Some stability properties for a BDF2-type scheme for parabolic equations

Athena Picarelli
joint work with O. Bokanowski and C. Reisinger

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Introduction

We consider second order Hamilton-Jacobi-Bellman (HJB) equations:

$$\begin{cases} v_t + \sup_{a\in\Lambda} \mathcal{L}^a(t,x,v,D_x v, D_x^2 v) = 0 & x \in \mathbb{R}^d, t \in (0,T) \\ v(0,x) = \psi(x) & x \in \mathbb{R}^d, \end{cases}$$

where $\mathcal{L}^a : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathscr{S}^{d\times d} \to \mathbb{R}$ takes the form

$$\mathcal{L}^a(t,x,r,p,Q) = \Big\{ -b(t,x,a)\cdot p - \frac{1}{2} Tr[\sigma\sigma^T(t,x,a)Q] - f(t,x,a)r + \ell(t,x,a) \Big\}.$$

Assumptions:

- $\Lambda \subset \mathbb{R}^m$ (set of control values): compact set;
- $T > 0$: terminal time;
- b, σ, f, ℓ, ψ: Lipschitz in space, Hölder in time.

$\Rightarrow$ Existence and uniqueness of a viscosity solution of the HJB equation.

Approximation schemes

Consider $d = 1$.
Let us consider a space and time discretization $h, \tau > 0$:

$$t_k = k\tau \qquad x_i = ih \qquad k = 0, \dots, N \quad i \in \mathbb{I}.$$

$$u_i^k \sim v(t_k, x_i).$$

Let u be a numerical approximation of v defined by a scheme

$$\begin{aligned} \mathscr{S}(t_k, x_i, u_i^{k+1}, [u])_i &= 0 & k = 0, \dots, N-1, \quad i \in \mathbb{I} \\ u_i^0 &= \psi(x_i) & i \in \mathbb{I} \end{aligned}$$

Key properties of numerical schemes

- Stability: For any (τ, h) the scheme admits a bounded solution;
- Consistency: For any smooth function φ, there exists a function $\mathcal{E}$ such that $\mathcal{E}(\tau, h) \to 0$ as $(\tau, h) \to 0$ and

$$\left| \mathscr{S}(t, x, \varphi(t+\tau, x), [\varphi]) - \Big(\varphi_t + \sup_{a \in \Lambda} \mathcal{L}^a(t, x, \varphi, \varphi_x, \varphi_{xx}) \Big) \right| \leq \mathcal{E}(\tau, h).$$

  $\mathcal{E}(\tau, h)$: consistency error;
  $\mathcal{E}(\tau, h) \leq C(\tau^\rho + h^\rho)$: scheme of order ρ;

- Monotonicity: For any bounded functions ϕ and ψ such that $\phi \leq \psi$ one has

$$\mathscr{S}(t, x, r, [\phi]) \geq \mathscr{S}(t, x, r, [\psi]).$$

Key convergence result

Theorem (Barles-Souganidis ('91))

If the scheme $\mathscr{S}$ is stable, consistent and monotone, then its solution u converges to the unique viscosity solution of the HJB equation.

Error estimates for monotone schemes: Krylov ('97,'00), Barles-Jakobsen ('05,'07).

Theorem (Godunov ('59))

Monotone linear schemes have order of consistency at most $\rho = 1$.

Second order schemes

Crank-Nicolson scheme:

$$\frac{u_i^{n+1}-u_i^n}{\tau} + \sup_{a\in\Lambda}\Bigg\{\frac{1}{2}\Big(-\tfrac{1}{2}(\sigma_i^{n+1,a})^2\ D^2u_i^{n+1} - b_i^{n+1,a}\ D^1u_i^{n+1}\Big) + \frac{1}{2}\Big(-\tfrac{1}{2}(\sigma_i^{n,a})^2\ D^2u_i^n - b_i^{n,a}\ D^1u_i^n\Big)\Bigg\} = 0,$$

where

$$D^2u_i := \frac{u_{i-1}-2u_i+u_{i+1}}{h^2} \quad \text{and} \quad D^1u_i := \frac{u_{i+1}-u_{i-1}}{2h}.$$

BDF2 scheme:

$$\frac{3u_i^{n+1}-4u_i^n+u_i^{n-1}}{2\tau} + \sup_{a\in\Lambda}\Bigg\{-\frac{1}{2}(\sigma_i^{n+1,a})^2\ D^2u_i^{n+1} + (b_i^{n+1,a})^+\ D^{1,-}u_i^{n+1} - (b_i^{n+1,a})^-\ D^{1,+}u_i^{n+1}\Bigg\} = 0,$$

where

$$D^{1,-}u_i := \frac{3u_i-4u_{i-1}+u_{i-2}}{2h} \quad \text{and} \quad D^{1,+}u_i := \frac{3u_i-4u_{i+1}+u_{i+2}}{2h}.$$

Example: Uncertain volatility model

An underlying asset is assumed to follow

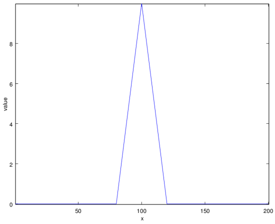
$$\mathrm{d}X(s) = rX(s)\,dt + \sigma_s X(s)\,\mathrm{d}B(s),$$

where we only assume $\sigma_s \in [\sigma_{\min}, \sigma_{\max}]$, an "uncertain" volatility.

The lowest arbitrage-free price is given by

$$\begin{cases} v_t + \sup_{\sigma\in\{\sigma_{\min},\sigma_{\max}\}} \left(-\frac{\sigma^2}{2} x^2 v_{xx} \right) - rxv_x + rv = 0, \\ v(0,x) = \psi(x), \end{cases}$$

where ψ is a European option payoff, and similar for the upper bound.
Let us consider the following ψ:



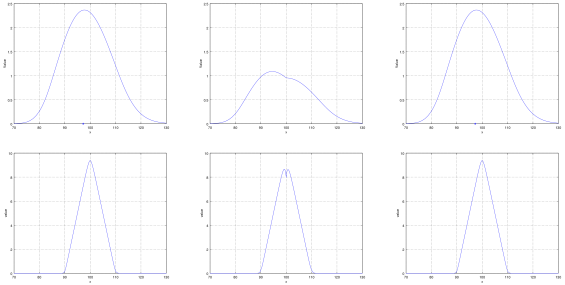
Example: Uncertain volatility model

Pooley-Forsyth-Vetzal ('01): in absence of a CFL condition the Crank-Nicolson scheme may converge to a wrong solution.

Bokanowski-AP-Reisinger ('17): the BDF scheme show good perfomances.

Value function $t = t_N = 0.1$:

Value function $t = t_1$:



Monotone scheme | Crank-Nicolson (no CFL) | BDF

Aim of our work: theoretically investigate stability and convergence properties of the BDF scheme.

Linear equation

$E_i^k := v(t_k, x_i) - u_i^k$ satisfies:

$$\frac{3E^k - 4E^{k-1} + E^{k-2}}{2\tau} + D_k A E^k + F_k B E^k = \mathcal{E}^k, \quad 2 \le k \le N, \qquad (1)$$

where

$$A = \frac{1}{h^2} tridiag(-1, 2, -1)$$

$$D_k = diag\left(\frac{(\sigma_i^k)^2}{2}\right)$$

$$B = \frac{1}{2h}\begin{pmatrix} 3 & 0 & & & \\ -4 & 3 & 0 & & \\ 1 & -4 & \ddots & \ddots & \\ 0 & \ddots & \ddots & \ddots & 0 \\ \ddots & \ddots & 1 & -4 & 3 \end{pmatrix}$$

$$F_k = diag(b_i^k) \quad (b_i^k \ge 0 \ \forall k, i)$$

Linear equation: L^2-norm

Scalarly multiplying (1) by E_k:

$$\underbrace{3(E^k,E^k)-4(E^{k-1},E^k)+(E^{k-2},E^k)}_{(I)} = -2\tau\underbrace{(D_kAE^k,E^k)}_{(II)}-2\tau\underbrace{(F_kBE^k,E^k)}_{(III)}+2\tau(\mathcal{E}^k,E^k)$$

$$\begin{aligned}(I) &= 4(E^k-E^{k-1},E^k)-(E^k-E^{k-2},E^k)\\ &= 2\left(\|E^k\|^2+\|E^k-E^{k-1}\|^2-\|E^{k-1}\|^2\right)-\frac{1}{2}\left(\|E^k\|^2+\|E^k-E^{k-2}\|^2-\|E^{k-2}\|^2\right)\\ &\geq \frac{3}{2}\|E^k\|^2-2\|E^{k-1}\|^2+\frac{1}{2}\|E^{k-2}\|^2+\|E^k-E^{k-1}\|^2-\|E^{k-1}-E^{k-2}\|^2.\end{aligned}$$

Linear equation: L^2-norm

Assumptions (H1):

- $\|b\|_\infty < \infty$;
- $\exists \eta > 0 : \quad \sigma^2(t,x) \geq \eta \qquad \forall x \in \mathbb{R}, t \in [0,T]$;
- $\exists L \geq 0 : \quad |\sigma^2(t,x) - \sigma^2(t,y)| \leq L|x-y| \quad \forall x, y \in \mathbb{R}, t \in [0,T]$.

Let

$$N(x) = \Big(\sum_{i \in \mathbb{I}} (x_i - x_{i-1})^2 \Big)^{1/2}.$$

One has

$$\begin{aligned}(II) + (III) &\geq \frac{\eta}{2h^2} N(E^k)^2 - \frac{L}{2h} N(E^k)\|E^k\| - \frac{2\|b\|_\infty}{h} N(E^k)\|E^k\| \\ &\geq -\frac{(L + 4\|b\|_\infty)^2}{4\eta}\|E^k\|^2.\end{aligned}$$

Linear equation: H^1-norm

Assumptions (H2):

- $\|b\|_\infty < \infty$;
- $\exists \eta > 0 : \quad \sigma^2(t,x) \geq \eta \qquad \forall x \in \mathbb{R}, t \in [0,T]$.

Let us define

$$|x|_A^2 = (x, Ax) = \sum_{i \in \mathbb{I}} \left(\frac{x_i - x_{i-1}}{h} \right)^2 = h^2 N(x)^2.$$

Scalarly multiplying (1) by AE_k.

$$\underbrace{3(E^k, AE^k) - 4(E^{k-1}, AE^k) + (E^{k-2}, AE^k)}_{(I)} =$$

$$- 2\tau \underbrace{(D_k AE^k, AE^k)}_{(II)} - 2\tau \underbrace{(F_k BE^k, AE^k)}_{(III)} + 2\tau (\mathcal{E}^k, AE^k)$$

Linear equation: H^1-norm

One has

$$(I) \geq \frac{3}{2}|E^k|_A^2 - 2|E^{k-1}|_A^2 + \frac{1}{2}|E^{k-2}|_A^2 + |E^k - E^{k-1}|_A^2 - |E^{k-1} - E^{k-2}|_A^2$$

and

$$\begin{aligned}(II) + (III) &= (D_k AE^k, AE^k) + \frac{1}{2h}\Big(F_k\big(3(e_i^k - e_{i-1}^k) - (e_{i-1}^k - e_{i-2}^k)\big)_i, AE^k\Big) \\ &\geq \frac{\eta}{2}\|AE^k\|^2 - 2\|b\|_\infty \|AE^k\| |E^k|_A \\ &\geq -\frac{4\|b\|_\infty^2}{\eta}|E^k|_A^2.\end{aligned}$$

Main recursion

Lemma

Let $|\cdot|$ be a given vectorial norm. If there exists $C \geq 0$ such that for any $k \geq 2$

$$\frac{1}{2}\Big(3|E^k|^2 - 4|E^{k-1}|^2 + |E^{k-2}|^2\Big) + |E^k - E^{k-1}|^2 - |E^{k-1} - E^{k-2}|^2 \leq \tau |E^k|\,|\mathcal{E}^k| + C\tau |E^k|^2$$

then there exists some other constant $C_1 \geq 0$ such that if τ small enough one has for any $n \in \mathbb{N}$:

$$\max_{2\leq k\leq n} |E^k|^2 \leq C_1\Big(|E^0|^2 + |E^1|^2 + \tau \sum_{2\leq j\leq n} |\mathcal{E}^j|^2\Big). \qquad (*)$$

Theorem

Under assumption (H1) (resp. (H2)), estimate $(*)$ holds for $|\cdot| = \|\cdot\|_2$ (resp. $|\cdot| = |\cdot|_A$).

Nonlinear case

u solution of the scheme:

$$\frac{3u^k - 4u^{k-1} + u^{k-2}}{2\tau} + \sup_{a \in \Lambda} \left\{ D_k^a A u^k + F_k^a B u^k \right\} = 0, \quad 2 \le k \le N$$

v solution of the equation:

$$\frac{3v^k - 4v^{k-1} + v^{k-2}}{2\tau} + \sup_{a \in \Lambda} \left\{ D_k^a A v^k + F_k^a B v^k \right\} = \mathcal{E}_k, \quad 2 \le k \le N.$$

Let

- $\bar{a} = (\bar{a}_i^k)_{i,k}$: optimal control associated to v;
- $\bar{b} = (\bar{b}_i^k)_{i,k}$:optimal control associated to u;
- Error $E = v - u$.

$$\frac{3E^k - 4E^{k-1} + E^{k-2}}{2\tau} + \left\{ D_k^{\bar{a}} A E_k + F_k^{\bar{a}} B E^k \right\} \ge \mathcal{E}_k$$

$$\frac{3E^k - 4E^{k-1} + E^{k-2}}{2\tau} + \left\{ D_k^{\bar{b}} A E^k + F_k^{\bar{b}} B E^k \right\} \le \mathcal{E}_k, \quad 2 \le k \le N.$$

Nonlinear case

It follows that there exist weights $\gamma_i^k = \gamma_i^k(\bar{a}, \bar{b}) \in [0, 1]$ such that

$$\gamma_i^k \left(\frac{3E_i^k - 4E_i^{k-1} + E_i^{k-2}}{2\tau} + \left\{ D_k^{\bar{a}} AE_k + F_k^{\bar{a}} BE^k \right\}_i - \mathcal{E}_{k,i} \right)$$
$$+ (1 - \gamma_i^k) \left(\frac{3E_i^k - 4E_i^{k-1} + E_i^{k-2}}{2\tau} + \left\{ D_k^{\bar{b}} AE_k + F_k^{\bar{b}} BE^k \right\}_i - \mathcal{E}_{k,i} \right) = 0.$$

Defined $\Gamma^k = diag(\gamma_i^k)$ we obtain

$$\frac{3E_i^k - 4E_i^{k-1} + E_i^{k-2}}{2\tau} + \Gamma^k \left\{ D_k^{\bar{a}} AE_k + F_k^{\bar{a}} BE^k \right\} + (I - \Gamma^k) \left\{ D_k^{\bar{b}} AE_k + F_k^{\bar{b}} BE^k \right\} = \mathcal{E}_k$$

and then

$$\frac{3E_i^k - 4E_i^{k-1} + E_i^{k-2}}{2\tau} + \widetilde{D}_k AE_k + \widetilde{F}_k BE^k = \mathcal{E}_k$$

where

$$\widetilde{D}_k := \Gamma^k D_k^{\bar{a}} + (I - \Gamma^k) D_k^{\bar{b}};$$
$$\widetilde{F}_k := \Gamma^k F_k^{\bar{a}} + (I - \Gamma^k) F_k^{\bar{b}}.$$

At this point we can apply the result

Conclusions

We have considered a second order (non monotone) BDF scheme in $1d$:

- Stability proof in $|\cdot|_A$ and $\|\cdot\|_2$ for the linear equation;
- Generalised to nonlinear problems;
- Error estimates for piecewise smooth solutions.

Further ongoing work:

- Degenerate diffusion;
- Multiple dimensions.

Thank you for your attention!