t)$.
- If $\xi(t)$ is a point mass, $\mathcal{M}$ is simply $m\mathcal{I}$, where m is the mass.
- The action functional:

$$\mathcal{F}(\xi(\cdot)) \doteq \int_0^t \tfrac{1}{2}\dot{\xi}^T(r)\mathcal{M}\dot{\xi}(r) - V(\xi(r))\, dr.$$

- Hamilton hypothesized that conservative systems moved along paths that minimized the action functional.
- Feynman: "The average kinetic energy minus the average potential energy is minimized along the true path. Here for 'average', we can think of [...] the integral over time." (orig. Hamilton)

Action Functional

- Feynman: "In fact, it doesn't really have to be a minimum... the fundamental principle was that for any *first-order variation* away from the optical path, the *change* in time was zero."
- One seeks a **stationary** point of the action functional.
- This is the quantum viewpoint expanded to a larger domain. More later.
- Conservation of momentum and conservation of energy follow from stationarity of the action functional.
- The action functional (revised arguments):

$$\mathcal{F}(\xi(0), \dot{\xi}(\cdot)) \doteq \int_0^t \tfrac{1}{2}\dot{\xi}^T(r)\mathcal{M}\dot{\xi}(r) - V(\xi(r))\, dr.$$

Action Functional

- Write dynamics as $\dot{\xi}_r = u_r$ for $r \in (0, t)$, with initial condition $\xi(0) = x$.
- The action functional:

$$\mathcal{F}(x, u(\cdot)) \doteq \int_0^t \tfrac{1}{2} u^T(r) \mathcal{M} u(r) - V(\xi(r))\, dr.$$

- For short time durations, the stationary point is a minimum.
- Note second term has an integrator, which builds up over time, destroying convexity.
- If V sufficiently smooth, the Fréchet derivative with respect to $u \in L_2(0, t)$ has Riesz representation

$$[\mathcal{F}_u(x, u)](r) = \mathcal{M} u(r) - \int_r^t V_x(\xi(\rho))\, d\rho \qquad \text{a.e. } r \in (0, t).$$

(I.e., $F(x, u + \delta) - F(x, u) - \langle \mathcal{F}_u(x, u), \delta \rangle = o(\|\delta\|)$.)

  - Second derivative representation: $[\mathcal{F}_{uu}(x, u)](r, \rho) = m - \int_{r \vee \rho}^t V_{xx}(\xi(\sigma))\, d\sigma$ a.e.

Staticization

- Need to search for stationary (static) points of the action functional.
- Terminology: Staticization, statica (analogous to minimization, minima).
- For longer durations, min over u is replaced by stat over u.
- Let $\bar{y} \in \mathcal{G}_{\mathcal{Y}}$ where $\mathcal{G}_{\mathcal{Y}}$ is an open subset of a Hilbert space. We say

$$\bar{y} \in \operatorname*{argstat}_{y \in \mathcal{G}_{\mathcal{Y}}} F(y) \quad \text{if} \quad \limsup_{y \to \bar{y}, y \in \mathcal{G}_{\mathcal{Y}}} \frac{|F(y) - F(\bar{y})|}{|y - \bar{y}|} = 0,$$

and

$$\overline{\mathrm{stat}}_{y \in \mathcal{G}_{\mathcal{Y}}} F(y) \doteq \left\{ F(\bar{y}) \,\middle|\, \bar{y} \in \operatorname*{argstat}_{y \in \mathcal{G}_{\mathcal{Y}}} \{F(y)\} \right\}$$

if $\mathrm{argstat}\{F(y) \,|\, y \in \mathcal{G}_{\mathcal{Y}}\} \neq \emptyset$. If $\exists$ a s.t. $\overline{\mathrm{stat}}_{y \in \mathcal{G}_{\mathcal{Y}}} F(y) = \{a\}$, then

$$\operatorname*{stat}_{y \in \mathcal{G}_{\mathcal{Y}}} F(y) \doteq a.$$

- If f Fréchet differentiable and $\mathcal{G}_{\mathcal{Y}}$ is open,

$$\operatorname*{argstat}_{y \in \mathcal{G}_{\mathcal{Y}}} F(y) = \{y \in \mathcal{G}_{\mathcal{Y}} \,|\, F_y(y) = 0\}.$$

General Theory for Staticization

One can generate an entire theory for stat that is analogous to standard optimal control theory. We consider the action-functional case. General system:

$$\dot{\xi}_r = u_r, \quad \xi_0 = x \in \boldsymbol{R}^n$$
$$J(t,x,u,z) = \int_0^t \tfrac{1}{2} u_r^T \mathcal{M} u_r - V(\xi_r)\,dr + \psi(\xi_t, z),$$
$$W(t,x,z) \doteq \operatorname*{stat}_{u\in\mathcal{U}} J(t,x,u,z),$$

Dynamic Programming Principle:

Suppose the stationary value, $W(t,x,z)$ exists for $0 \le t \le T < \infty$ and $x, z \in \boldsymbol{R}^n$, and that there are stationary trajectories Hölder in x with constant greater than $1/2$. Then, for $s \in (0,t)$,

$$W(t,x,z) = \operatorname*{stat}_{u\in\mathcal{U}} \left\{ \int_0^s \tfrac{1}{2} u_r^T \mathcal{M} u_r - V(\xi_r)\,dr + W(t-s, \xi_s, z) \right\}.$$

HJ-Stat PDE Problem:

$$0 = \operatorname*{stat}_{v \in \mathbf{R}^m} \left\{ \tfrac{1}{2} v^T \mathcal{M} v - V(x) - W_r(r,x,z) + W_x(r,x,z) \cdot f(x,v) \right\},$$

for $(r,x) \in (0,t) \times \mathbf{R}^n$, and by the convexity,

$$0 = \min_{v \in \mathbf{R}^m} \left\{ \tfrac{1}{2} v^T \mathcal{M} v - V(x) - W_r(r,x,z) + W_x(r,x,z) \cdot f(x,v) \right\},$$

$$W(0,x,z) = \psi(x,z), \qquad x \in \mathbf{R}^n.$$

A Stat Representation Theorem:

Suppose $(0,t) \setminus \mathcal{A}$ consists only of isolated points. Suppose that for $s \in \mathcal{A}$, the stationary-action value exists for all $x \in \mathbf{R}^n$,and that the above Hölder continuity in x holds there. Then, the stationary-action value function satisfies the HJ-Stat PDE on $\mathcal{A} \times \mathbf{R}^n$.

Fundamental Solutions to TPBVPs

- Shorthand: TPBVP = two-point boundary value problem.
- The action functional approach will allow us to obtain a fundamental solution for classes of TPBVPs (e.g., n-body and wave equation).
- The fundamental solution may be computed offline.
- By *fundamental solution* for a TPBVP here, we mean an object, that once computed, allows for solution of TPBVPs for a variety of boundary data *without* re-solution/re-integration of the TPBVP problem.
- Given specific boundary conditions, staticization of the fundamental solution with an appropriate appended (terminal payoff) functional will generate the solution to the TPBVP of interest.

Action Functional Approach

- Formulate control problem. Dynamics:

$$\dot{\xi} = u, \qquad \xi(0) = x, \qquad u \in \mathcal{U} = L_2^{loc}$$

- Payoff/Value:

$$J^0(t,x,u) = \int_0^t -V(\xi(r)) + \tfrac{1}{2} u^T(r) \mathcal{M} u(r)\, dr,$$
$$W^0(t,x) \doteq \operatorname*{stat}_{u \in \mathcal{U}} J^0(t,x,u).$$

- HJB PDE (*forward* HJB PDE):

$$\begin{aligned}
0 &= -\frac{\partial}{\partial t} W(r,x) + \operatorname*{stat}_{v \in R^n} \left\{ v \cdot \nabla_x W(r,x) + \tfrac{1}{2} v^T \mathcal{M} v \right\} - V(x) \\
&= -\frac{\partial}{\partial t} W(r,x) + \inf_{v \in R^n} \left\{ v \cdot \nabla_x W(r,x) + \tfrac{1}{2} v^T \mathcal{M} v \right\} - V(x) \\
&= -\frac{\partial}{\partial t} W(r,x) - V(x) - \frac{1}{2} [\nabla_x W(r,x)]^T \mathcal{M}^{-1} \nabla_x W(r,x).
\end{aligned}$$

Action Functional Approach to TPBVPs

- Use terminal cost to effect boundary condition at termination.
- Payoff (now with terminal cost):

$$\bar{J}(t, x, u) = \int_0^t \tfrac{1}{2} u^T(r) \mathcal{M} u(r) - V(\xi(r))\, dr + \psi(\xi(t)).$$

- If $\psi(x) = -\bar{v}^T \mathcal{M} x$, we obtain boundary conditions (via either characteristic equations for HJB or Pontryagin):

$$\xi(0) = x, \quad p(t) = \nabla_x \psi(\xi(t)) \qquad \Rightarrow \qquad \xi(0) = x, \quad \dot{\xi}(t) = -\mathcal{M}^{-1} p(t) = \bar{v}.$$

- Solution of the control problem with this terminal cost yields solution of the desired TPBVP. $\dot{\xi}(0) = u(0)$ is required second initial condition.
- If $\psi(x) = \psi^\infty(x; z) = \delta_0^-(x - z)$ where

$$\delta_0^-(y) \doteq \begin{cases} 0 & \text{if } y = 0 \\ +\infty & \text{otherwise,} \end{cases}$$

we obtain boundary conditions:

$$\xi(0) = x, \qquad \xi(t) = z.$$

Fundamental Solutions (easier short-duration case)

- Using terminal cost to effect boundary condition at termination - short horizon case.
- Payoff and Value:

$$J^\infty(t,x,u;z) = \int_0^t \tfrac{1}{2}u^T(r)\mathcal{M}u(r) - V(\xi(r))\,dr + \psi^\infty(\xi(t),z)$$

$$W^\infty(t,x) = W^\infty(t,x,z) = \operatorname*{stat}_{u\in\mathcal{U}} J^\infty(t,x,u;z)) = \inf_{u\in\mathcal{U}} J^\infty(t,x,u;z).$$

- If replace ψ^∞ with $\bar\psi(x) = -\bar v^T\mathcal{M}x$, obtain value, $\bar W(t,x) = \mathrm{stat}_{u\in\mathcal{U}}\,\bar J(t,x,u)$,

$$\bar W(t,x) = \operatorname*{stat}_{u\in\mathcal{U}}\left\{\int_0^t \tfrac{1}{2}u^T(r)\mathcal{M}u(r) - V(\xi(r))\,dr + \bar\psi(\xi(t))\right\}$$

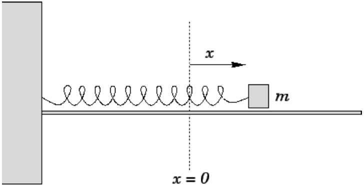
$$= \inf_{u\in\mathcal{U}}\inf_{z\in R^n}\left\{\int_0^t \tfrac{1}{2}u^T(r)\mathcal{M}u(r) - V(\xi(r))\,dr + \psi^\infty(\xi(t),z) + \bar\psi(z)\right\}$$

$$= \inf_{z\in R^n}\left\{W^\infty(t,x;z) + \bar\psi(z)\right\} = \int_{R^n}^{\oplus} W^\infty(t,x;z)\otimes\bar\psi(z).$$

- Min-plus convolution of W^∞ with various terminal costs yields solution of TPBVPs. W^∞ **is the fundamental solution.**

Motivational Example

- Simple problem: Mass, m, and spring-constant, K.
  Newton form: $\ddot{\xi} = -(K/m)\xi$.



- Two-point boundary value problem (TPBVP) from x to z in time $[s,t]$ with velocity/control $u \in \mathcal{U} \doteq \mathcal{L}_2^{loc}(0,\infty)$, is

$$J^\infty(t,x,u,z) = \int_0^t \frac{m}{2}u^2(r) - \frac{K}{2}\xi^2(r)\,dr + \psi^\infty(\xi(t),z),$$
$$\dot{\xi}(r) = u(r), \qquad \xi(0) = x,$$
$$\psi^\infty(x,z) \doteq \begin{cases} 0 & \text{if } x = z, \\ +\infty & \text{otherwise.} \end{cases}$$

- ψ^∞ forces solution to hit terminal state $\xi(t) = z$.
- One seeks $W^\infty(t,x;z) = \operatorname{stat}_{u\in\mathcal{U}} J^\infty(t,x,u,z)$.

Motivational Example

- The associated HJ PDE problem is

$$
\begin{aligned}
0 &= -\operatorname*{stat}_{v\in\boldsymbol{R}}\left[\tfrac{m}{2}v^2-\tfrac{K}{2}x^2+W_s(s,x)+vW_x(s,x)\right]\\
&= -\min_{v\in\boldsymbol{R}}\left[\tfrac{m}{2}v^2-\tfrac{K}{2}x^2+W_s(s,x)+vW_x(s,x)\right]\\
&= \tfrac{1}{2m}[W_x(s,x)]^2+\tfrac{K}{2}x^2-W_s(s,x) \qquad s\in(0,t),\ x\in\boldsymbol{R}\\
W(t,x) &= \psi^\infty(x,z) \qquad x\in\boldsymbol{R}.
\end{aligned}
$$

- Note the min even though problem is stat.
- Suggests optimal control $u^*(r)=\hat{u}^*(r,x)\doteq(-1/m)W_x(r,\xi^*(r))$.
- Try $W(t,x,z)=\frac{1}{2}\left[P(t)x^2+2Q(t)xz+R(t)z^2\right]$.
- Then the solution is given by

$$P(t)=R(t)=\cot(t), \quad Q(t)=-1/\sin(t).$$

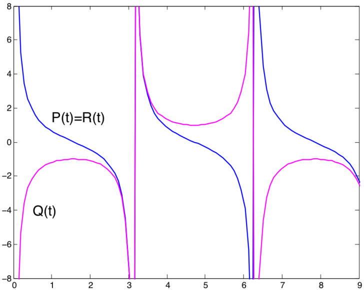
Motivational Example: Mass-Spring

- One finds velocity/control, u^*, and trajectory, ξ^*, given by (modulo sign errors)

$$u^*(r) = -[P(t-r)\xi^*(r) + Q(t-r)z] = \frac{-\cos(t-r)}{\sin(t-r)}\xi^*(r) + \frac{1}{\sin(t-r)}z,$$

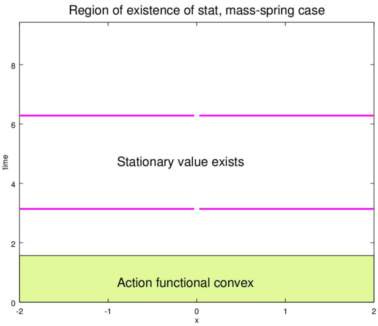
$$\xi^*(r) = x\cos(r) + \frac{z - x\cos(t)}{\sin(t)}\sin(r).$$

- Note that one loses convexity of J^∞ in u, and one must seek a staticum rather than a minimum.
- Asymptotes correspond to times when the quadratic in $u(\cdot)$ becomes purely linear in one direction.
- stat **propagates past asymptotes.**
- At $t = \pi$, either no solution or infinite number of solutions, depending on x, z.



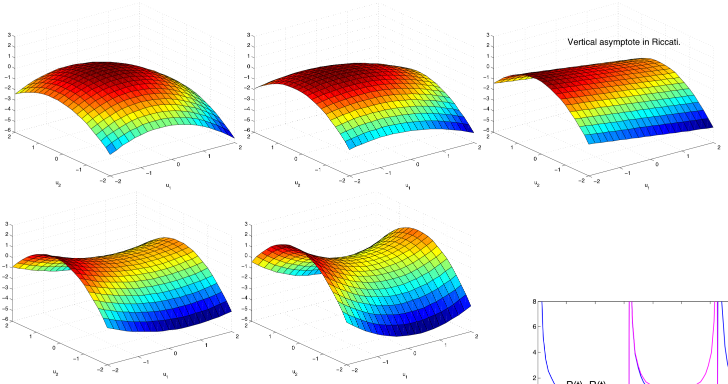
Mass-Spring Example

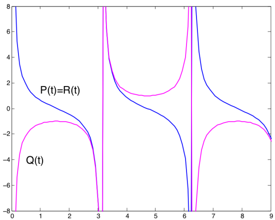
- Only have convexity of action for a short period.
- Stationary value exists except at isolated points.
- stat propagates past asymptotes/times where domain contracts.
- Time axis is vertical in this plot.



Mass-Spring Example

$$W^{\infty}(t,x;z) = \operatorname*{stat}_{u\in\mathcal{U}_{0,t}} J(t,x,u,z) = \left[x^T P_t^{\infty} x + 2z^T Q_t^{\infty} x + z^T R_t^{\infty} z\right] \quad \text{(when fini}$$



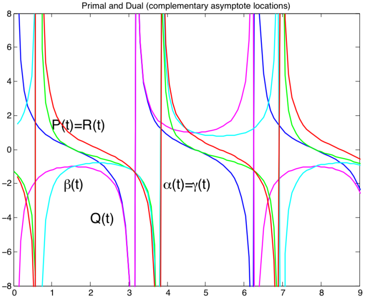


- Asymptotes correspond to times when the quadratic in $u(\cdot)$ becomes purely linear in one direction.
- Need to propagate stat past asymptotes.

Propagation Through Asymptotes

- Propagation through stat duality:
- Dual satisfies:

$$\begin{aligned}\dot{\alpha}_t &= -\alpha_t[D^{-1} + C^{-1}BC^{-1}]\alpha_t,\\ \dot{\beta}_t &= -\alpha_t[D^{-1} + C^{-1}BC^{-1}]\beta_t\\ &\quad + BC^{-1}\beta_t,\\ \dot{\gamma}_t &= -\beta_t^T[D^{-1} + C^{-1}BC^{-1}]\beta_t.\end{aligned}$$



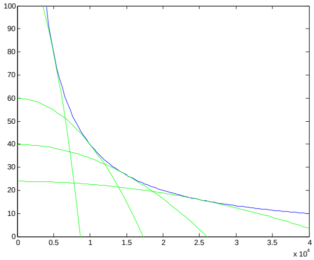
- Locations of asymptotes may be different between primal and dual.
- Propagation recipe:
  1. Propagate primal [dual] Riccati until approaching asymptote.
  2. Switch to dual [primal] Riccati until approaching dual [primal] asymptote, and return to step 1.
- (Symplectic methods provide alternative, of course.)

The n-body Problem (Fundamental Solutions)

- Recall classic gravitational potential for two bodies at x^i and x^j with masses m_i and m_j:

$$-V(x^i, x^j) = \frac{G m_i m_j}{|x^i - x^j|}.$$

- Inverse norm is difficult.



- Additive inverse of potential as optimized quadratic (with $\widehat{G} \doteq (3/2)^{3/2} G$).

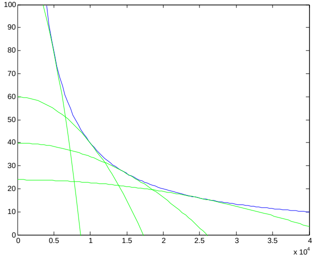
$$\frac{\widehat{G} m_i m_j}{|x^i - x^j|} = \widehat{G} \sup_{\alpha^{i,j} \in [0,\infty)} \left\{ m_i m_j \alpha^{i,j} \left[1 - \frac{(\alpha^{i,j} |x^i - x^j|)^2}{2} \right] \right\}.$$

- Total potential (for many bodies):

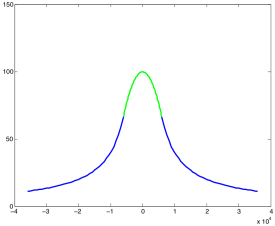
$$-V(x) = \sum_{(i,j) \in \mathcal{I}} \widehat{G} \sup_{\alpha^{i,j} \in [0,\infty)} \left\{ m_i m_j \alpha^{i,j} \left[1 - \frac{(\alpha^{i,j} |x^i - x^j|)^2}{2} \right] \right\}.$$

The n-body Problem

- Physical bodies have positive radius.
- This implies there exists maximum possible relevant α.
- One can also obtain an a priori bound on maximal separation of bodies, yielding minimum possible relevant α.
- Total potential:



$$-V(x) = \sum_{(i,j)\in\mathcal{I}} \widehat{G} \sup_{\alpha^{i,j}\in(\varepsilon_\alpha, \sqrt{2/3}\bar{\delta}^{-1})} \left\{ m_i m_j \alpha^{i,j} \left[1 - \frac{(\alpha^{i,j}|x^i - x^j|)^2}{2} \right] \right\}.$$



(Actually also valid for uniform density spherical body - inside and outside the object.)

The n-body Problem

- Total potential:

$$-V(x) = \sum_{(i,j)\in\mathcal{I}} \widehat{G} \sup_{\alpha^{i,j}\in(\varepsilon_\alpha,\sqrt{2/3}\bar{\delta}^{-1})} \left\{ m_i m_j \alpha^{i,j} \left[1 - \frac{(\alpha^{i,j}|x^i - x^j|)^2}{2} \right] \right\}$$
$$= \sup_{\alpha\in(\varepsilon_\alpha,\sqrt{2/3}\bar{\delta}^{-1})^{\#\mathcal{I}}} \left\{ \tfrac{1}{2} x^T \beta(\alpha) x + \lambda(\alpha) \right\} \qquad \text{(concave in } \alpha)$$

- Dynamics: $\dot{\xi} = u, \;\; \xi(0) = x, \;\; u \in \mathcal{U} = L_2^{loc}$.
- Action functional:

$$\bar{J}^\infty(t,x,u;z) = \int_0^t \tfrac{m}{2}|u(r)|^2 + \sup_{\alpha\in(\varepsilon_\alpha,\sqrt{2/3}\bar{\delta}^{-1})^{\#\mathcal{I}}} \left\{ \tfrac{1}{2}\xi^T(r)\beta(\alpha)\xi(r) + \lambda(\alpha) \right\} dr$$
$$+ \psi^\infty(\xi(t),z)$$
$$= \sup_{\alpha(\cdot)\in\mathcal{A}} \left\{ \int_0^t \tfrac{m}{2}|u(r)|^2 + \tfrac{1}{2}\xi^T(r)\beta(\alpha(r))\xi(r) + \lambda(\alpha(r))\, dr \right.$$
$$\left. + \psi^\infty(\xi(t),z) \right\} \qquad \text{(concave in } \alpha(\cdot))$$

The n-body Problem

- Action functional:

$$\overline{J}^{\infty}(t,x,u;z) = \sup_{\alpha\in\mathcal{A}} \left\{ \int_0^t \tfrac{m}{2}|u(r)|^2 + \tfrac{1}{2}\xi^T(r)\beta(\alpha(r))\xi(r) + \lambda(\alpha(r))\,dr \right.$$
$$\left. + \psi^{\infty}(\xi(t),z) \right\} \qquad \text{(concave in } \alpha(\cdot)\text{)}$$

- Value function (short horizon case):

$$\overline{W}^{\infty}(t,x;z) = \inf_{u\in\mathcal{U}} \sup_{\alpha\in\mathcal{A}} \left\{ \int_0^t \tfrac{m}{2}|u(r)|^2 + \tfrac{1}{2}\xi^T(r)\beta(\alpha(r))\xi(r) + \lambda(\alpha(r))\,dr \right.$$
$$\left. + \psi^{\infty}(\xi(t),z) \right\},$$

and letting α^* be the optimal α,

$$= \inf_{u\in\mathcal{U}} J^{\infty}(t,x,u,\alpha^*;z) \doteq \mathcal{W}^{\alpha^*,\infty}(t,x;z).$$

- If $t \leq \bar{t} = \bar{t}(\bar{\delta})$, then $J^{\infty}(t,x,\cdot,\alpha^*;z)$ is convex (i.e., in u).

The n-body Problem

- If $t \leq \bar{t} = \bar{t}(\bar{\delta})$, then $J^{\infty}(t, x, u, \alpha^{*}; z)$ is convex in u.
- $J^{\infty}(t, x, u, \alpha; z)$ is concave in α.
- Using above, one finds value function satisfies

$$\begin{aligned}\overline{W}^{\infty}(t, x; z) =& \inf_{u \in \mathcal{U}} \sup_{\alpha \in \mathcal{A}} \left\{ \int_0^t \tfrac{m}{2}|u(r)|^2 + \tfrac{1}{2}\xi^T(r)\beta(\alpha(r))\xi(r) + \lambda(\alpha(r))\, dr \right. \\ &\left. + \psi^{\infty}(\xi(t), z) \right\} \\ =& \inf_{u \in \mathcal{U}} \sup_{\alpha \in \mathcal{A}} J^{\infty}(t, x, u, \alpha; z) \\ =& \sup_{\alpha \in \mathcal{A}} \inf_{u \in \mathcal{U}} J^{\infty}(t, x, u, \alpha; z) \qquad \text{(surprising, work required)} \\ =& \sup_{\alpha \in \mathcal{A}} \mathcal{W}^{\alpha,\infty}(t, x; z).\end{aligned}$$

- For each $\alpha \in \mathcal{A}$, $\mathcal{W}^{\alpha,\infty}(t, x; z)$ is solution of an LQ control problem.

The n-body Fundamental Solution as a Set

- We have

$$\mathcal{W}^{\alpha,\infty}(t,x;z) = \frac{1}{2}\left[x^T P_t^\infty(\alpha)x + 2z^T Q_t^\infty(\alpha)x + z^T R_t^\infty(\alpha)z + r_t^\infty(\alpha)\right]$$

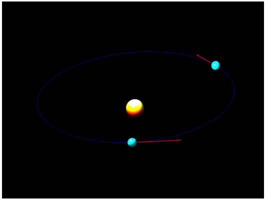
  where $P_t^\infty, Q_t^\infty, R_t^\infty$ are solutions of Riccati equations and r_t^∞ is a simple integral.
- Keep in mind $\alpha = \alpha(\cdot)$.
- Value is $\overline{W}^\infty(t,x;z) = \sup_{\alpha\in\mathcal{A}} \mathcal{W}^{\alpha,\infty}(t,x;z)$

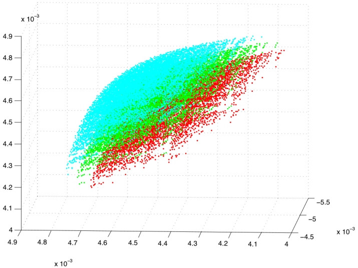
The n-body Problem Fundamental Solution as a Set

- The sets

$$\{P_t^\infty(\alpha), Q_t^\infty(\alpha), R_t^\infty(\alpha), r_t^\infty(\alpha) \,|\, \alpha \in \mathcal{A}\}$$

  represent the fundamental solution of n-body TPBVPs.
- Each quadruple obtained by Riccatis.
- Sets are indexed by the body masses and the length of the time interval.
- A two-body fundamental solution is depicted in figure.

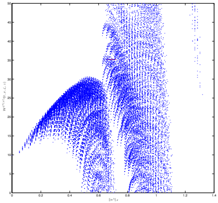


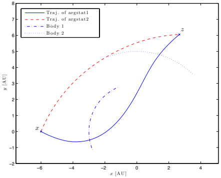


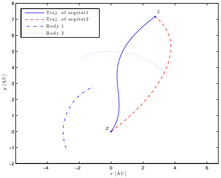
- TPBVPs are converted to initial value problems via a max-plus convolution of the fundamental solution with the appropriate terminal cost (as in previous example).

Orbital Mechanics Application

- Special case of a small body moving among large bodies in known orbits.
- One constructs fundamental solution as a finite-dimensional set (similar to set in n-body case).
- The same fundamental solution set may be applied to different TPBVPs, with different x, z points.
- For each problem, multiple solutions of the TPBVP were found.



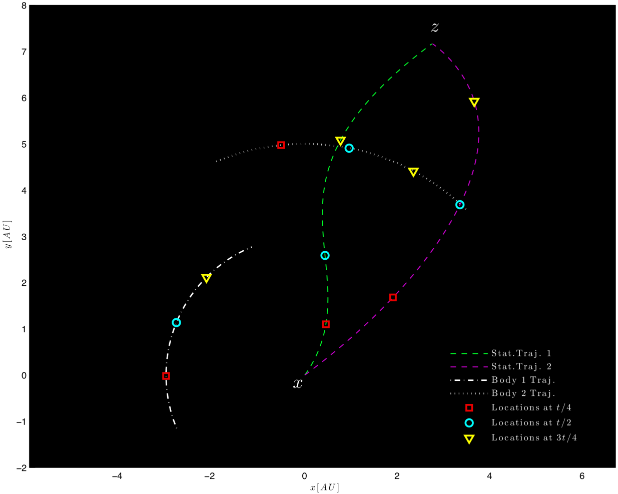




- First plot is projection of fundamental solution; second and third display multiple solutions of the TPBVPs.

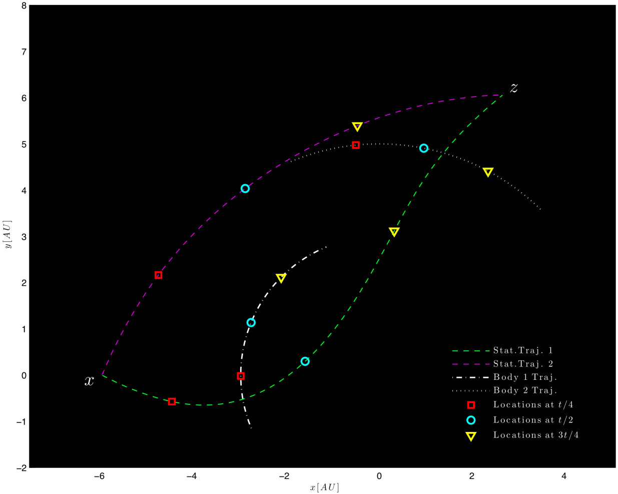
Orbital Mechanics Application

- Application of the TPBVP fundamental solution to a single small object moving among two large bodies.
- Boundary data is initial and terminal position.
- Fundmntl. solution specific to masses and duration.
- Multiple solutions unexpected.



Orbital Mechanics Application

- Application of the TPBVP fundamental solution to a single small object moving among two large bodies.
- SAME fundamental solution applies to multiple TPBVPs.
- Fundmntl. solution specific to masses and duration.
- Multiple solutions unexpected.



Orbital Mechanics Application

- Dynamics movie not included.

Orbital Mechanics Application

- Dynamics movie not included.