POD-Based Model Predictive Control with control and state constraints

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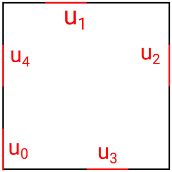
Outline

Assumptions and Notations



Let be:

- $\Omega \subset \mathbb{R}^d$ a bounded domain with Lipschitz-continuous boundary $\Gamma = \Gamma_c \cup \Gamma_{out}$
- $Q = (0, T) \times \Omega$, $\Sigma_c = (0, T) \times \Gamma_c$ and $\Sigma_{out} = (0, T) \times \Gamma_{out}$ for $T > 0$
- $u = (u_i)_{1 \le i \le m} \in \mathscr{U} = L^2(0, T; \mathbb{R}^m)$ a control function
- $b_i : \Gamma_c \to \mathbb{R}$, $1 \le i \le m$ given control shape functions
- $V = H^1(\Omega)$ and $H = L^2(\Omega)$
- $W(0, T) = \left\{\varphi \in L^2(0, T; V)) \,\middle|\, \varphi_t \in L^2(0, T; V')\right\}$.

State Equation

Then, for any control u in

$$\mathscr{U}_{\text{ad}} = \left\{ u \in \mathscr{U} \;\middle|\; u_{\text{a}i}(t) \le u_i(t) \le u_{\text{b}i}(t) \text{ for } i = 1, \ldots, m \text{ and } t \in [0, T] \right\}$$

the state $y \in W(0, T)$ is governed by the following Convection-Diffusion Equation:

$$\begin{aligned}
y_t(t, \mathbf{x}) - \lambda \Delta y(t, \mathbf{x}) + \mathbf{v}(t, \mathbf{x}) \cdot \nabla y(t, \mathbf{x}) &= f(t, \mathbf{x}), && \text{in } Q, \\
\lambda \frac{\partial y}{\partial \mathbf{n}}(t, \mathbf{s}) + \gamma_{\text{c}} y(t, \mathbf{s}) &= \gamma_{\text{c}} \sum_{i=1}^{m} u_i(t) b_i(\mathbf{s}), && \text{on } \Sigma_{\text{c}}, \\
\lambda \frac{\partial y}{\partial \mathbf{n}}(t, \mathbf{s}) + \gamma_{\text{out}} y(t, \mathbf{s}) &= \gamma_{\text{out}} y_{\text{out}}(t), && \text{on } \Sigma_{\text{out}}, \\
y(0, \mathbf{x}) &= y_\circ(\mathbf{x}), && \text{in } \Omega
\end{aligned} \tag{1}$$

with scalar parameters $\gamma_{\text{out}} \ge 0$ and $\gamma_{\text{c}}, \lambda > 0$.

Weak Formulation of State Equation

We can write the State Equation (1) in the following weak formulation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle y(t),\varphi\rangle_H + a(t;y(t),\varphi) = \langle \mathcal{F}(t)+\gamma_\mathsf{c}\mathcal{B}(u(t)),\varphi\rangle_{V',V}$$

$$y(0) = y_\circ \qquad \text{in } H \tag{2}$$

for all $\varphi \in V$ a.e. in $(0, T]$, where:

$$a(t;\varphi,\phi) = \int_\Omega \lambda\nabla\varphi\cdot\nabla\phi + \big(\mathbf{v}(t)\cdot\nabla\varphi\big)\phi\,\mathrm{d}\mathbf{x} + \gamma_\mathsf{c}\int_{\Gamma_\mathsf{c}}\varphi\phi\,\mathrm{d}\mathbf{s} + \gamma_\mathsf{out}\int_{\Gamma_\mathsf{out}}\varphi\phi\,\mathrm{d}\mathbf{s}$$

$$\langle \mathcal{F}(t),\varphi\rangle_{V',V} = \int_\Omega f(t)\varphi\,\mathrm{d}\mathbf{x} + \gamma_\mathsf{out}y_\mathsf{out}(t)\int_{\Gamma_\mathsf{out}}\varphi\,\mathrm{d}\mathbf{s}$$

$$\langle \mathcal{B}u,\varphi\rangle_{V',V} = \sum_{i=1}^m u_i\int_{\Gamma_\mathsf{c}} b_i\varphi\,\mathrm{d}\mathbf{s}$$

for all $\varphi,\phi \in V$ a.e. in $(0, T]$.

State Constraints

We set $\mathscr{W} = L^2(0, T; H) \simeq L^2(Q)$. We deal with pointwise state constraints of the following type:

$$y_\mathsf{a}(t, \mathbf{x}) \leq y(t, \mathbf{x}) \leq y_\mathsf{b}(t, \mathbf{x}) \quad \text{a.e. in } Q, \tag{3}$$

where $y_a, y_b \in \mathscr{W}$ are given. To gain regular Lagrange multipliers we utilize a Lavrentiev regularization. Let $\varepsilon > 0$ be chosen and $w \in \mathscr{W}$ an additional (artificial) control. Then, (3) is replaced by the mixed constraints

$$y_\mathsf{a}(t, \mathbf{x}) \leq y(t, \mathbf{x}) + \varepsilon w(t, \mathbf{x}) \leq y_\mathsf{b}(t, \mathbf{x}) \quad \text{a.e. in } Q.$$

Optimal Control Problem

The quadratic cost functional $J : \mathscr{X} \to \mathbb{R}$ is given by

$$
\begin{aligned}
J(x) = \frac{\sigma_Q}{2} \int_0^T \|y(t) - y_Q(t)\|_H^2 \,\mathrm{d}t + \frac{\sigma_T}{2} \|y(T) - y_T\|_H^2 \\
+ \frac{1}{2} \sum_{i=1}^m \sigma_i \|u_i\|_{L^2(0,T)}^2 + \frac{\sigma_w}{2} \|w\|_{\mathscr{W}}^2
\end{aligned}
$$

for $x = (y, u, w) \in \mathscr{X} = W(0, T) \times \mathscr{U} \times \mathscr{W}$, where $y_Q \in L^2(0, T; H)$ and $y_T \in H$ are desired states. Furthermore, σ_Q, $\sigma_T \geq 0$, and $\sigma_1, \ldots, \sigma_m, \sigma_w > 0$. The optimal control problem is given by

$$
\min J(x) \quad \text{s.t.} \quad x \in \mathscr{X}_{\mathrm{ad}} \text{ and } y \text{ solves (1)} . \qquad \textbf{(P)}
$$

with

$$
\mathscr{X}_{\mathrm{ad}} = \big\{(x = (y, u, w) \in \mathscr{X} \;\big|\; y_{\mathrm{a}} \leq y + \varepsilon w \leq y_{\mathrm{b}} \text{ and } u \in \mathscr{U}_{\mathrm{ad}}\big\}.
$$

Theorem (Tröltzsch '10, Gubisch-Volkwein '14)

Suppose that the feasible set $\mathscr{X}_{\mathsf{ad}} \neq \emptyset$ and that $\bar{x} = (\bar{y}, \bar{u}, \bar{w}) \in \mathscr{X}_{\mathsf{ad}}$ is the solution to $(\mathbf{P})$*. Then, there exist unique Lagrange multipliers $\bar{p} \in W(0,T)$ and $\bar{\beta} \in \mathscr{W}$, $\bar{\mu} = (\bar{\mu}_i)_{1 \le i \le m} \in \mathscr{U}$ satisfying the dual equations*

$$
\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t} \langle \bar{p}(t), \varphi \rangle_H + a(t; \varphi, \bar{p}(t)) + \langle \bar{\beta}(t), \varphi \rangle_H &= \sigma_Q \langle (y_Q - \bar{y})(t), \varphi \rangle_H, \\
\bar{p}(T) &= \sigma_T \big(y_T - \bar{y}(T) \big)
\end{aligned}
\tag{4}
$$

for all $\varphi \in V$ and a.e. in $[0,T]$ and the optimality system

$$
\begin{aligned}
\sigma_i \bar{u}_i - \int_{\Gamma_c} b_i \bar{p} \, \mathrm{d}\mathbf{s} + \bar{\mu}_i = 0 &\quad \text{in } L^2(0,T) \text{ for } i = 1, \ldots, m, \\
\sigma_w \bar{w} + \varepsilon \bar{\beta} = 0 &\quad \text{in } \mathscr{W}.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
\bar{\beta} &= \max\big\{0, \bar{\beta} + \gamma(\bar{y} + \varepsilon \bar{w} - y_{\mathsf{b}})\big\} + \min\big\{0, \bar{\beta} + \gamma(\bar{y} + \varepsilon \bar{w} - y_{\mathsf{a}})\big\}, \\
\bar{\mu}_i &= \max\big\{0, \bar{\mu}_i + \gamma_i(\bar{u}_i - u_{\mathsf{b}i})\big\} + \min\big\{0, \bar{\mu}_i + \gamma_i(\bar{u}_i - u_{\mathsf{a}i})\big\}
\end{aligned}
$$

for $i = 1, \ldots, m$ and for arbitrarily chosen $\gamma, \gamma_1, \ldots, \gamma_m > 0$.

Starting from the previous Theorem and choosing $\gamma_i = \sigma_i > 0$, $i = 1, \ldots, m$ and $\gamma = \sigma_w/\varepsilon^2 > 0$, after several computations, we found that:

$$\begin{aligned} 0 = \int_{\Gamma_c} b_i \bar{p}\, \mathrm{d}\mathbf{s} - \sigma_i \bar{u}_i - \max\left\{0, \int_{\Gamma_c} b_i \bar{p}\, \mathrm{d}\mathbf{s} - \sigma_i u_{bi}\right\} \\ - \min\left\{0, \int_{\Gamma_c} b_i \bar{p}\, \mathrm{d}\mathbf{s} - \sigma_i u_{ai}\right\} \end{aligned} \tag{5}$$

a.e. in $[0, T]$ for $i = 1, \ldots, m$.

$$0 = -\frac{\sigma_w}{\varepsilon}\,\bar{w} - \max\left\{0, \frac{\sigma_w}{\varepsilon^2}(\bar{y} - y_b)\right\} - \min\left\{0, \frac{\sigma_w}{\varepsilon^2}(\bar{y} - y_a)\right\} \tag{6}$$

a.e. in Q.

For $i = 1, \ldots, m$ we introduce the mappings $\mathcal{H}_i : \mathscr{Z} = \mathscr{U} \times \mathscr{W} \to \mathscr{U}$ and $\mathcal{H} : \mathscr{Z} \to \mathscr{W}$ by

$$\mathcal{H}_i(z) = \int_{\Gamma_\mathsf{c}} b_i p(z)\,\mathrm{d}\mathbf{s} - \sigma_i u_i - \max\left\{0, \int_{\Gamma_\mathsf{c}} b_i p(z)\,\mathrm{d}\mathbf{s} - \sigma_i u_{\mathsf{b}i}\right\}$$

$$- \min\left\{0, \int_{\Gamma_\mathsf{c}} b_i p(z)\,\mathrm{d}\mathbf{s} - \sigma_i u_{\mathsf{a}i}\right\} \quad \text{a.e. in } [0, T],$$

$$\mathcal{H}(z) = -\frac{\sigma_w}{\varepsilon}\, w - \max\left\{0, \frac{\sigma_w}{\varepsilon^2}(y(z) - y_\mathsf{b})\right\} - \min\left\{0, \frac{\sigma_w}{\varepsilon^2}(y(z) - y_\mathsf{a})\right\}$$

a.e. in Q, for $z = (u, w) \in \mathscr{Z}$ with $y(z)$ solution of the State Equation and $p(z)$ solution of the Adjoint Equation. Then, we set

$$\mathcal{G} = \big((\mathcal{H}_1, \ldots, \mathcal{H}_m), \mathcal{H}\big)^\top : \mathscr{Z} \to \mathscr{Z}.$$

Now the nonsmooth operator equations (5) and (6) become

$$\mathcal{G}(\bar{z}) = 0 \quad \text{in } \mathscr{Z}.$$

PDASS Algorithm

a) We apply the semi-smooth Newton Method, which is locally superlinearly convergent (Hintermüller,Ito,Kunisch '02),

b) The next iterate is given by the solution of

$$\begin{pmatrix} \mathcal{A}_{11}^k & \mathcal{A}_{12}^k \\ \mathcal{A}_{21}^k & \mathcal{A}_{22}^k \end{pmatrix} \begin{pmatrix} y^{k+1} \\ p^{k+1} \end{pmatrix} = \begin{pmatrix} \mathcal{Q}_1(z^k; u_\mathrm{a}, u_\mathrm{b}, b_i, f, y_\mathrm{out}) \\ \mathcal{Q}_2(z^k; y_Q, y_\mathrm{a}, y_\mathrm{b}, \varepsilon, \sigma_w) \end{pmatrix} \tag{7}$$

which allows to compute controls and Lagrange multipliers afterwards.

c)
1: Choose starting value $z^0 = (u^0, w^0) \in \mathscr{Z}_\mathrm{ad}$ and set $k = 0$;
2: Determine y^0 and p^0 by solving the state and the adjoint equations;
3: **repeat**
4: Determine $\mathscr{A}_{\mathrm{a}i}^{\mathscr{U}}(z^k)$, $\mathscr{A}_{\mathrm{b}i}^{\mathscr{U}}(z^k)$, $\mathscr{I}_i^{\mathscr{U}}$ for $i = 1, \ldots, m$ and $\mathscr{A}_\mathrm{a}^{\mathscr{W}}(z^k)$, $\mathscr{A}_\mathrm{b}^{\mathscr{W}}(z^k)$, $\mathscr{I}^{\mathscr{W}}$;
5: Compute the solution (y^{k+1}, p^{k+1}) by solving (7);
6: Compute $z^{k+1} = (u^{k+1}, w^{k+1}) \in \mathscr{Z}_\mathrm{ad}$ and set $k = k + 1$;
7: **until** stopping criterium is satisfied.

Why POD?

Observation

Since we use Finite Elements Method (with N_x nodes) and Implicit Euleur Scheme (with N_t time steps) for solving the State and Adjoint Equations, the previous linear system is $(2N_xN_t) \times (2N_xN_t)$, that can be huge!

In order to speed-up the computation, we use Proper Orthogonal Decomposition as in, for example, Kunisch-Volkwein '01, Gubisch-Volkwein '13, Alla-Hinze '15, ...
In this way, we obtain a *reduced system* $(2\ell N_t) \times (2\ell N_t)$, where $\ell << N_x$, so we gain a good speed-up in terms of computational time for solving the linear systems.

How does POD work for an Euclidian Space X?

- **Setting:** $X = \mathbb{R}^N$, $\{y_j\}_{j=0}^M \subset \mathbb{R}^N$, $Y = [y_0] \dots [y_M] \in \mathbb{R}^{N\times(M+1)}$
- **Inner Product:** $\langle \psi, \widetilde{\psi} \rangle_X = \langle \psi, \widetilde{\psi} \rangle_W = \psi^T W \widetilde{\psi}$ with $W = W^T > 0$
- **POD:** for any $\ell \in \{1, \dots, L\}$ solve

$$\min_{\{\psi_i\}_{i=1}^{\ell} \subset \mathbb{R}^N} \sum_{j=0}^{M} \alpha_j \left\| y_j - \sum_{i=1}^{\ell} \langle y_j, \psi_i \rangle_W \psi_i \right\|_W^2$$

  subject to $\langle \psi_i, \psi_j \rangle_W = \delta_{ij}$ for $1 \le i, j \le \ell$.
- **Equivalent Eigenvalue Problem:** Define $D = \operatorname{diag}(\alpha_0, \dots, \alpha_M) \in \mathbb{R}^{(M+1)\times(M+1)}$ and $\overline{Y} = W^{1/2} Y D^{1/2}$ and solve

$$\overline{Y}^T \overline{Y} \widetilde{\psi}_i = \widetilde{\lambda}_i \widetilde{\psi}_i,\ \widetilde{\lambda_1} \ge \widetilde{\lambda_2} \ge \dots \ge \widetilde{\lambda}_\ell > 0$$

  and set $\psi_i = \sqrt{\widetilde{\lambda}_i}^{-1} Y D^{1/2} \widetilde{\psi}_i$ for $i = 1, \dots, \ell$.

In our case, we can extend the previous strategy to our problem in an Hilbert Space:

- We take the snapshots for y and p from the step 2 of PDASS Algorithm, so $M+1 = 2N_t$ and $N = N_x$.
- We set $W_{ij} = \langle \phi_i, \phi_j \rangle_V$ with ϕ_i Finite Element basis functions for $i, j = 0, \dots, N_x - 1$.
- We choose $\alpha_0 = \alpha_M = \Delta t/2$ and $\alpha_j = \Delta t$ for $j = 1, \dots, M-1$
- We solve the Eigenvalue Problem in order to have the POD Basis matrix $\Psi = [\psi_1] \dots [\psi_\ell] \in \mathbb{R}^{N_x \times \ell}$
- We project the Optimality System (7) in to the POD subspace.

Example

If we want to project a Finite Element matrix K and a vector $Q \in \mathbb{R}^{N_x}$ in to the POD subspace: $K^\psi = \Psi^T K \Psi$ and $Q^\psi = \Psi^T W Q$

Why MPC?

- We would like to keep the temperature in the room as long as possible inside the constraints range,
- It can be easily applied to a wide range of optimization problems,
- MPC can react to the changes of variables and parameters, i.e. air flow, outside temperature, etc.,
- It is an online optimization method

Reference: *Nonlinear Model Predictive Control:Theory and Algorithms*, L. Grüne - J. Pannek, 2011, Springer.

MPC and POD

Ideas:

- **Method 1:** We can compute the snapshots for the POD Basis with starting arbitrary values (u^0, w^0).
- **Method 2:** We can solve the first open-loop optimization problem with FE and take from that the snapshots for the POD Basis.
- **Method 3:** We can do Method 2 and refresh the POD Basis with an open-loop FE solves after a fixed (or random) number of MPC steps.
- **Method 4:** We can do Method 2 and refresh the POD Basis with an open-loop FE solves according to a posteriori error estimator. (Still in progress)

Outline

State Equation and Cost Functional

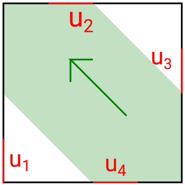
State Equation:

$$
\begin{aligned}
y_t(t,\mathbf{x}) - \lambda \Delta y(t,\mathbf{x}) + \mathbf{v}(t,\mathbf{x}) \cdot \nabla y(t,\mathbf{x}) &= f(t,\mathbf{x}), && \text{in } Q,\\
\lambda \frac{\partial y}{\partial \mathbf{n}}(t,\mathbf{s}) + \gamma_{\mathsf{c}} y(t,\mathbf{s}) &= \gamma_{\mathsf{c}} \sum_{i=1}^{m} u_i(t) b_i(\mathbf{s}), && \text{on } \Sigma_{\mathsf{c}},\\
\lambda \frac{\partial y}{\partial \mathbf{n}}(t,\mathbf{s}) + \gamma_{\mathsf{out}} y(t,\mathbf{s}) &= \gamma_{\mathsf{out}} y_{\mathsf{out}}(t), && \text{on } \Sigma_{\mathsf{out}},\\
y(0,\mathbf{x}) &= y_\circ(\mathbf{x}), && \text{in } \Omega
\end{aligned}
$$

Cost Functional:

$$
\begin{aligned}
J(x) = {} & \frac{\sigma_Q}{2} \int_0^T \|y(t) - y_Q(t)\|_H^2 \, \mathrm{d}t + \frac{\sigma_T}{2} \|y(T) - y_T\|_H^2 \\
& + \frac{1}{2} \sum_{i=1}^{m} \sigma_i \|u_i\|_{L^2(0,T)}^2 + \frac{\sigma_w}{2} \|w\|_{\mathscr{W}}^2
\end{aligned}
$$

with $x \in \mathscr{X}_{\mathsf{ad}} = \left\{ x = (y,u,w) \middle| \; y_{\mathsf{a}} \le y + \varepsilon w \le y_{\mathsf{b}}, u_{\mathsf{a}i} \le u_i \le u_{\mathsf{b}i} \right\}$.



$\Omega = [0,1] \times [0,1],\ T = 1.0,\ m = 4$ controls,
$\Gamma_{c1} = \{x_1 = 0, x_2 \in [0, 0.25]\}$,
$\Gamma_{c2} = \{x_1 \in [0.25, 0.5], x_2 = 1.0\}$,
$\Gamma_{c3} = \{x_1 = 1.0, x_2 \in [0.5, 0.75]\}$,
$\Gamma_{c4} = \{x_1 \in [0.5, 0.75], x_2 = 0\}$,
Grid 25×25, $T_{OL} = 0.5$, $\Delta t = 0.01$,

$\lambda = 1.0,\ \gamma_c = 1.0,\ \gamma_{out} = 0.03,\ y_{out}(t) = 1.0,\ f(t,x) = 0,$
$\mathbf{v}(t,\mathbf{x}) = (v_1(t,\mathbf{x}), v_2(t,\mathbf{x}))$ with:

$$v_1 = \begin{cases} -1.0 \text{ if } t \leq 0.5, \mathbf{x} \in \mathcal{V}_{\mathcal{F}}, \\ -0.6 \text{ if } t > 0.5, \mathbf{x} \in \mathcal{V}_{\mathcal{F}}, \\ 0 \text{ otherwise} \end{cases} v_2 = \begin{cases} 1.0 \text{ if } t \leq 0.5, \mathbf{x} \in \mathcal{V}_{\mathcal{F}}, \\ 0.6 \text{ if } t > 0.5, \mathbf{x} \in \mathcal{V}_{\mathcal{F}}, \\ 0 \text{ otherwise} \end{cases}$$

with $\mathcal{V}_{\mathcal{F}} = \{\mathbf{x} = (x_1, x_2) | x_1 + x_2 \geq 0.5, x_1 + x_2 \leq 1.5\}$.

Data for Test 1 II

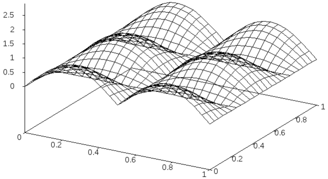


Figure 1 : $y_0(\mathbf{x}) = |\sin(2\pi x_1) cos(2\pi x_2)|$

$y_Q(t, \mathbf{x}) = \min(2.0 + t, 3.0)$, $y_T(\mathbf{x}) = 3.0$
$\sigma_Q = 10^{-4}$, $\sigma_T = 10^{-4}$, $\sigma_w = 1$, $\sigma_i = 5 \times 10^{-3}$
$y_a(t) = 0.5 + \min(1.5t, 1.5)$, $y_b = 2.5$, $\varepsilon = 0.1$, $u_{ai} = 0$, $u_{bi} = 10.0$,
$u_i^0 = 5.0$

Test 1: Results

Scheme	POD-B.	J	$\|\varepsilon w\|$	Time	Speed-up
FE	·	0.03288	0.01319	6203 s	·
POD-M1	4	0.03448	0.01338	1127 s	5.34
POD-M1	12	0.03327	0.01304	1099 s	5.64
POD-M1	24	0.03297	0.01301	1255 s	4.94
POD-M2	4	0.04494	0.02072	1014 s	6.12
POD-M2	12	0.03744	0.01660	1149 s	5.39
POD-M2	24	0.03693	0.01626	1223 s	5.07
POD-M3	4	0.04056	0.01854	1548 s	4,01
POD-M3	12	0.03506	0.01494	1574 s	3.94
POD-M3	24	0.03445	0.01449	1811 s	3.42

Remark

We have: $\varepsilon w = (y_\mathrm{a} - y)\,\chi_{\mathscr{A}_\mathrm{a}^\mathscr{W}(z)} + (y_\mathrm{b} - y)\,\chi_{\mathscr{A}_\mathrm{b}^\mathscr{W}(z)}.$

Test 1: Results II

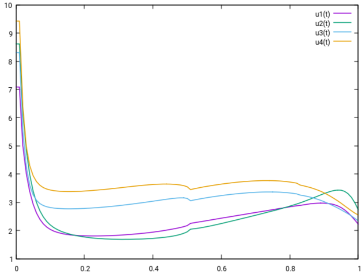
Scheme	POD-B.	err-rel(T)	err-rel	$\|u^{FE} - u^{POD}\|$
POD-M1	4	0.01026	0.04026	2.23370
POD-M1	12	0.00661	0.01804	1.18958
POD-M1	24	0.00477	0.00773	0.52521
POD-M2	4	0.02188	0.04899	1.59847
POD-M2	12	0.01935	0.02153	1.20778
POD-M2	24	0.01568	0.02087	1.17320
POD-M3	4	0.01540	0.02576	1.24263
POD-M3	12	0.00743	0.01237	1.01840
POD-M3	24	0.00383	0.00967	0.91781

where:

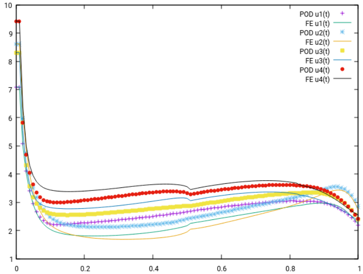
err-rel(T) $= \|y^{FE}(T) - y^{POD}(T)\| / \|y^{FE}(T)\|$,

err-rel$=\|y^{FE} - y^{POD}\| / \|y^{FE}\|$,

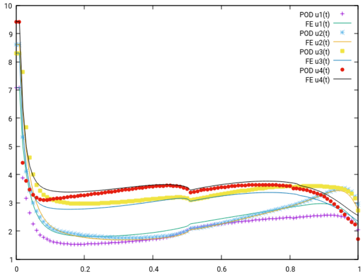
$\|u^{FE} - u^{POD}\| = \sum_{i=1}^{m} \|u_i^{FE} - u_i^{POD}\|_{L^2(0,T)}$



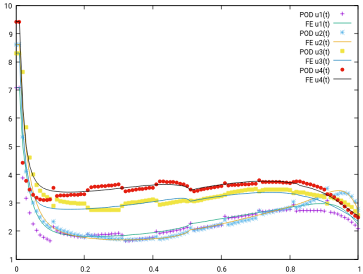
(a) FE



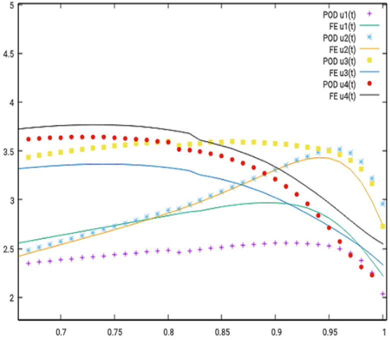
(b) POD-M1-12 Basis



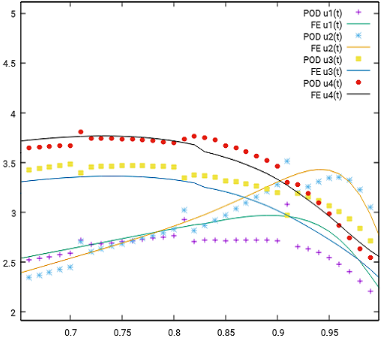
(c) POD-M2-12Basis



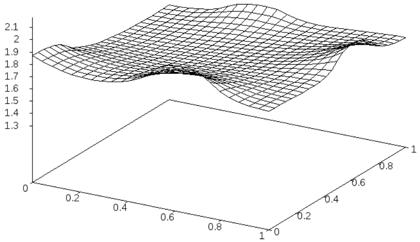
(d) POD-M3-12Basis



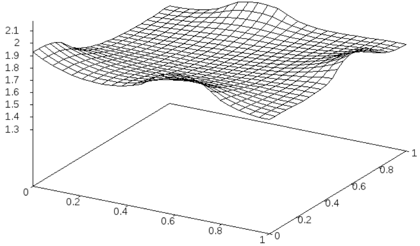
(e) POD-Method 2, 12 Basis



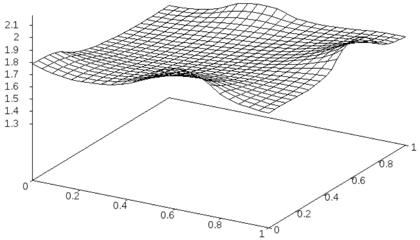
(f) POD-Method 3, 12 Basis



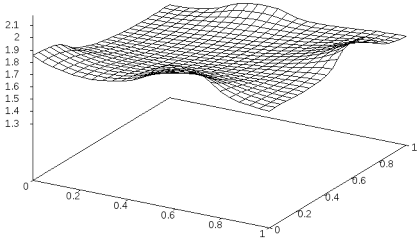
(a) FE, $t = 0.75$



(b) POD-M1-12Basis, $t = 0.75$



(c) POD-M2-12Basis, $t = 0.75$



(d) POD-M3-12Basis, $t = 0.75$

New Data for Test 2

All the data are the same of Test 1, except the ones in blue:

$y_Q(t, \mathbf{x}) = \min(2.0 + t, 3.0)$, $y_T(\mathbf{x}) = 3.0$
$\sigma_Q = 1$, $\sigma_T = 1$, $\sigma_w = 10^{-4}$, $\sigma_i = 5 \times 10^{-3}$
$y_a(t) = 0.5 + \min(1.5t, 1.5)$, $y_b = 2.5$, $\varepsilon = 0.1$, $u_{\mathrm{a}i} = 0$, $u_{\mathrm{b}i} = 7.0$,
$u_i^0 = 6.0$

Test 2: Results

Scheme	POD-B.	J	Time	Speed-up
FE	·	0.22034	4354 s	·
POD-M1	14	0.69116	871 s	5.00
POD-M1	24	0.22051	722 s	6.03
POD-M2	7	0.73813	802 s	5.43
POD-M2	14	0.22299	1095 s	3.98
POD-M2	24	0.22123	1157 s	3.76
POD-M3	7	0.73813	1024 s	4.25
POD-M3	14	0.22196	1414 s	3.08
POD-M3	24	0.22077	1515 s	2.87

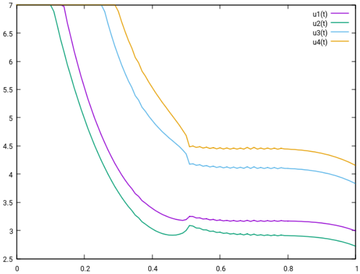
Test 2: Results II

Scheme	POD-B.	$\|y(T) - y_T\|$	$\|y - y_Q\|$	$\|\varepsilon w\|$
FE	·	0.15441	0.54679	0.15403
POD-M1	14	0.65252	0.90923	0.01295
POD-M1	24	0.15479	0.54706	0.15383
POD-M2	7	0.69531	0.92961	0.01287
POD-M2	14	0.16598	0.54932	0.14621
POD-M2	24	0.15841	0.54763	0.15104
POD-M3	7	0.69532	0.92972	0.01286
POD-M3	14	0.16190	0.54821	0.14927
POD-M3	24	0.15638	0.54720	0.15275

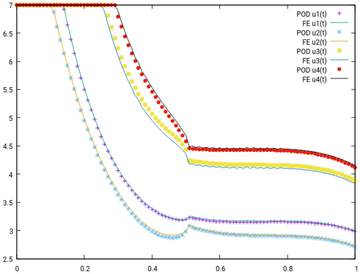
Test 2: Results III

Scheme	POD-B.	err-rel(T)	err-rel	$\|u^{\text{FE}} - u^{\text{POD}}\|$
POD-M1	14	0.17912	0.25349	15.07059
POD-M1	24	0.00100	0.00158	0.16298
POD-M2	7	0.19448	0.26201	15.22679
POD-M2	14	0.00473	0.00582	0.27088
POD-M2	24	0.00227	0.00300	0.22767
POD-M3	7	0.19437	0.26103	15.2267
POD-M3	14	0.00310	0.00376	0.27054
POD-M3	24	0.00076	0.00088	0.04767

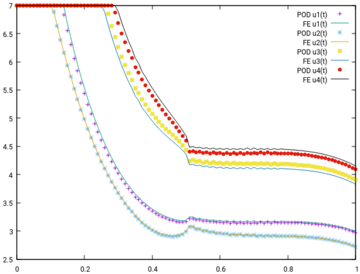
where:
err-rel$(T) = \|y^{\text{FE}}(T) - y^{\text{POD}}(T)\| / \|y^{\text{FE}}(T)\|$,
err-rel$=\|y^{\text{FE}} - y^{\text{POD}}\| / \|y^{\text{FE}}\|$,
$\|u^{\text{FE}} - u^{\text{POD}}\| = \sum_{i=1}^{m} \|u_i^{\text{FE}} - u_i^{\text{POD}}\|_{L^2(0,T)}$



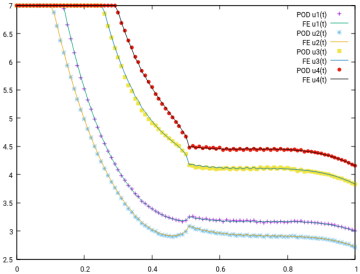
(a) FE



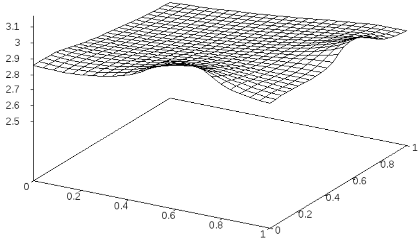
(b) POD-M1-24Basis



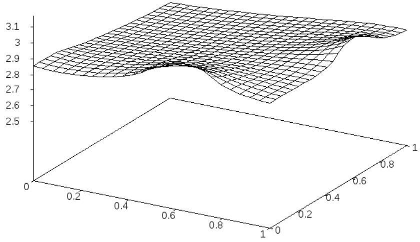
(c) POD-M2-24Basis



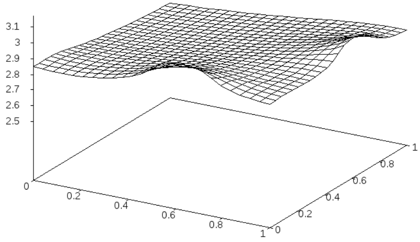
(d) POD-M3-24Basis



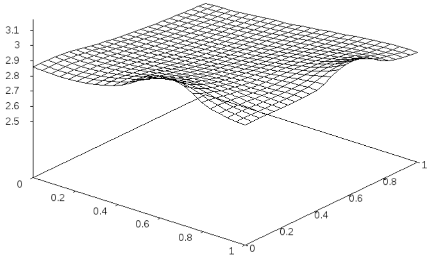
(a) FE, $t = 1.0$



(b) POD-M1-24Basis, $t = 1.0$



(c) POD-M2-24Basis, $t = 1.0$



(d) POD-M3-24Basis, $t = 1.0$

Conclusions and Outlook

Conclusions:

- We can replace the pointwise state constraints with mixed constraints, by an artificial control w (Lavrentiev regularization).
- We can solve the Open-Loop problem with PDASS, that has a superlinear rate of convergence.
- Due the 'long time horizon' we can apply MPC and combining it with POD, in order to speed up the computational time, utilizing different approaches.

Outlook:

- Introduce a posteriori Error Estimator for the POD Basis. (In progress)
- Compute $\mathbf{v}$, the velocity field, from Navier-Stokes Equation.
- Coupling Advection-Diffusion equation with Navier-Stokes. (Boussinesq Approximation)

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THANK YOU