Scalable Calculation of Reach Sets

A Mixed Implicit Explicit Formulation

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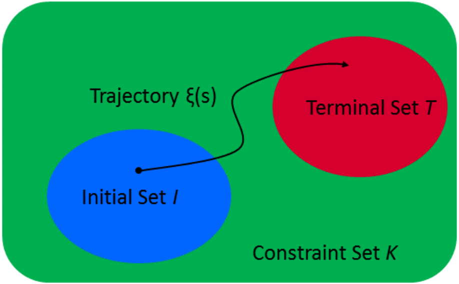
Reachability / Viability

Does there exist a trajectory and/or do all trajectories of dynamic system H lead from a state in initial set I to a state in terminal set T while staying within constraint set K?

- Depending on the formulation, some of these sets may be omitted.

An analysis of transients

- Verification of a **safety** property over a finite horizon.
- Not suitable for infinite horizon properties such as stability or liveness.



Is that a Good Thing or a Bad Thing?

If target is desirable

- Design is validated by satisfying trajectory.

Image: Jan Erik Paulsen

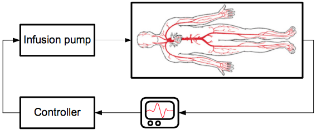
If target is undesirable

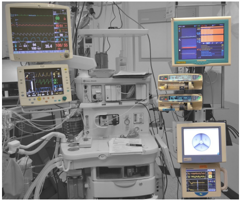
- Design is falsified by counter-example trajectory.
- Proof of correctness requires demonstrating that behaviour is impossible.

Image: NASA

Motivation

Closed-loop control of anesthesia





Shared control of vehicles

Image: Ascending Technologies

System dynamics are often nonlinear, nondeterministic and/or poorly characterized.

Outline

1. Motivation
2. Constructs
3. Hamilton-Jacobi Formulation
4. The Mixed Implicit Explicit Formulation
5. Beyond the Double Integrator
6. Future Work & Conclusions

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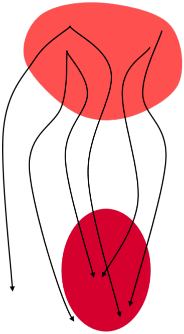
Backward Reachability

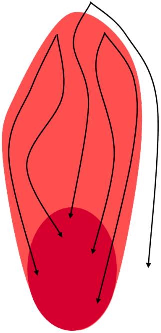
What states give rise to trajectories which will reach a target set?

- At a fixed time or over an interval?
- For all controls or for some control?

Reach$_+(t, \mathcal{T})$



Reach$_-([t_s, t_f], \mathcal{T})$

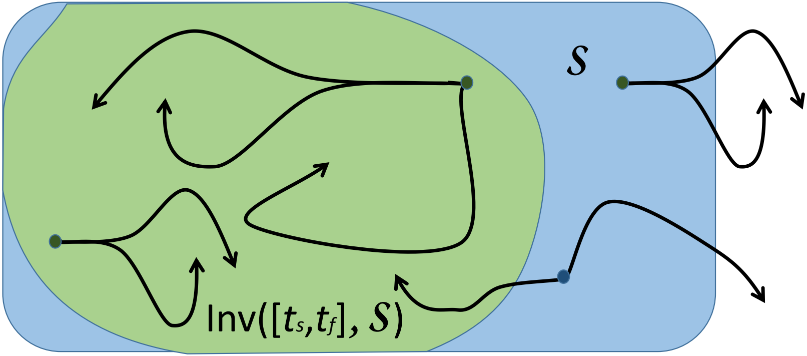


$$\text{Reach}_+(t, \mathcal{T}) \triangleq \{x(0) \mid \exists u(\cdot), x(t) \in \mathcal{T}\}$$

$$\text{Reach}_-([t_s, t_f], \mathcal{T}) \triangleq \{x(0) \mid \forall u(\cdot), \exists t \in [t_s, t_f], x(t) \in \mathcal{T}\}$$

Invariance Kernel

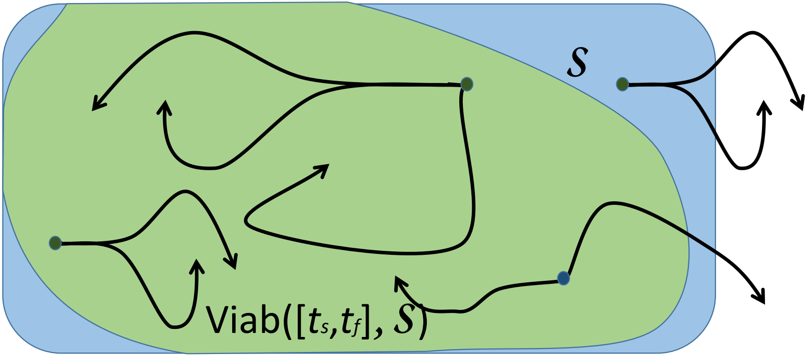
$$\text{Inv}\left([t_s, t_f], \mathcal{S}\right) \triangleq \{x(t_s) \in \mathcal{S} \mid \forall u(\cdot), \forall t \in [t_s, t_f], x(t) \in \mathcal{S}\},$$



- What states will remain safe despite input uncertainty.
- Inputs treated in a worst-case fashion.

Viability Kernel

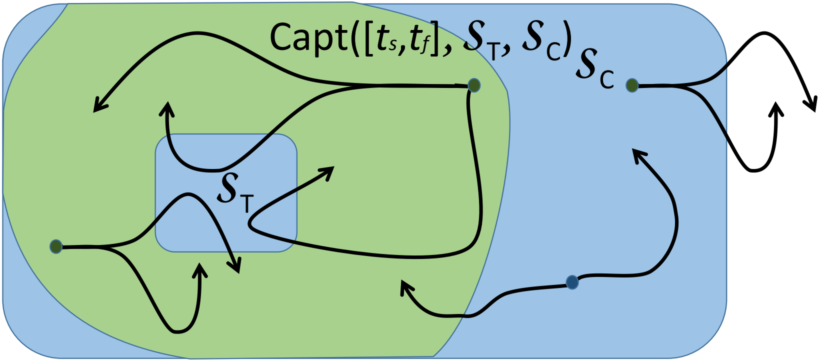
$$\text{Viab}\left([t_s, t_f], \mathcal{S}\right) \triangleq \{x(t_s) \in \mathcal{S} \mid \exists u(\cdot), \forall t \in [t_s, t_f], x(t) \in \mathcal{S}\},$$



- Also called controlled invariant set.
- Inputs treated in a best-case fashion.

Capture Basin

$$\text{Capt}\left([t_s,t_f],\mathcal{S}_T,\mathcal{S}_C\right) \triangleq \left\{ x(t_s)\in\mathcal{S}_C \;\middle|\; \begin{array}{c} \exists u(\cdot), \exists t_T\in[t_s,t_f], \forall t\in[t_s,t_T], \\ x(t)\in\mathcal{S}_C \,\wedge\, x(t_T)\in\mathcal{S}_T \end{array} \right\},$$



- Trajectories must stay inside constraint $\mathcal{S}_C$ until they reach target $\mathcal{S}_T$
- Inputs treated in a best-case fashion.

Discriminating Kernel

$$\mathsf{Disc}\left([t_s, t_f], \mathcal{S}\right) \triangleq \{x(t_s) \in \mathcal{S} \mid \exists u(\cdot), \forall v(\cdot), \forall t \in [t_s, t_f], x(t) \in \mathcal{S}\},$$

That is hard to draw...

- Also called robust controlled invariant set.
- Two inputs "control" $u(\cdot)$ and "disturbance" $v(\cdot)$ treated adversarially.

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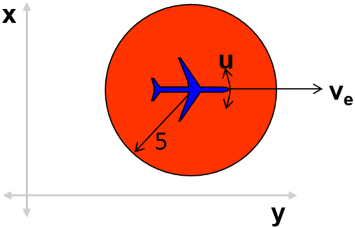
Game of Two (Identical) Vehicles

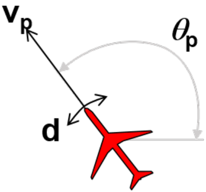
A classical collision avoidance problem

- Vehicle state: position in plane plus heading.
- Vehicle dynamics: unicycle (Dubin's / Reed-Shepp car).
- Collision if vehicles get within five units of each other.
- Fixed (equal) linear velocity v_e and v_p.
- Bounded angular velocity u and d.



dynamics (pursuer)

$$\frac{d}{dt}\begin{bmatrix} x_p \\ y_p \\ \theta_p \end{bmatrix} = \begin{bmatrix} v_p \cos\theta_p \\ v_p \sin\theta_p \\ d \end{bmatrix}$$



Work in a Relative Coordinate Frame

Reformulate the problem with the evader fixed at the origin.

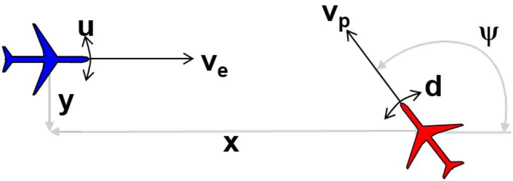
- State variables are relative planar location x and y and relative heading ψ.

Dynamics:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \psi \end{bmatrix} = \begin{bmatrix} -v_e + v_p \cos\psi - uy \\ v_p \sin\psi - ux \\ d - u \end{bmatrix}$$

Target set

$$h(x) = \sqrt{x^2 + y^2} - 5$$

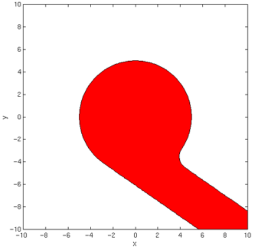


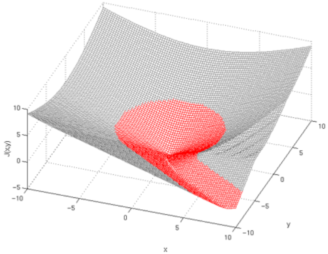
Implicit Surface Functions

Sets are represented by sublevel sets of scalar function

- State space dimension does not matter conceptually.
- Surfaces automatically merge and/or separate.
- Geometric quantities (such as surface normal) are easy to calculate.

$$\mathcal{S}(t) = \{x \in \mathbb{R}^n \mid \phi(t,x) \le 0\}$$





Hamilton-Jacobi Formulation

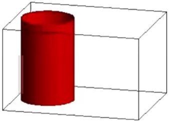
Implicit surface function for desired reachable tube / discriminating kernel given by value function of Hamilton-Jacobi (HJ) PDE for finite horizon differential game with optimal stopping time:

$$D_t\phi(t,x) + \min[0, H(x, D_x\phi(t,x))] = 0$$

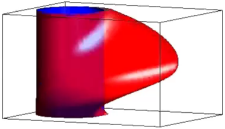
where

$$H(x,p) = \max_u \min_d f(x,u,d) \cdot p \qquad \phi(x,0) = h(x)$$

dynamics: $\dot{x} = f(x,u,d)$ target set: $\{x \in \mathbb{R}^n \mid h(x) \le 0\}$



growth of reachable set



final reachable set

The Challenge: Scalability

Approximating sets, tubes or kernels using HJ PDE is

- Conceptually straightforward
- Computationally tractable for systems with 2–4 state space dimensions.

But most real systems have 6+ state space dimensions.

In contrast, algorithms using parametric representations for reachable sets are widely available and scalable: Interval arithmetic, polygons, zonotopes, ellipsoids, support functions / vectors. . .

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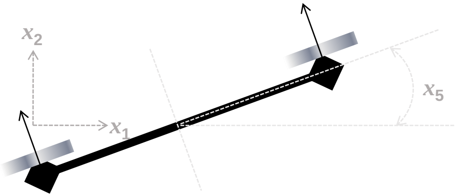
Longitudinal Model of a Quadrotor

- From [Bouffard 2012]

$$\begin{aligned}\dot{x}_1 &= x_3,\\ \dot{x}_2 &= x_4,\\ \dot{x}_3 &= u_1 K \sin x_5,\\ \dot{x}_4 &= -g + u_1 K \cos x_5,\\ \dot{x}_5 &= x_6,\\ \dot{x}_6 &= -d_0 x_5 - d_1 x_6 + n_0 u_2,\end{aligned}$$

- Inputs: total thrust u_1 and desired roll angle u_2



Systems with Terminal Integrators

Common form of system dynamics

$$\begin{aligned} \dot{y} &= f(y, u) && \text{coupled states } y \in \mathbb{R}^{d_y}, \\ \dot{x}_i &= b_i(y) && \text{terminal integrators } x_i \in \mathbb{R} \text{ for } i = 1, \dots, d_x \end{aligned}$$

Straightforward decoupled representation

- Run HJ formulation to find implicit representation $\phi_i(t, x_i, y)$ for all states y plus each x_i individually

$$H(x_i, y, D_{x_i}\phi_i, D_y\phi_i) = \min_u f(y, u) \cdot D_y\phi_i + b(y)D_{x_i}\phi_i$$

- States are inside overall reach set only if inside every individual reach set

$$\mathcal{S}(t) = \{(x, y) \mid \phi_i(t, x_i, y) \leq 0 \; \forall i = 1, \dots, d_x\}$$

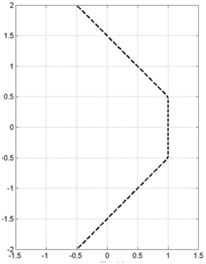
- Computational cost for solution (n nodes per dimension) drops from $\mathcal{O}\left((n^{(d_y + d_x)}\right)$ to $\mathcal{O}\left(d_x n^{(d_y+1)}\right)$.

Mixed Implicit Explicit Formulation

Consider a single terminal integrator $d_x = 1$. Represent sets as an interval in x for every y

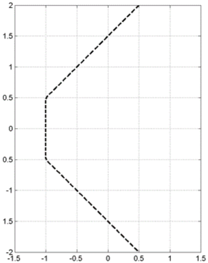
$$\mathcal{S} = \left\{ (x, y) \;\middle|\; \underline{\psi}(y) \leq x \leq \overline{\psi}(y) \right\}$$

A **mixed implicit explicit** (MIE) formulation of the set.



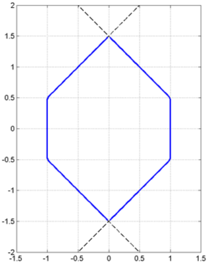
upper bound

$x \leq \overline{\psi}(y)$



lower bound

$\underline{\psi}(y) \leq x$



overall set $\mathcal{S}$

MIE HJ PDE for Terminal Integrators

- If $x(t,y) = \overline{\psi}(t,y)$ is the upper boundary of the reach set then formally

$$b(y) = \tfrac{d}{dt}x(t,y) = \tfrac{d}{dt}\overline{\psi}(t,y) = D_t\overline{\psi}(t,y) + D_y\overline{\psi}(t,y) \cdot f(y,u)$$

- Rearrange to find a terminal value HJ PDE

$$D_t\overline{\psi}(t,y) + H\left(t, y, D_y\overline{\psi}(t,y)\right) = 0$$

$$\overline{\psi}(y,0) = \overline{\psi}_0(y) \quad \text{and} \quad H(x,y,p) = \max_u f(x,u) \cdot p - b(y)$$

- Repeat with $x(t,y) = \underline{\psi}(t,y)$ for lower boundary
- Given target set

$$\left\{(x,y) \;\middle|\; \underline{\psi}_0(y) \le x \le \overline{\psi}_0(y)\right\}$$

  we can compute backward reach set at time t

$$\left\{(x,y) \;\middle|\; \underline{\psi}(t,y) \le x \le \overline{\psi}(t,y)\right\}$$

- Computational cost for solution (n nodes per dimension) drops from $\mathcal{O}\left(d_x n^{(d_y+1)}\right)$ to $\mathcal{O}\left(2 d_x n^{d_y}\right)$.

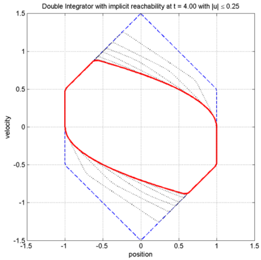
Example: Standard Double Integrator

Dynamics

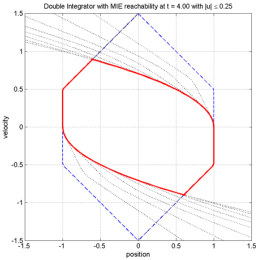
$$\dot{y} = f(y,u) = u \qquad \dot{x} = y \qquad |u| \leq u_{\max}$$

yields terminal integrator Hamiltonian (for upper bound)

$$H(t,y,p) = \max_{|u| \leq u_{\max}} (p \cdot u - y) = (|p| u_{\max} - y)$$



Regular implicit surface formulation
One HJ PDE in 2D



MIE formulation
Two HJ PDEs in 1D

Proof by Finite Horizon Optimal Control

- Terminal integrator's dynamics (for $t < 0$) are

$$x(0, y(0)) = x(t, y(t)) + \int_t^0 b(y(s))\, ds$$

or

$$x(t, y(t)) = \int_t^0 -b(y(s))\, ds + x(0, y(0))$$

- Can be interpreted as a finite horizon optimal control problem with associated HJ PDE

$$D_t\overline{\psi}(t, y) + H\left(t, y, D_y\overline{\psi}(t, y)\right) = 0$$

$$\overline{\psi}(y, 0) = \overline{\psi}_0(y) \quad \text{and} \quad H(x, y, p) = \max_u f(x, u) \cdot p - b(y)$$

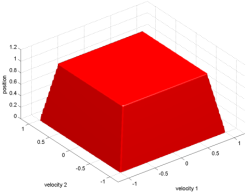
- Solution $\overline{\psi}(t, y)$ provides the smallest $x(t, y(t)$ giving rise to a trajectory which reaches the upper boundary $x(0, y(0)) = \overline{\psi}_0(y(0))$ of the target set at $t = 0$.

Rotating Double Integrator

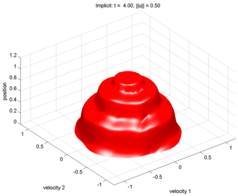
- Let $u \in U = \{u \in \mathbb{R}^2 \mid \|u\|_2 \leq u_{\max}\}$ and

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} \begin{bmatrix} -y_2 \\ y_1 \end{bmatrix} + \mu(\|y\|_2) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \dot{x} = \|y\|_2$$

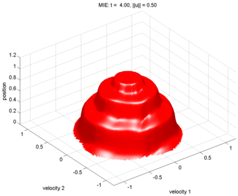
- If $\mu(\alpha) = 1$ it behaves radially like the first quadrant of a traditional double integrator.
- For this experiment $\mu(\alpha) = 2\sin(4\pi\alpha)$.



Target set



Standard Implicit



MIE

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MIE for the Game of Two Vehicles?

Not possible to formulate in terminal integrator form.

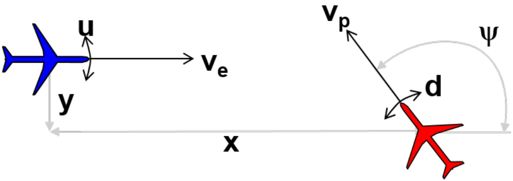
- All states are coupled for $u \neq 0$.
- Target set couples x and y.

Dynamics:

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ \psi \end{bmatrix} = \begin{bmatrix} -v_e + v_p \cos\psi - uy \\ v_p \sin\psi - ux \\ d - u \end{bmatrix}$$

Target set

$$h(x) = \sqrt{x^2 + y^2} - 5$$

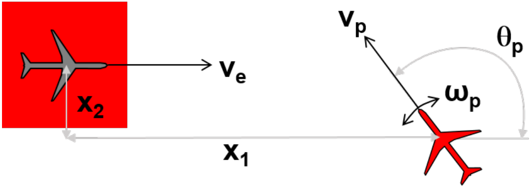


Pursuit of an Oblivious Evader

- Oblivious evader has no input, so relative position variables are terminal integrators.
- Decouple target set using a box.
- Allow pursuer to adjust both heading and speed: Bounded inputs are ω_p and a_p

Dynamics:

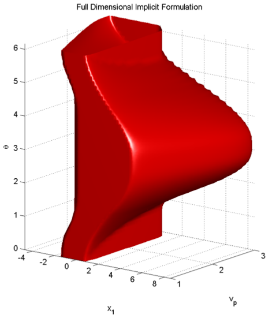
$$\frac{d}{dt}\begin{bmatrix} x \\ x_2 \\ \theta_p \\ v_p \end{bmatrix} = \begin{bmatrix} -v_e + v_p \cos\theta_p \\ v_p \sin\theta_p \\ \omega_p \\ a_p \end{bmatrix}$$



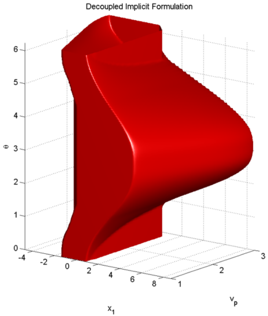
Dynamics and target set suitable for an aerial refueling scenario [Ding & Tomlin, CDC 2010].

Pursuit of the Oblivious Evader Results

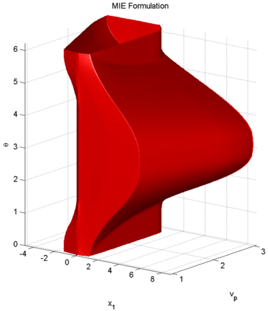
Projection into (x_1, v_p, θ)



Full Dimensional Implicit



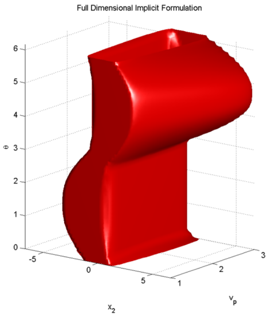
Decoupled Implicit



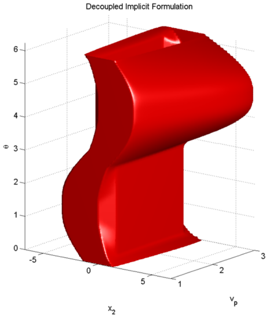
MIE

Pursuit of the Oblivious Evader Results

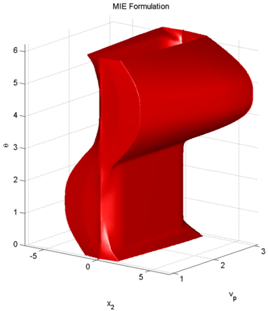
Projection into (x_2, v_p, θ)



Full Dimensional Implicit



Decoupled Implicit

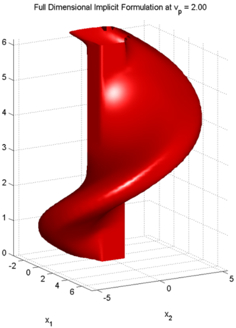


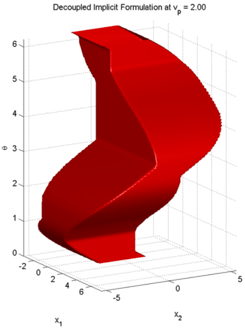
MIE

Pros & Cons

- Much faster
  - Full dimensional implicit: $151^2 \times 65 \times 100$ grid takes 30 hours (est).
  - Decoupled implicit: $151 \times 65 \times 100$ grid takes 541 seconds.
  - MIE: 65×100 grid takes 3.1 seconds.
- Projection formulations cannot represent true reach tube.
  - They must overapproximate.
  - For example: slices of reach tube for $v_p = 2$.

Full dimensional implicit


Full Dimensional Implicit Formulation at v_p = 2.00


Decoupled Implicit Formulation at v_p = 2.00

Decoupled implicit

or

MIE

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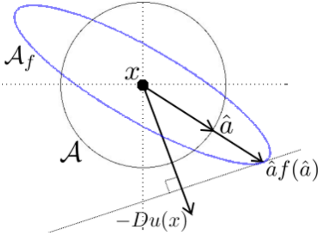
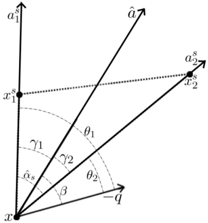
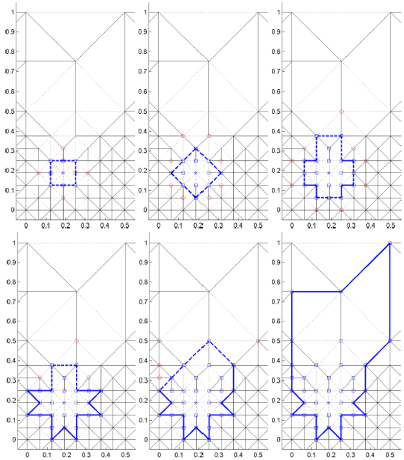
No Time For. . .

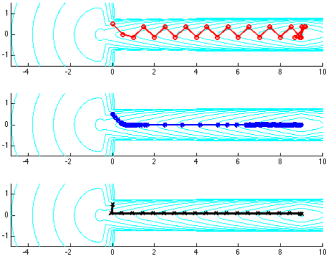




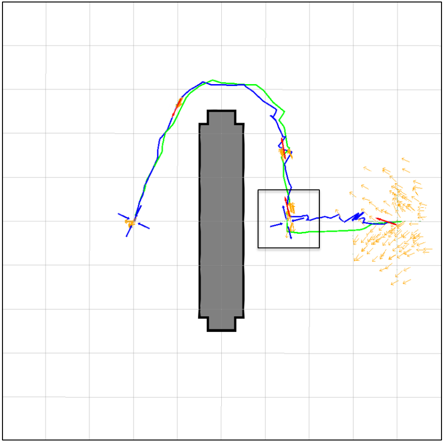


Monotone acceptance ordered upwind method [Alton & Mitchell, JSC 2012].

No Time For. . .

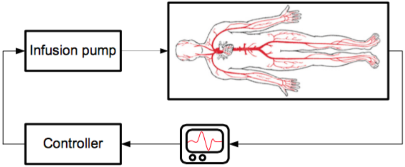


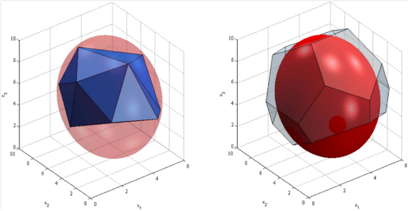
Top: Fixed stepsize explicit (forward Euler).
Middle: Adaptive stepsize implicit (`ode15s`).
Bottom: Sampled gradient algorithm.

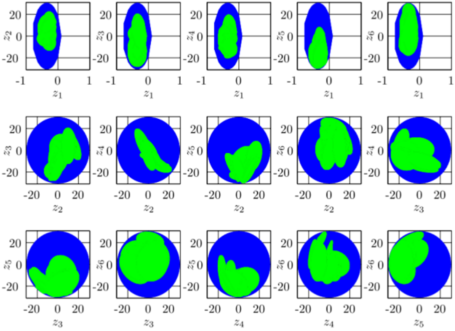


Sampled gradient particle filter [Traft & Mitchell, CDC 2016] for planning with run-time uncertainty.

No Time For. . .



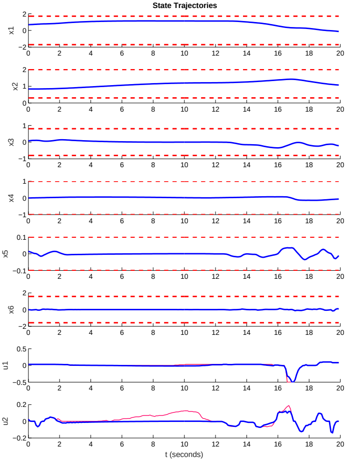




Parametric approximations of viability and discriminating kernels with applications to verification of automated anesthesia delivery [Maidens et al, Automatica 2013; Yousefi et al, ACC 2016].

No Time For. . .

Shared control smart wheelchair for older adults with cognitive impairments [Viswanathan et al, Autonomous Robots 2016].



Ensuring safety for human-in-the-loop flight control of low sample rate indoor quadrotors [Mitchell et al, CDC 2016].

Reproducible Research

Central concept of the scientific method: Results should be independently replicable, given appropriate resources.

> *[a]n article about computational science in a scientific publication is not the scholarship itself, it is merely advertising of the scholarship. The actual scholarship is the complete software development environment and the complete set of instructions which generated the figures.*
>
> *[Jon Claerbout, as quoted by Buckheit & Donoho, 1995]*

Computational science faces some challenges:

- Appropriate IP licensing standards.
- Disciplined software development processes.
- Availability of software (and increasingly hardware) infrastructure and tools.
- Weak (but strengthening) community norms.

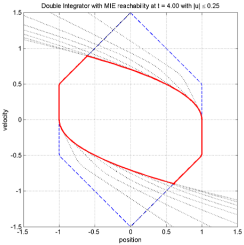
Some Suggestions for Doing It Better

- Use a (modern) version control system.
  - Online repositories (eg: bitbucket, github) include wikis and issue trackers.
- Document in the data (and code is data).
  - You will forget how and why you did things.
  - Files and directories will get separated or lost.
- Write tests first and run them often.
  - Bugs are inevitable and “static” code isn’t.
- If you do it twice, automate it
  - Computers are better at repetition, you can automate a person with a checklist, and automation is documentation.
- Plan to release your code
  - If you think “I would be embarrassed by this bit of code” then either fix it now or realize that everybody writes embarrassing code.
- **Improve your process gradually by continually.**
  - Every little bit helps.

[Wilson et al, “Best Practices for Scientific Computing” in PLOS Biology, 2014].

Mixed Implicit Explicit Formulation: Open Questions

- Inputs are calculated separately in each projection and for upper and lower bounds.
  - Pessimistic but sound for presented examples.
  - Potentially "leaky corners" in the refueling scenario.
  - What about adversarial inputs?
- What are appropriate "boundary conditions"?
  - Implicit level set formulation can work around non-physical boundary conditions.
  - What should be done in MIE formulation when $\overline{\psi}(y) < \underline{\psi}(y)$?



Double integrator MIE

- What class(es) of systems are amenable to MIE formulation?
  - Terminal integrators with unstable linear self-coupling.
  - Systems with monotone dynamics?
  - Must all implicit states be decoupled?
- Is it possible to handle discontinuous $\overline{\psi}(y)$ and $\underline{\psi}(y)$?
- Can we guarantee the sign of the approximation error?

Mixed Implicit Explicit Formulation: Conclusions

HJ PDE formulation of reachable sets / discriminating kernels:

- Is a power approach to verification of nonlinear dynamic systems.
- Has had limited success due to curse of dimensionality.

Poor scaling may be overcome in some circumstances.

- Dynamics may be partially decoupled [Mitchell & Tomlin, JSC 2003; Chen & Tomlin, CDC 2015].
- Mixed implicit explicit formulation may permit further reduction of dimension [Mitchell, HSCC 2011].

Numerous questions in PDE and numerical analysis remain to be answered.