Reduced Basis Method for Parametric $\mathscr{H}_2$-Optimal Control Problems

NUMOC2017

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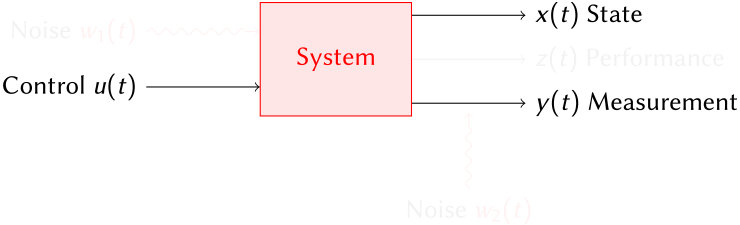
22nd June, 2017

Outline

Introduction

Goal: Set up a general framework for realistic control problems.



In a more mathematical way: LTI System

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + B_u\mathbf{u}(t) + B_w w_1(t) && \text{State equation} \\
\mathbf{y}(t) &= C_y\mathbf{x}(t) + D_{yw} w_2(t) && \text{Measurements} \\
\mathbf{z}(t) &= C_z\mathbf{x}(t) + D_{zu}\mathbf{u}(t) && \text{Performance output} \\
\mathbf{x}(0) &= \mathbf{x}_0.
\end{aligned}
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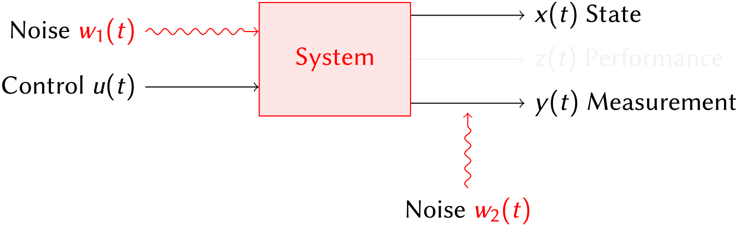

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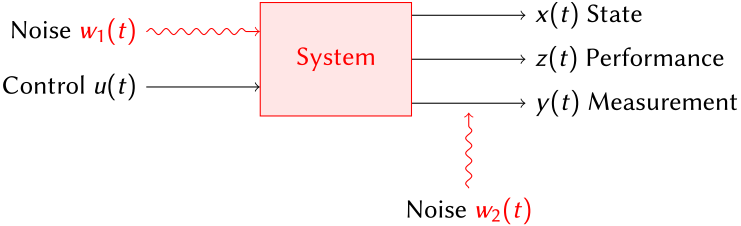

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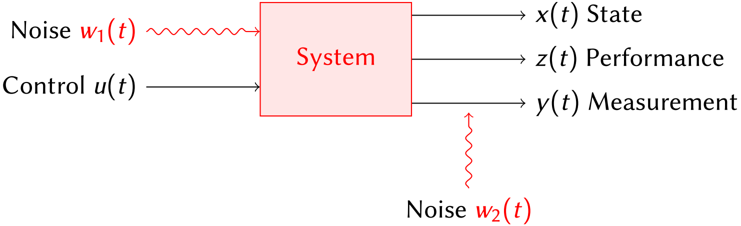

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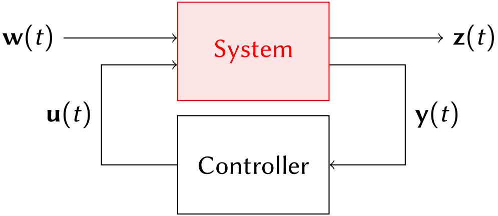
The Control Problem
What should the controller do?



Control objective

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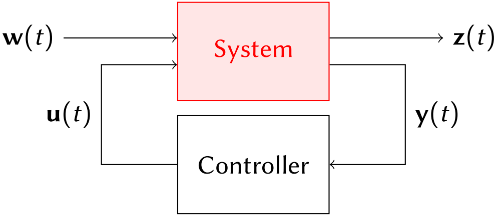

Controller:

$$\dot{\mathbf{x}}_K = A_K \mathbf{x}_K + B_K \mathbf{y}$$
$$\mathbf{u} = C_K \mathbf{x}_K$$

Closed-loop LTI:

$$\dot{h} = \mathscr{A} h + \mathscr{B} \mathbf{w}$$
$$\mathbf{z} = \mathscr{C} h$$

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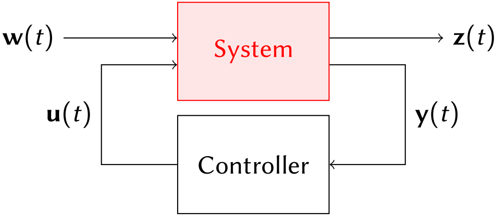

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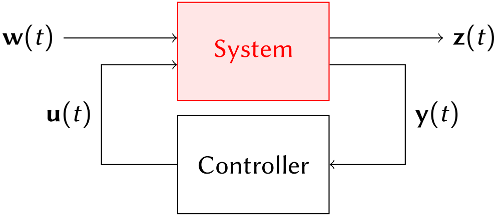

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- System is stable
- Effect of noise $\mathbf{w}(t)$ on output $\mathbf{z}(t)$ is minimized in $\mathscr{H}_2$ norm

$$\|G\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} \|G(i\omega)\|_F^2 \, d\omega}, \text{ where } G(s) = \mathscr{C}(sI - \mathscr{A})^{-1} \mathscr{B}$$

The Control Problem

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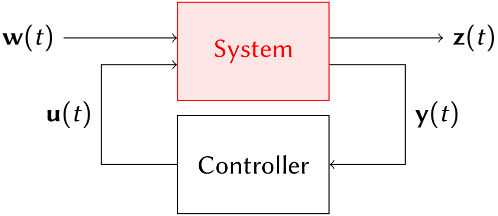

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Theorem 1 ([Zhou, Doyle, Glover 1996])

Let $P, Q \in \mathbb{R}^{n\times n}$ be the unique stabilizing solutions to the algebraic Riccati equations (AREs)

$$A^T P + PA - PB_u(D_{zu}^T D_{zu})^{-1} B_u^T P + C_z^T C_z = 0 \tag{1}$$

$$AQ + QA^T - QC_y^T(D_{yw} D_{yw}^T)^{-1} C_y Q + B_w B_w^T = 0. \tag{2}$$

Define the control gain $K := (D_{zu}^T D_{zu})^{-1} B_u^T P$ and the observer gain $L := QC_y^T(D_{yw} D_{yw}^T)^{-1}$. Then the $\mathcal{H}_2$ optimal controller is given by

$$\mathbf{u}(t) := -K\mathbf{x}_K(t), \tag{3}$$

where $\mathbf{x}_K(t) \in \mathbb{R}^n$ is the solution to the observer-equation:

$$\dot{\mathbf{x}}_K(t) = (A - BK)\mathbf{x}_K(t) + L[\mathbf{y}(t) - C_y\mathbf{x}_K(t)]. \tag{4}$$

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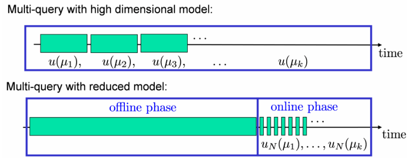
Reduced Basis Approximation

Reduced Basis Method in a Nutshell

- Calculation is split in **offline and online phase**



- Online efficiency through **parameter separability**
- Basis generation (usually) by **Greedy procedure**
- Many **applications**: Elliptic/parabolic PDEs, variational inequalities, dyn. systems, Kalman filter, …

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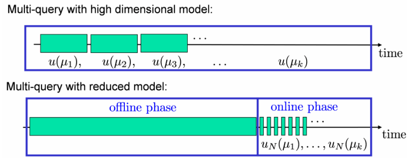

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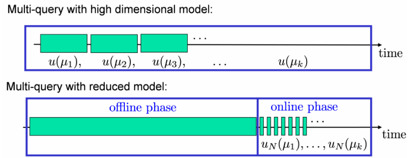

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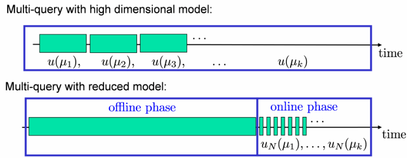

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Reduction of the ARE

How to do that?

Recall: The ARE is a nonlinear matrix-valued equation for $X(\mu) \in \mathbb{R}^{n \times n}$:

$$A(\mu)^T X(\mu) + X(\mu) A(\mu) - X(\mu) B(\mu) R(\mu)^{-1} B(\mu)^T X(\mu) + C(\mu)^T C(\mu) = 0.$$

SVD/Eigenvalue decomposition $X(\mu) = V \Sigma V^T$

Idea:

[S., Haasdonk 2017]

Approximate

$$X(\mu) \approx \widehat{X}(\mu) := V_X X_N(\mu) V_X^T$$

Projection leads to small $N \ll n$ dimensional ARE

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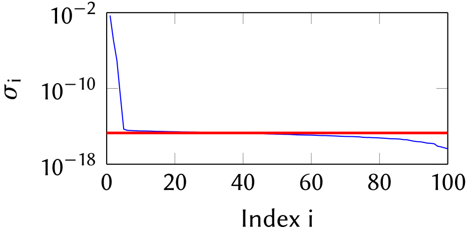

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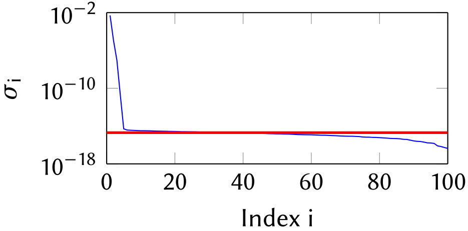

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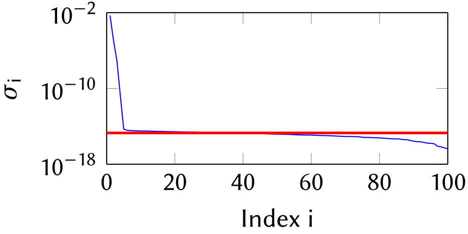

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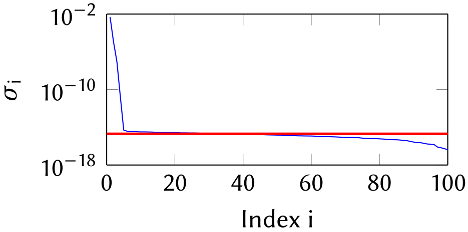

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Low-Rank Factor Greedy Algorithm

Algorithm 1: Low-Rank Factor Greedy Algorithm (LRFG) for the basis generation.

Data: Initial basis matrix V_0, training set $\mathscr{P}_{\text{train}} \subset \mathscr{P}$, tolerance ε, inner tolerance $tol_i \in [0, 1]$, error indicator $\Delta(V, \mu)$

Set $V := V_0$.

while $\max_{\mu \in \mathscr{P}_{train}} \Delta(V, \mu) > \varepsilon$ **do**

- $\mu^* := \arg\max_{\mu \in \mathscr{P}_{\text{train}}} \Delta(V, \mu)$
- Solve the full dimensional ARE for the low rank factor $Z(\mu^*)$
- Set $Z_\perp := (I_n - VV^T)Z(\mu^*)$
- $\widehat{Z} = \text{POD}(Z_\perp, \text{tol}_i)$
- Extend the current basis matrix $V = (V, \widehat{Z})$

Return V.

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$$\Delta(W,\mu) := \frac{\|\mathscr{R}(\widehat{P}(\mu))\|_F}{\|C(\mu)^T Q(\mu) C(\mu)\|_F}$$

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Algorithm 4: Low-Rank Factor Greedy Algorithm (LRFG) for the basis generation.

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  $\mu^* := \arg\max_{\mu \in \mathscr{P}_{\text{train}}} \Delta(V,\mu)$

  Solve the full dimensional ARE for the low rank factor $Z(\mu^*)$

  Set $Z_\perp := (I_n - VV^T)Z(\mu^*)$

  $\widehat{Z} = \text{POD}(Z_\perp, \text{tol}_i)$

  Extend the current basis matrix $V = (V, \widehat{Z})$

Return V.

$$\Delta(W,\mu) := \frac{\|\mathscr{R}(\widehat{P}(\mu))\|_F}{\|C(\mu)^T Q(\mu) C(\mu)\|_F}$$

Low-Rank Factor Greedy Algorithm

Algorithm 5: Low-Rank Factor Greedy Algorithm (LRFG) for the basis generation.

Data: Initial basis matrix V_0, training set $\mathscr{P}_{\text{train}} \subset \mathscr{P}$, tolerance ε, inner tolerance $tol_i \in [0,1]$, error indicator $\Delta(V,\mu)$

Set $V := V_0$.

while $\max_{\mu \in \mathscr{P}_{train}} \Delta(V,\mu) > \varepsilon$ **do**

- $\mu^* := \arg\max_{\mu \in \mathscr{P}_{\text{train}}} \Delta(V,\mu)$
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$$\Delta(W,\mu) := \frac{\|\mathscr{R}(\widehat{P}(\mu))\|_F}{\|C(\mu)^T Q(\mu) C(\mu)\|_F}$$

$\text{POD}(X, \text{tol}_i)$ returns first l SVs, such that:

$$\frac{\sum_{i=1}^{l} \sigma_i^2}{\sum_{i=1}^{\text{rank}(X)} \sigma_i^2} \geq \text{tol}_i (> 0.99)$$

What is the overall idea?
The observer equation

The state estimation involves an n-dimensional ODE:

$$\dot{\mathbf{x}}_K(t;\mu) = (A(\mu) - B(\mu)K(\mu))\mathbf{x}_K(t;\mu) + L(\mu)[\mathbf{y}(t) - C_y(\mu)\mathbf{x}_K(t;\mu)].$$

Apply **POD** by calculating snapshots for some $\mu \in \mathscr{P}$ and inputs $\mathbf{y}(t)$:

$$X := [\mathbf{x}_K(t_1;\mu_1), \mathbf{x}_K(t_2;\mu_1), \ldots, \mathbf{x}_K(t_l;\mu_P)]$$

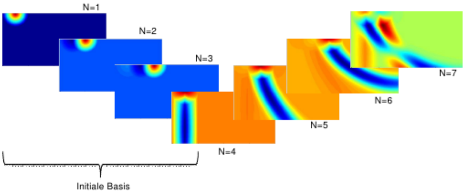

Figure: *Some bases for an advection diffusion equation.*

S. Volkwein. *Lecture Notes: Proper Orthogonal Decomposition: Theory and Reduced-Order Modelling.* 2013.

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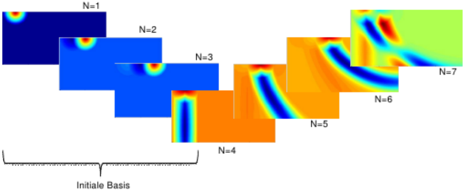

Figure: *Some bases for an advection diffusion equation.*

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The Reduced Controller

Or: Why is now everything faster?

Build three different bases V_P, V_Q and $V_{\mathbf{x}_K}$ (independently, offline). See[S., Haasdonk 2016].

Definition 2 (Reduced controller)

Let P_{N_P} and Q_{N_Q} be the solutions of the reduced AREs and define the reduced gains$\widehat{K} := (D_{zu}^T D_{zu})^{-1} B_u^T \widehat{P}$, $\widehat{L} := \widehat{Q} C_y^T (D_{yw} D_{yw}^T)^{-1}$, where

$$\widehat{P} = V_P P_{N_P} V_P^T, \quad \widehat{Q} = V_Q Q_{N_Q} V_Q^T,$$

and the control signal

$$\widehat{\mathbf{u}}(t) := -\widehat{K}\widetilde{\mathbf{x}}(t),$$

where $\widetilde{\mathbf{x}}(t) \in \mathbb{R}^{N_{\widetilde{x}}}$ satisfies the reduced observer equation

$$\dot{\widetilde{\mathbf{x}}} = V_{\mathbf{x}_K}^T (A - B\widehat{K} - \widehat{L}C_y) V_{\mathbf{x}_K} \widetilde{\mathbf{x}} + V_{\mathbf{x}_K}^T \widehat{L}\mathbf{y}$$

A. Schmidt and B. Haasdonk. 'Reduced basis method for H2 optimal feedback control problems'. *IFAC-PapersOnLine* (2016). 2nd IFAC Workshop on Control of Systems Governed by Partial Differential Equations CPDE 2016.

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Numerical Results

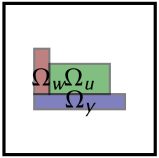
Distributed Control of Damped 2D Wave

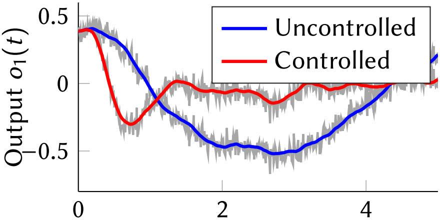
Consider for $\Omega := [0,1]^2$ the following damped wave-equation with parameters $\mu_1, \mu_2 \in [0.1, 2] \times [1, 100]$.

$$
\begin{aligned}
f_{tt} - 0.1\Delta f + \mu_1 f_t &= 10 \cdot \mathbf{1}_{\Omega_u} u + 3 \cdot \mathbf{1}_{\Omega_w} w_1, && t > 0,\\
f(0,\xi;\mu) &= \sin(\xi_1 \pi)\sin(\xi_2 \pi), && \xi \in \Omega\\
f_t(0,\xi;\mu) &= 0, && \xi \in \Omega,\\
s(t;\mu) &= \frac{1}{|\Omega_y|}\int_{\Omega_y} f(t,\xi;\mu)d\xi + 0.05 w_2(t).
\end{aligned}
$$

Consider two performance outputs:

$$
o_1(t) = \frac{1}{|\Omega_y|}\int_{\Omega_y} \mu_2 f(t,\xi;\mu)d\xi, \quad o_2(t) = 0.1u(t)
$$





The equations are discretized in space by using FD: $n = 800$.
We construct the three bases: V_P, V_Q and $V_{\hat{\mathbf{x}}}$. The observer system was simulated with $\mathbf{y}(t) = \sin(t^2)$ with $t \in [0, 2\pi]$.

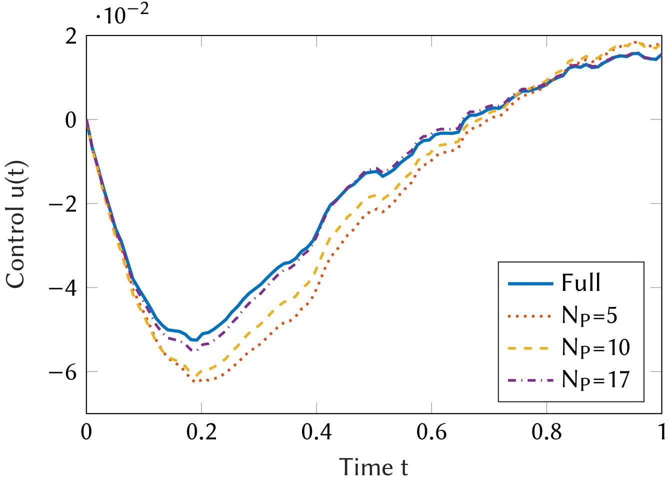
	$t_{\text{full}}[s]$	$t_{\text{red}}[s]$	Basis size
Feedback ARE P	15.2	0.002	17
Observer ARE Q	14.3	0.004	30
State Estimation ODE $\hat{\mathbf{x}}$	$4.1 \cdot 10^{-4}$	$2.0 \cdot 10^{-5}$	25

Table: Basis generation results.

Overall RB approximation

Overall controller is ODE of dimension 25 instead of 800 and the speed up for the AREs is in the magnitude of **5.000** (using `care`). By using parameter separability, e.g. $A(\mu) = \sum_{i=1}^{Q_A} \theta_i(\mu) A_i$ the online simulation can be implemented **independent** of n.

Setting: Full basis for observer gain basis and state estimation basis $\widehat{x}$.



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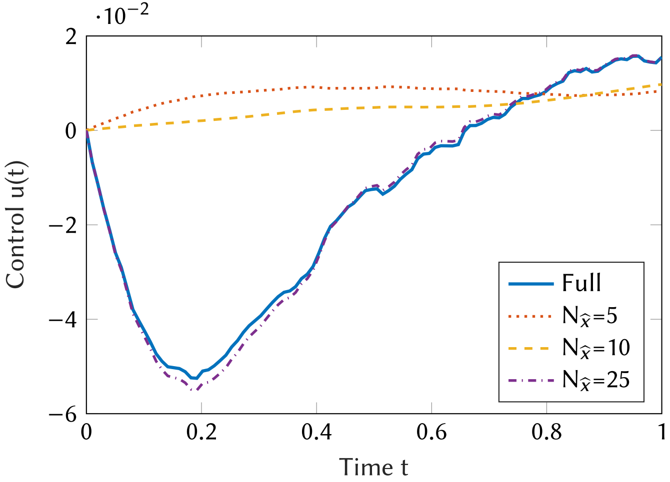

Relative maximum error over test set $\mathscr{P}_{\text{test}}$ with $|\mathscr{P}_{\text{test}}| = 50$.

$$\max_{\mu \in \mathscr{P}_{\text{test}}} \frac{\| G(\cdot;\mu) - \widehat{G}(\cdot;\mu) \|_2}{\| G(\cdot;\mu) \|_2}. \tag{5}$$

Table: Relative error in the transfer functions from **w** to **z** in the closed loop systems.

		N_P			
		5	10	12	17
N_Q	5	2.78e-02	2.63e-02	2.66e-02	2.88e-02
	10	6.63e-03	7.20e-03	6.49e-03	5.23e-03
	15	1.44e-03	1.94e-03	1.40e-03	8.29e-04
	30	1.31e-03	1.83e-03	1.27e-03	6.34e-04

Surprise: No instabilities occured! See famous one-page article[Doyle '78].

J. Doyle. 'Guaranteed margins for LQG regulators'. *Automatic Control, IEEE Transactions on* (Aug. 1978).

Conclusion and Outlook

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We have seen: **Model order reduction for $\mathscr{H}_2$ optimal control problems.**

- Full parametric and realistic control setup
- Expensive due to two AREs and state estimation
- $\Rightarrow$ Model reduction for AREs and the state estimation
- $\Rightarrow$ Large speed-up

What next?

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- Robustification?
- Same approach for $\mathscr{H}_\infty$-control problems

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Thank you for your attention!
Questions? Comments?

References I

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K. Zhou, J. Doyle and K. Glover. *Robust and Optimal Control*. 1996.

The Control Problem

Mathematical model of the controller

Consider the open-loop system

$$\dot{\mathbf{x}} = A\mathbf{x} + B_u\mathbf{u} + B_w\mathbf{w}$$
$$\mathbf{z} = C_z\mathbf{x} + D_{zu}\mathbf{u}$$
$$\mathbf{y} = C_y\mathbf{x} + D_{yw}\mathbf{w}$$

Ansatz for controller

$$\dot{\mathbf{x}}_K = A_K\mathbf{x}_K + B_K\mathbf{y}$$
$$\mathbf{u} = C_K\mathbf{x}_K$$

The interconnection (i.e. the closed-loop system) for $h^T = (\mathbf{x}, \mathbf{x}_K)$ can be written as

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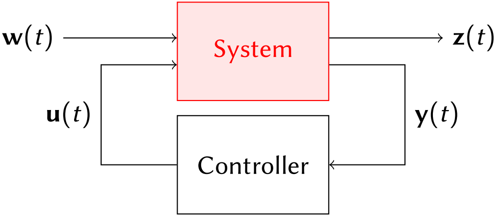

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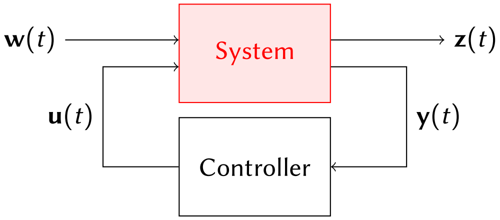

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