On economic model predictive control for time-varying systems

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Outline

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Introduction

Model predictive control

MPC convergence results

Application

Conclusion

Setting

- Consider a discrete-time, time-varying system

$$x(k+1) = f(k, x(k), u(k)),\ x(0) = x_0$$

with $x(k) \in X$, $u(k) \in U$.
$\rightsquigarrow$ solution trajectory $x_u(\cdot, x_0)$

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- Example: Simplified building model:

$$x(k+1) = \underbrace{x(k)}_{\text{inside temperature}} + \underbrace{u(k)}_{\text{heating/cooling}} + \underbrace{w(k)}_{\text{outside temperature}}$$

- Goal: Keep temperature (x) within a certain range $\mathbb{X}(k)$, using as little energy (u) as possible.

Setting

- Infinite horizon optimal control problem

$$\min_{u\in\mathbb{U}^{\infty}(k,x_0)} J_{\infty}(k,x_0,u)=\sum_{j=0}^{\infty}\ell(k+j,x_u(j,x_0),u(j)) \qquad (1)$$

with stage cost $\ell:\mathbb{N}_0\times X\times U\to\mathbb{R}$, and where $\mathbb{U}^{\infty}(k,x_0)$ is the set of admissible control sequences.

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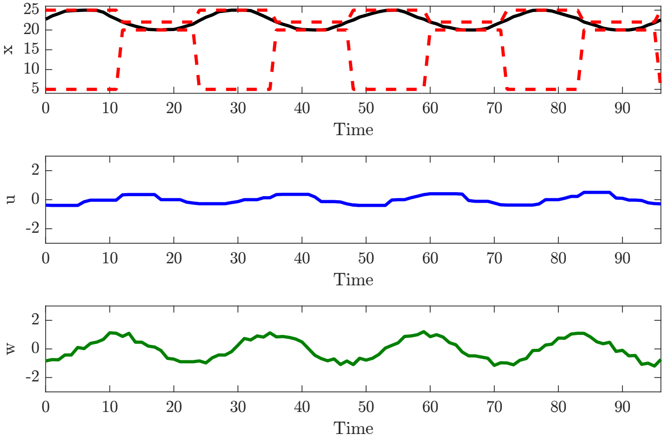
- In the example:
  Stage cost:

$$\ell(k, x, u) = u^2$$

Example: optimal trajectory?



What is optimal?

- $J_\infty(k, x_0, u)$ may not be finite for any u.
  $\rightsquigarrow J_\infty(k, x_0, u^*) \leq J_\infty(k, x_0, u)$ is not a meaningful definition of optimality.

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- Concept of overtaking optimality: (x_{u^*}, u^*) is called *overtaking optimal*[a] if

$$\liminf_{K\to\infty} \sum_{k=0}^{K-1} \ell(k, x_u(k, x_0), u(k)) - \ell(k, x_{u^*}(k, x_0), u^*(k)) \geq 0 \qquad (2)$$

  holds for all pairs (x_u, u).

[a] Joël Blot and Naïla Hayek. *Infinite-horizon optimal control in the discrete-time framework*. Springer, 2014.

Optimal operation

- For what trajectory does the system perform best?

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- In time-varying setting: more general optimal reference: system optimally operated at (x^*, u^*) if

$$\liminf_{K\to\infty} \sum_{k=0}^{K-1} \ell(k, x_u(k, x_0), u(k)) - \ell(k, x^*(k), u^*(k)) \geq 0 \qquad (3)$$

(Initial value of $x^* = x_{u^*}(\cdot; k, x)$ is free!)

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Model predictive control

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- $\rightsquigarrow$ Basic idea of MPC:
  Consider *finite horizon* optimization problems

$$\min_{u \in \mathbb{U}^{N}(k,x_0)} J_N(k, x_0, u) = \sum_{j=0}^{N-1} \ell(k+j, x_u(j, x_0), u(j)) \qquad (4)$$

where ℓ is the stage cost and $N \in \mathbb{N}$ the *horizon length*.

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  where ℓ is the stage cost and $N \in \mathbb{N}$ the *horizon length*.
- Solve (4) instead $\rightsquigarrow u_N^*$.
- Apply $\mu_N(x_0) := u_N^*(0)$ as a feedback to the system.

Why use *economic* MPC?

- Classical MPC: setpoint stabilization or tracking
  Stage cost penalizes distance to reference trajectory:

$$\ell(k, x, u) = \|x - x^*(k)\|^2 + \|u - u^*(k)\|^2$$

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  - Optimal reference is unknown
  - Tracking does not lead to optimal performance $\rightsquigarrow$ [Grüne, Stieler '14]

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- Two problems:
  - Optimal reference is unknown
  - Tracking does not lead to optimal performance $\rightsquigarrow$ [Grüne, Stieler '14]
- Solution: Use economic criterion in stage cost function of MPC.
  In the example $\rightsquigarrow$ $\ell(k, x, u) = u^2$

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- For now: optimality in a weak sense
  $\rightsquigarrow$ Look at relation between MPC closed loop cost

$$J_L^{\mathsf{cl}}(k, x_0, \mu_N) = \sum_{j=0}^{L-1} \ell(k+j, x_{\mu_N}(j, x_0), \mu_N(x_{\mu_N}(j, x_0)))$$

and some optimal value "$V_\infty(k, x_0) = \inf_{u \in \mathbb{U}^\infty(k, x_0)} J_\infty(k, x_0, u)$".

Some technicalities

- Modified cost function:

$$\hat{\ell}(k, x, u) = \ell(k, x, u) - \ell(k, x^*(k), u^*(k)) \quad (5)$$

$$\hat{J}_N(k, x_0, u) = \sum_{j=0}^{N-1} \hat{\ell}(k + j, x_u(j, x_0), u(j)) \quad (6)$$

for $N \in \mathbb{N}_0 \cup \{\infty\}$.

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- Optimal value function

$$\hat{V}_N(k, x_0) := \inf_{u \in \mathbb{U}^N(k, x_0)} \hat{J}_N(k, x_0, u) \tag{7}$$

for $N \in \mathbb{N}_0 \cup \{\infty\}$.

Main result

Theorem 1

Estimate for closed-loop cost:

$$\hat{J}_L^{cl}(k, x, \mu_N) + \hat{V}_\infty(k + L, x_{\mu_N}(L, x)) \leq \hat{V}_\infty(k, x) + L\delta(N) \qquad (8)$$

for some function $\delta \in \mathcal{L}$*.*

Requirements:

- Turnpike property
- Continuity of optimal value function

Turnpike property (finite horizon)

The turnpike property holds if optimal trajectories satisfy

$$|(x_{u_N^*}(j,x), u_N^*(j))|_{(x^*(k+j),u^*(k+j))} \leq \sigma(P) \quad \text{for all } j \in \{0,\dots,N\} \text{ with } j \notin \mathcal{Q}(k,x,P,N). \tag{9}$$

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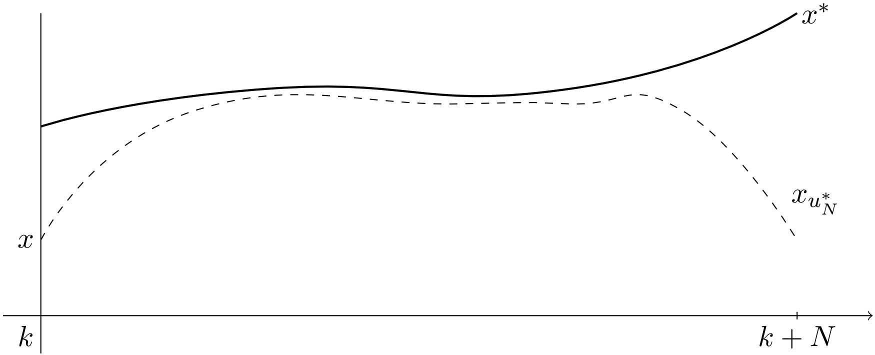

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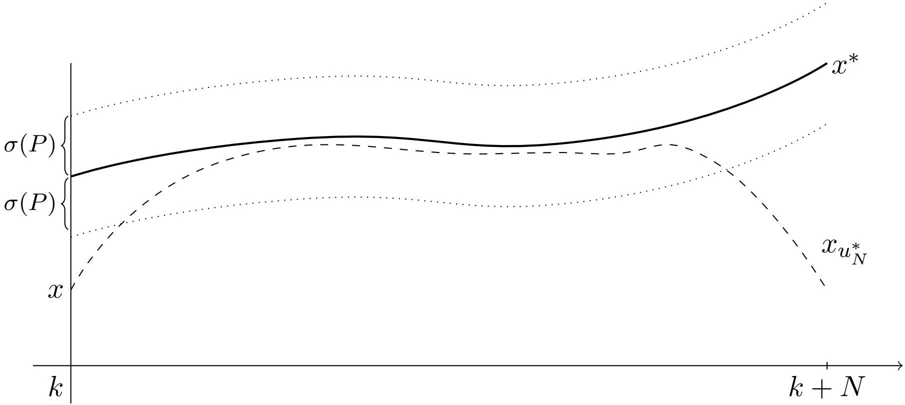

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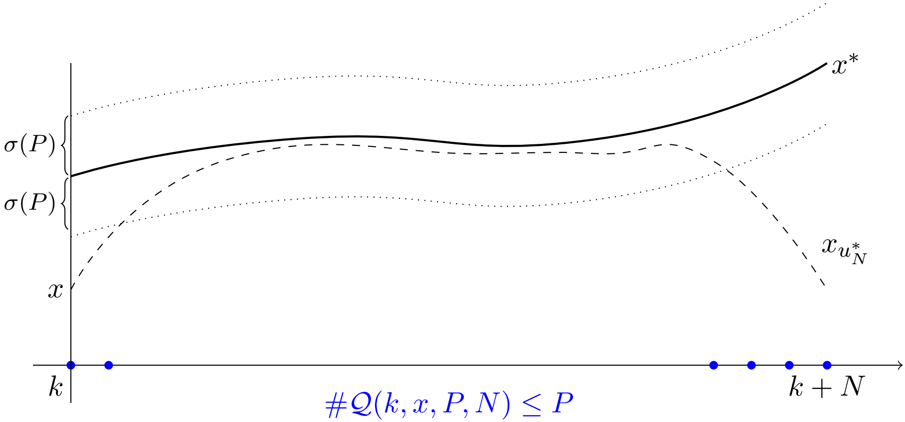

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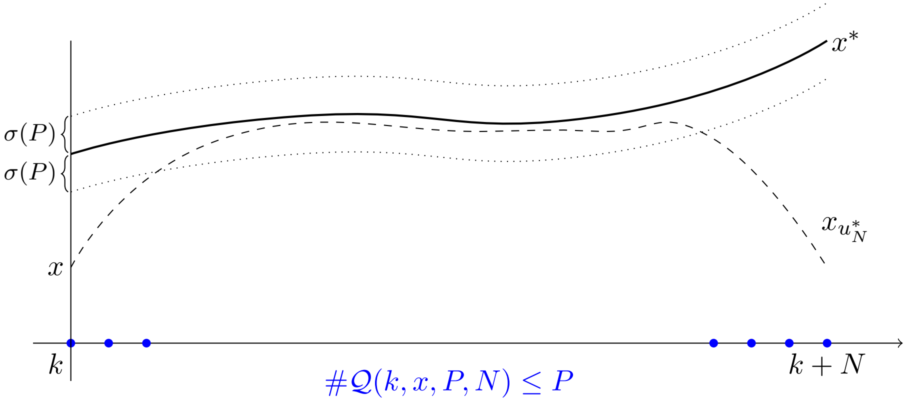

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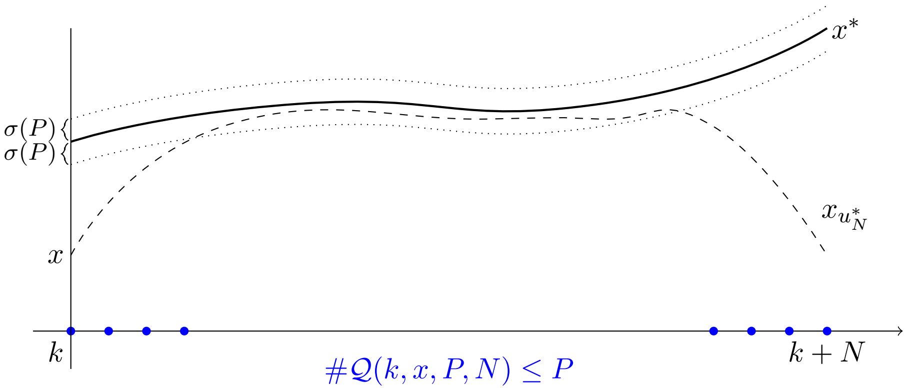
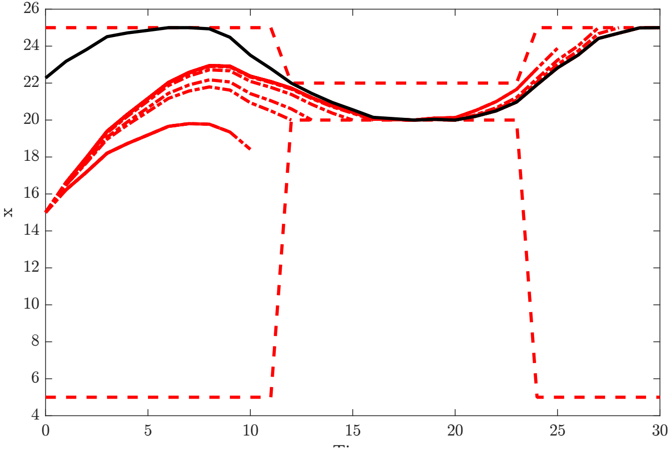
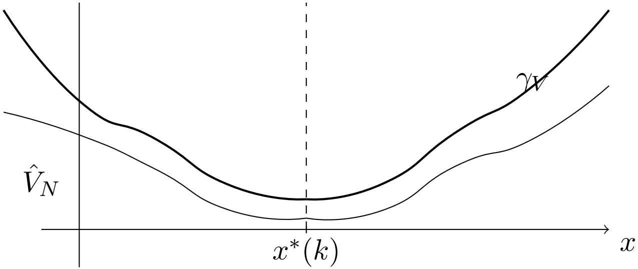


Example: turnpike property for different horizon length N



Continuity of $\hat{V}_N$ and $\hat{V}_\infty$



- We assume that the optimal value functions $\hat{V}_N$ and $\hat{V}_\infty$ are continuous in a neighbourhood of x^*:

$$|\hat{V}_N(k,x) - \hat{V}_N(k,x^*(k))| \leq \gamma_V(N, \|x - x^*(k)\|) \qquad (10)$$

for all $k \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$, $x \in \mathcal{B}_\varepsilon(x^*(k))$.

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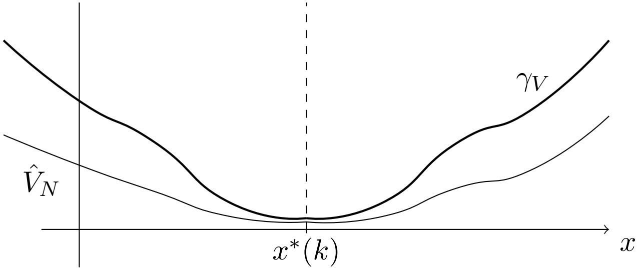

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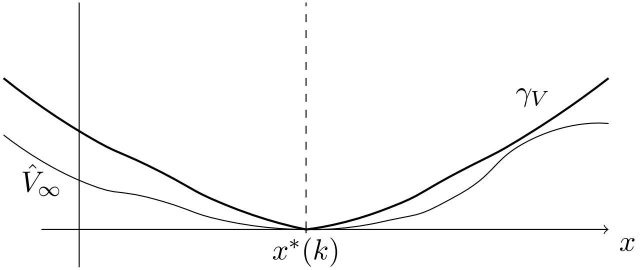

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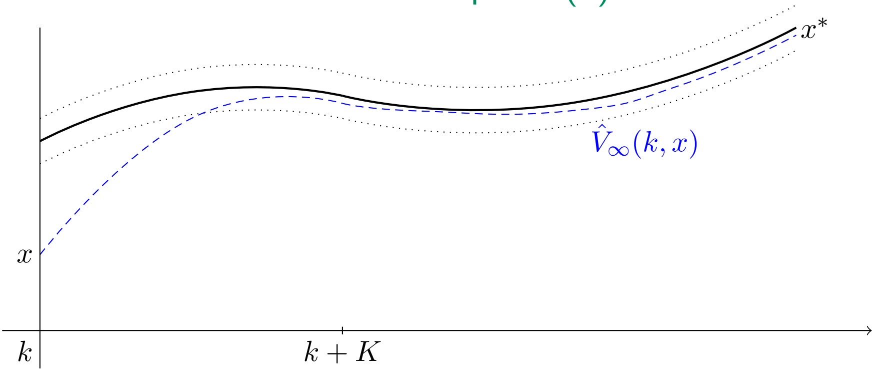
  for all $k \in \mathbb{N}$, $N \in \mathbb{N} \cup \{\infty\}$, $x \in \mathcal{B}_\varepsilon(x^*(k))$.
- Remark: for infinite horizon $N = \infty$: $\rightsquigarrow \hat{V}_\infty(k, x^*(k)) = 0$.

Outline of proof (1)



Show that

- $\hat{V}_\infty(k,x) = \hat{J}_K(k,x,u^*_\infty) + \text{"}\omega(N)\text{"}$
- $\hat{J}_K(k,x,u^*_\infty) = \hat{J}_K(k,x,u^*_N) + \text{"}\rho(N)\text{"}$

where $\omega, \rho \in \mathcal{L}$.

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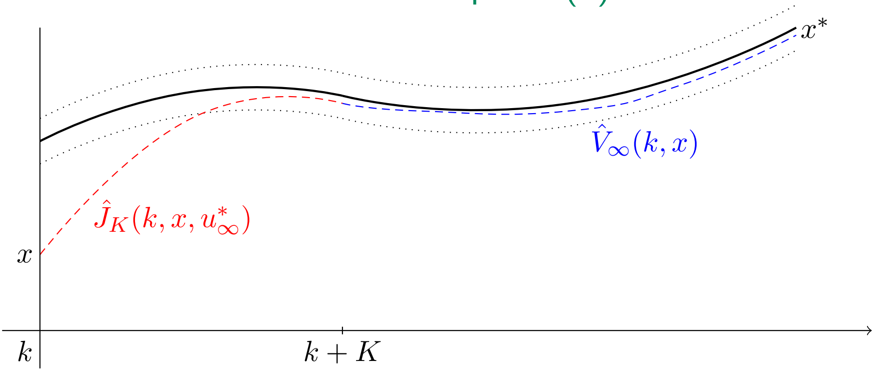

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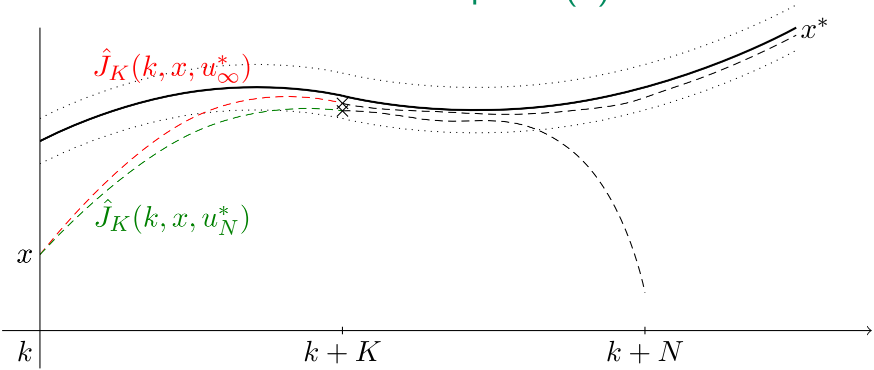

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Outline of proof (2)

- Closed-loop cost:

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$$\begin{aligned}\hat{\ell}(i, x, \mu_N(x)) &= \hat{J}_K(i, x, u_N^*) - \hat{J}_{K-1}(i+1, x^+, u_{N-1,x^+}^*) \\ &= \dots \\ &= \hat{V}_\infty(i, x) - \hat{V}_\infty(i+1, x^+) + \text{"}\delta(N)\text{"}\end{aligned} \quad (12)$$

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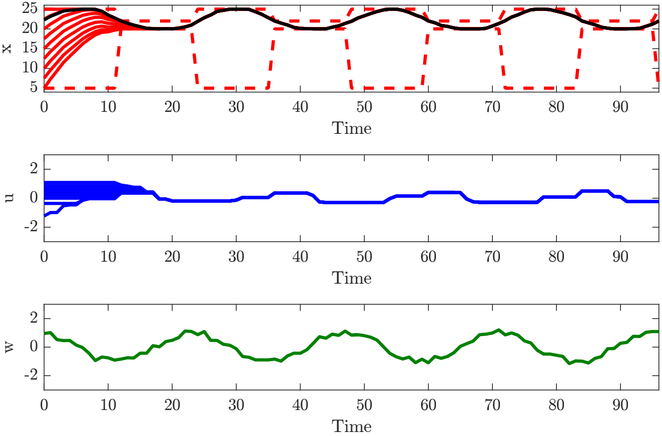
- Summing up yields:

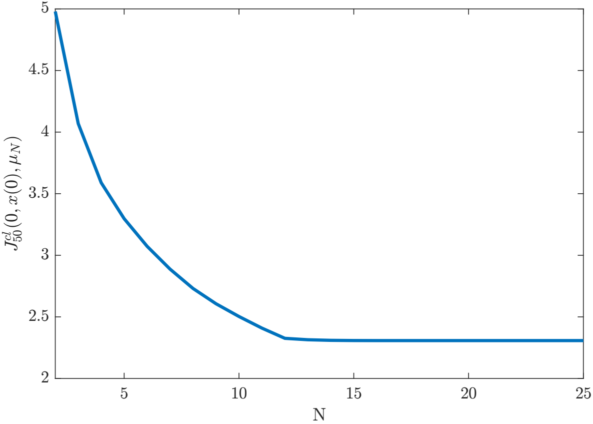
$$\hat{J}_L^{cl}(k, x, \mu_N) + \hat{V}_\infty(k + L, x_{\mu_N}(L, x)) \leq \hat{V}_\infty(k, x) + L\delta(N) \quad (13)$$

for $\delta \in \mathcal{L}$.

Results: MPC closed loop for different initial values



MPC closed loop cost for different horizon length



How to verify requirements?

- Consider strictly dissipative systems:
  There exists a storage function $\lambda : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ such that

$$\lambda(k+1, f(k,x,u)) - \lambda(k,x) \leq \hat{\ell}(k,x,u) - \alpha(|(x,u)|_{(x^*(k),u^*(k))}) \tag{14}$$

  for all $k \in \mathbb{N}_0$, $(x,u) \in \mathbb{X}(k) \times \mathbb{U}(k)$.

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- We can show:
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- Recent work (together with Marleen Stieler):
  Verified dissipativity for example system:
  - Stage cost needs to be strictly convex $\rightsquigarrow$ modification necessary because $\ell(k,x,u) = u^2$ is not convex w.r.t. x
  - Use $\ell(k,x,u) = u^2 + \varepsilon(x - x_{ref})^2$, for $\varepsilon \ll 1$

Outline

Application: Energy efficient building operation

Boussinesq approximation[1] of airflow and temperatures inside a building:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \Delta \mathbf{u} - \mathbf{g} \alpha (T - \tilde{T}) \quad (15)$$

$$\nabla \cdot \mathbf{u} = 0 \quad (16)$$

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \Delta T \quad (17)$$

+ time-varying boundary conditions.

where
$\mathbf{u} : \Omega \times [0, \infty) \to \mathbb{R}^d$ is air velocity
$p : \Omega \times [0, \infty) \to \mathbb{R}$ is pressure
$T : \Omega \times [0, \infty) \to \mathbb{R}$ is temperature

and parameters $\rho, \nu, \mathbf{g}, \alpha, \tilde{T}, \kappa$.

[1] David J Tritton. *Physical fluid dynamics*. Springer Science & Business Media, 2012.

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Summary, Remarks and open questions

Conclusion:

- Economic MPC solution approximates time-varying infinite horizon optimal solution
- (Strict) Dissipativity as a sufficient condition

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- Economic MPC solution approximates time-varying infinite horizon optimal solution
- (Strict) Dissipativity as a sufficient condition

Outlook:

- More realistic PDE model of heat flow inside a building
- Can we still observe turnpike property?

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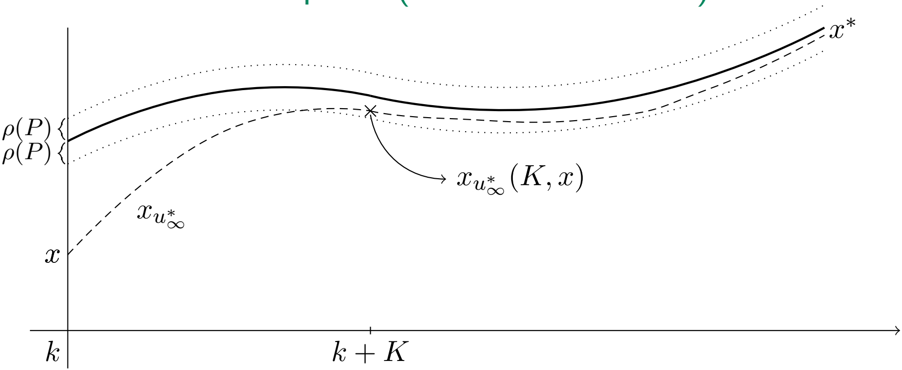
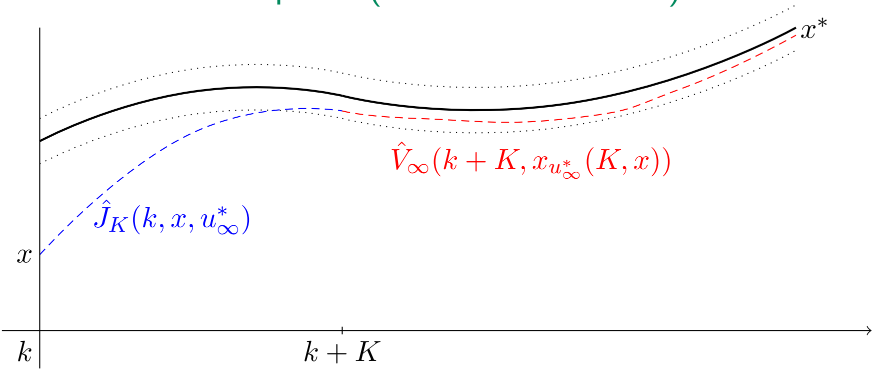
Idea of proof (for infinite horizon)



Pick $P \in \mathbb{N}$ such that $\rho(P) \leq \varepsilon$ (possible because of turnpike property).
$\rightsquigarrow$ For $K \notin \mathcal{Q}(k,x,P,\infty)$:

$$|(x_{u^*_\infty}(K,x), u^*_\infty(K))|_{(x^*(k+K),u^*(k+K))} \leq \rho(P) \leq \varepsilon$$

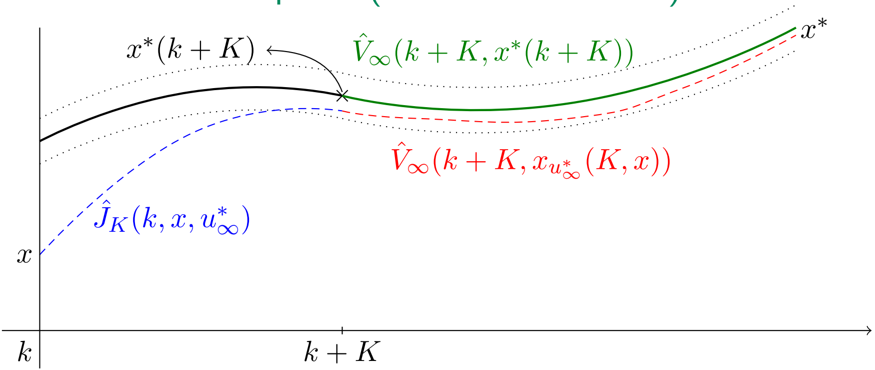
Idea of proof (for infinite horizon)



Dynamic programming principle:

$$\hat{V}_\infty(k,x) = \hat{J}_K(k,x,u^*_\infty) + \hat{V}_\infty(k+K, x_{u^*_\infty}(K,x))$$

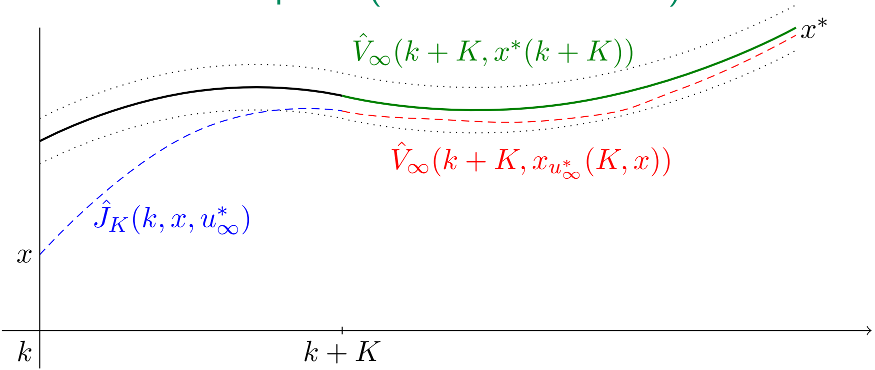
Idea of proof (for infinite horizon)



$\hat{V}_\infty(k+K, x^*(k+K)) = 0.$

$$\Rightarrow \hat{V}_\infty(k,x) = \hat{J}_K(k,x,u^*_\infty) + \underbrace{\hat{V}_\infty(k+K, x_{u^*_\infty}(K,x)) - \hat{V}_\infty(k+K, x^*(k+K))}_{=:R_1(k,x,K)}$$

Idea of proof (for infinite horizon)



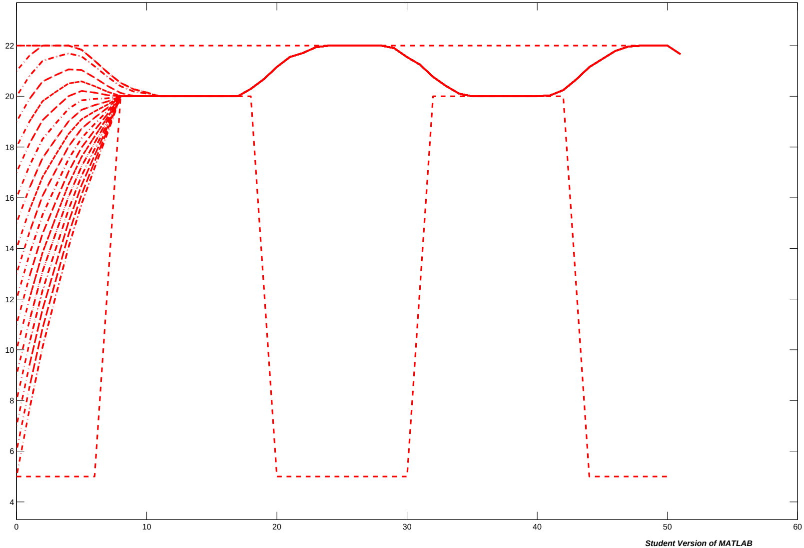
Continuity property:

$$|\hat{V}_\infty(k+K, x_{u^*_\infty}(K,x)) - \hat{V}_\infty(k+K, x^*(k+K))| \leq \omega_V(\rho(P))$$

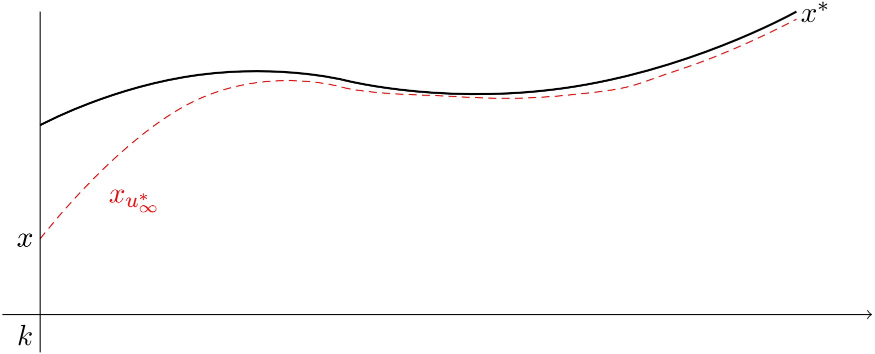
$$\Rightarrow \hat{V}_\infty(k,x) = \hat{J}_K(k,x,u^*_\infty) + R_1(k,x,K)$$

with $|R_1(k,x,K)| \leq \omega_V(\sigma(P))$.

Example: turnpike property for different initial value x_0

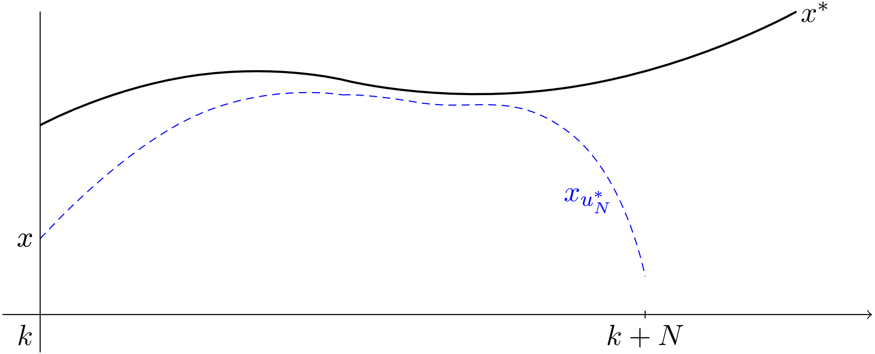


Idea of proof for Lemma 2



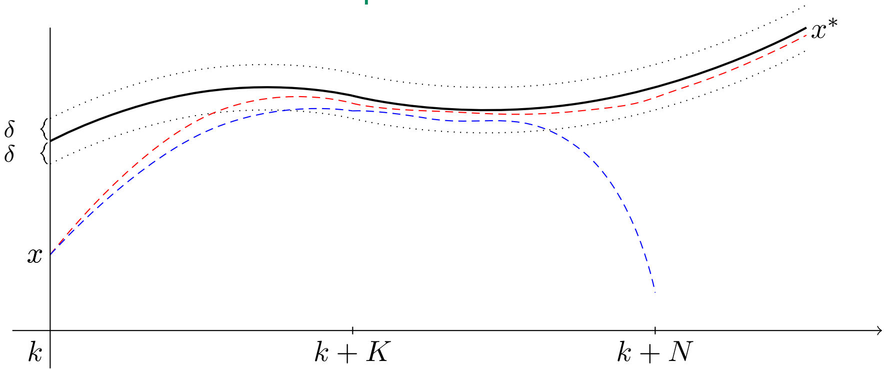
Open-loop on infinite horizon: $x_{u^*_\infty}$

Idea of proof for Lemma 2



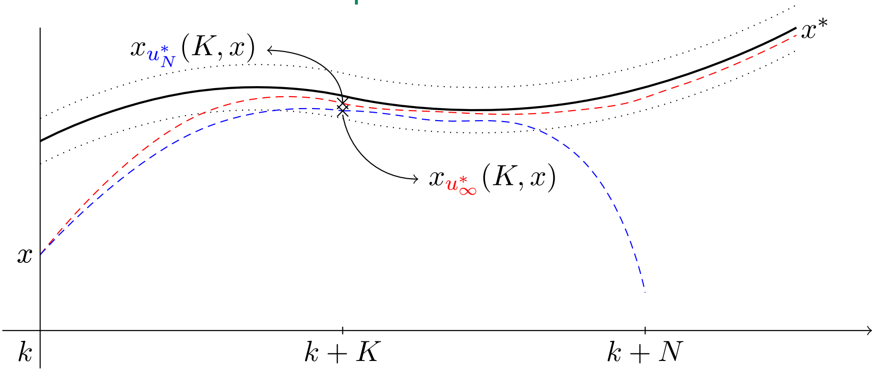
Open-loop on finite horizon: $x_{u_N^*}$

Idea of proof for Lemma 2



Choose P large enough, s.t. $\delta := \max\{\sigma(P), \rho(P)\} < \varepsilon$. Pick $K \in \{0, \dots, N\} \setminus (\mathcal{Q}(k, x, P, N) \cup \mathcal{Q}(k, x, P, \infty))$.

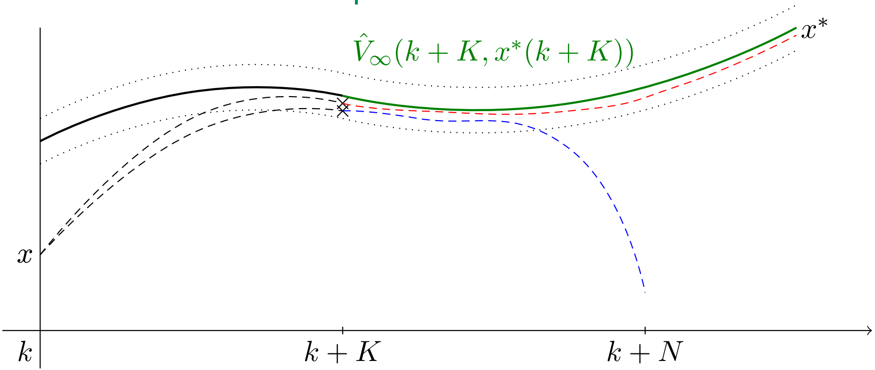
Idea of proof for Lemma 2



Optimality of u_∞^*:

$$\hat{J}_K(k,x,u_\infty^*) + \hat{V}_\infty(k+K, x_{u_\infty^*}(K,x)) \leq \hat{J}_K(k,x,u_N^*) + \hat{V}_\infty(k+K, x_{u_N^*}(K,x))$$

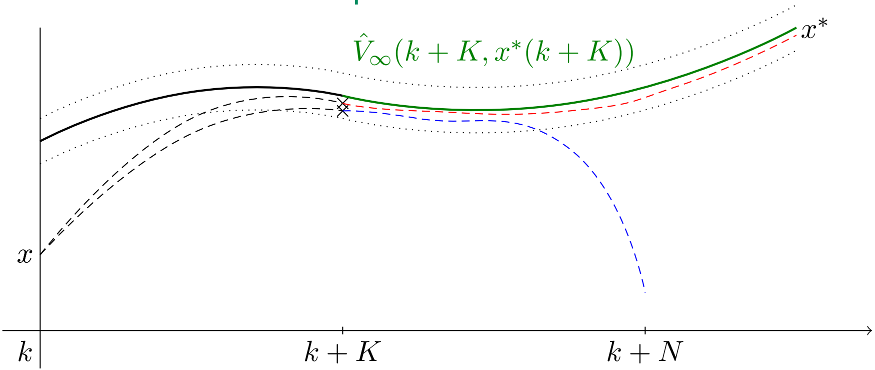
Idea of proof for Lemma 2



Apply Lemma 1:

$$\hat{J}_K(k,x,u^*_\infty) + \underbrace{\hat{V}_\infty(k+K, x_{u^*_\infty}(K,x))}_{R_1(k,x,K)+\hat{V}_\infty(k+K,x^*(k+K))} \leq \hat{J}_K(k,x,u^*_N) + \underbrace{\hat{V}_\infty(k+K, x_{u^*_N}(K,x))}_{\tilde{R}_2(k,x,K,N)+\hat{V}_\infty(k+K,x^*(k+K))}$$

Idea of proof for Lemma 2

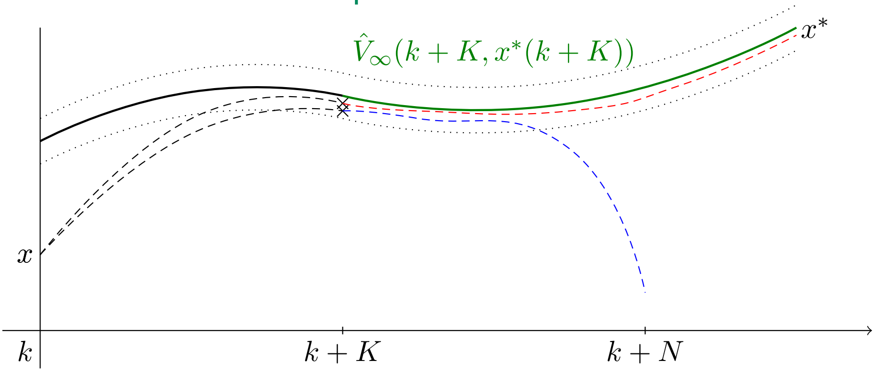


Apply Lemma 1:

$$\hat{J}_K(k,x,u^*_\infty) + \underbrace{\hat{V}_\infty(k+K, x_{u^*_\infty}(K,x))}_{R_1(k,x,K)+\hat{V}_\infty(k+K,x^*(k+K))} \leq \hat{J}_K(k,x,u^*_N) + \underbrace{\hat{V}_\infty(k+K, x_{u^*_N}(K,x))}_{\tilde{R}_2(k,x,K,N)+\hat{V}_\infty(k+K,x^*(k+K))}$$

$$\Rightarrow \hat{J}_K(k,x,u^*_\infty) \leq \hat{J}_K(k,x,u^*_N) - R_1(k,x,K) + \tilde{R}_2(k,x,K,N)$$

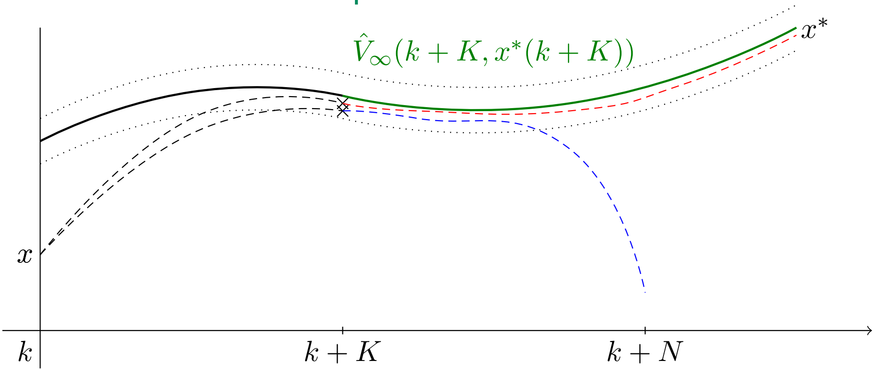
Idea of proof for Lemma 2



Similarly: converse equality from Lemma 1 (2):

$$\hat{J}_K(k, x, u_N^*) \leq \hat{J}_K(k, x, u_\infty^*) - R_2(k, x, K, N) + \tilde{R}_1(k, x, K, N)$$

Idea of proof for Lemma 2

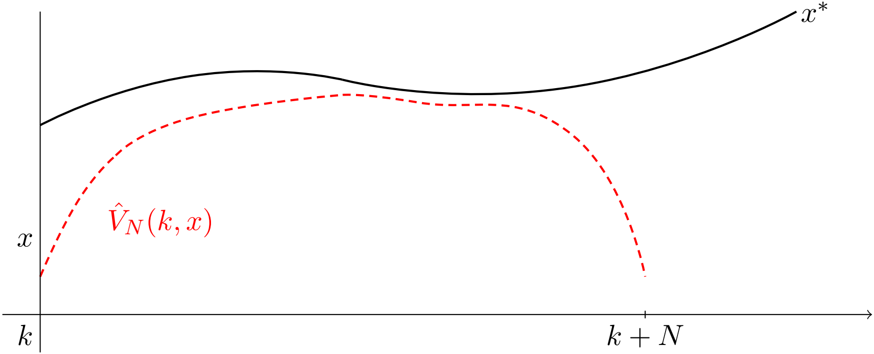


Similarly: converse equality from Lemma 1 (2):

$$\hat{J}_K(k,x,u_N^*) \leq \hat{J}_K(k,x,u_\infty^*) - R_2(k,x,K,N) + \tilde{R}_1(k,x,K,N)$$

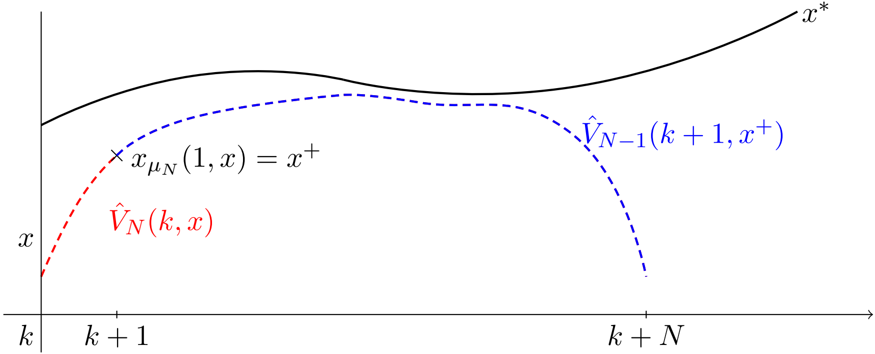
$$\Rightarrow \hat{J}_K(k,x,u_\infty^*) = \hat{J}_K(k,x,u_N^*) + R_3(k,x,K,N)$$

Idea of proof for Theorem 1



- Optimal trajectory starting in x with horizon N. $\rightsquigarrow$ $\hat{V}_N(k,x)$

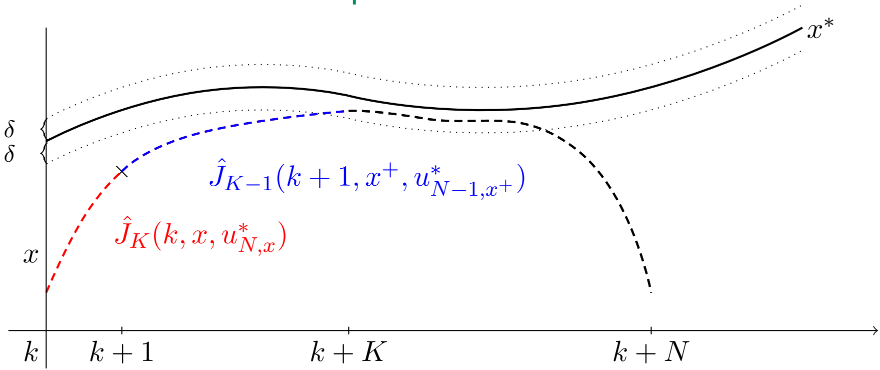
Idea of proof for Theorem 1



- Optimal trajectory starting in x^+ with horizon $N-1$.
  $\rightsquigarrow \hat{V}_{N-1}(k+1, x^+)$
- Stage cost along the MPC closed loop solution:

$$\hat{\ell}(k, x, \mu_N(x)) = \quad \hat{V}_N(k, x) - \hat{V}_{N-1}(k+1, x^+)$$

Idea of proof for Theorem 1



- Optimal trajectory starting in x^+ with horizon $N-1$.
  $\rightsquigarrow \hat{V}_{N-1}(k+1, x^+)$
- Stage cost along the MPC closed loop solution:

$$\begin{aligned} \hat{\ell}(k, x, \mu_N(x)) &= \hat{V}_N(k, x) - \hat{V}_{N-1}(k+1, x^+) \\ &= \hat{J}_K(k, x, u^*_{N,x}) - \hat{J}_{K-1}(k+1, x^+, u^*_{N-1,x^+}) \end{aligned}$$

Idea of proof for Theorem 1

- Stage cost along the MPC closed loop solution:

$$\hat{\ell}(k,x,\mu_N(x)) = \hat{J}_K(k,x,u^*_{N,x}) - \hat{J}_{K-1}(k+1,x^+,u^*_{N-1,x^+})$$
$$\overset{\text{Lemma 2}}{=} \hat{J}_K(k,x,u^*_{\infty,x}) - \hat{J}_{K-1}(k+1,x^+,u^*_{\infty,x^+}) + R_3(\ldots) - R_3(\ldots)$$

Idea of proof for Theorem 1

- Stage cost along the MPC closed loop solution:

$$\begin{aligned}
\hat{\ell}(k,x,\mu_N(x)) = \quad & \hat{J}_K(k,x,u^*_{N,x}) - \hat{J}_{K-1}(k+1,x^+,u^*_{N-1,x^+}) \\
\overset{\text{Lemma 2}}{=} & \color{blue}{\hat{J}_K}\color{black}(k,x,u^*_{\infty,x}) - \color{blue}{\hat{J}_{K-1}}\color{black}(k+1,x^+,u^*_{\infty,x^+}) + R_3(\dots) - R_3(\dots) \\
\overset{\text{Lemma 1}}{=} & \color{red}{\hat{V}_\infty}\color{black}(k,x) - \color{red}{\hat{V}_\infty}\color{black}(k+1,x^+) \\
& + \underbrace{R_1(\dots) - R_1(\dots) + R_3(\dots) - R_3(\dots)}_{=:\color{green}{R_4(k,x,K,N)}}
\end{aligned}$$

Idea of proof for Theorem 1

- Stage cost along the MPC closed loop solution:

$$
\begin{aligned}
\hat{\ell}(k,x,\mu_N(x)) = \quad & \hat{J}_K(k,x,u^*_{N,x}) - \hat{J}_{K-1}(k+1,x^+,u^*_{N-1,x^+}) \\
\overset{\text{Lemma 2}}{=} & \hat{J}_K(k,x,u^*_{\infty,x}) - \hat{J}_{K-1}(k+1,x^+,u^*_{\infty,x^+}) + R_3(\ldots) - R_3(\ldots) \\
\overset{\text{Lemma 1}}{=} & \hat{V}_\infty(k,x) - \hat{V}_\infty(k+1,x^+) \\
& + \underbrace{R_1(\ldots) - R_1(\ldots) + R_3(\ldots) - R_3(\ldots)}_{=:R_4(k,x,K,N)}
\end{aligned}
$$

- One can show that $|R_4(k,x,K,N)| \leq \delta(N)$ for some $\delta \in \mathcal{L}$.

Idea of proof for Theorem 1

- Stage cost along the MPC closed loop solution:

$$\begin{aligned}\hat{\ell}(k,x,\mu_N(x)) = & \quad \hat{J}_K(k,x,u^*_{N,x}) - \hat{J}_{K-1}(k+1,x^+,u^*_{N-1,x^+}) \\ & \overset{\text{Lemma 2}}{=} \hat{J}_K(k,x,u^*_{\infty,x}) - \hat{J}_{K-1}(k+1,x^+,u^*_{\infty,x^+}) + R_3(\dots) - R_3(\dots) \\ & \overset{\text{Lemma 1}}{=} \hat{V}_\infty(k,x) - \hat{V}_\infty(k+1,x^+) \\ & \qquad + \underbrace{R_1(\dots) - R_1(\dots) + R_3(\dots) - R_3(\dots)}_{=:R_4(k,x,K,N)}\end{aligned}$$

- One can show that $|R_4(k,x,K,N)| \leq \delta(N)$ for some $\delta \in \mathcal{L}$.
- Sum $\hat{\ell}(k+j,x,\mu_N(x))$ along the closed loop:

$$\begin{aligned}\hat{J}^{\mathsf{cl}}_L(k,x,\mu_N) &= \sum_{j=0}^{L-1} \hat{\ell}(k+j,x_{\mu_N}(j,x),\mu_N(x_{\mu_N}(j,x))) \\ &= \sum_{j=0}^{L-1} \hat{V}_\infty(j,x) - \hat{V}_\infty(j+1,x^+) + R_4(k,x,K,N) \\ &\leq \hat{V}_\infty(k,x) - \hat{V}_\infty(k+L,x_{\mu_N}(L,x)) + L\delta(N)\end{aligned}$$