1-D cubic NLS with several Diracs as initial data and consequences

Valeria Banica (Univ. Pierre et Marie Curie)

joint work with **Luis Vega** (BCAM)

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Plan of the talk

- The 1-D cubic NLS with rough data
- The 1-D cubic NLS with several Dirac data
  (existence result and Talbot effect)
- About the binormal flow and vortex filament dynamics
- The results transferred to the binormal flow

Results for 1-D cubic NLS with rough data

The 1-D cubic NLS

$$iu_t + u_{xx} \pm |u|^2 u = 0$$

is well-posed in H^s, $s \geq 0$ (Ginibre-Velo 79, Cazenave-Weissler 90).

If $s < 0$ it is ill-posed (Kenig-Ponce-Vega 01, Christ-Colliander-Tao 03).

Well-posedness holds for data with Fourier transform in L^p spaces (Vargas-Vega 01, Grünrock 05, Christ 07).

Methods of proving existence : fixed points arguments relying on Strichartz type spaces.

Results for 1-D cubic NLS with Dirac data

For $a\delta_0$ as initial data, the 1-D cubic NLS is ill-posed: when looking for a (unique) solution, by using Galilean invariance, one obtains $e^{ia^2 \log t} \frac{a}{\sqrt{t}} e^{i\frac{x^2}{4t}}$ which has no limit at $t = 0$ (Kenig-Ponce-Vega 01).

A natural change to do is to consider the perturbed cubic 1DNLS

$$i\psi_t + \psi_{xx} \pm \left(|\psi|^2 - \frac{a^2}{t}\right)\psi = 0,$$

and get as an explicit solution $\frac{a}{\sqrt{t}} e^{i\frac{x^2}{4t}} = ae^{it\Delta}\delta_0(x)$.

The problem is however ill-posed, as smooth perturbations of the solution $\frac{a}{\sqrt{t}} e^{i\frac{x^2}{4t}}$ at time $t = 1$ behave near $t = 0$ as $e^{ia^2 \log t} f(x)$ (B.-Vega 09).

Some notations

For a sequence $\{\alpha_k\}$ and $s \geq 0$ we denote

$$\|\{\alpha_k\}\|^2_{l^{2,s}} := \sum_{k\in\mathbb{Z}} (1+|k|)^{2s}|\alpha_k|^2, \quad \|\{\alpha_k\}\|^2_{l^{\infty,s}} := \sup_{k\in\mathbb{Z}} (1+|k|)^{2s}|\alpha_k|^2.$$

We consider distributions

$$u = \sum_{k\in\mathbb{Z}} \alpha_k \delta_k.$$

Their Fourier transform on $\mathbb{R}$ writes

$$\hat{u}(\xi) = \sum_{k\in\mathbb{Z}} \alpha_k e^{-ik\xi},$$

and in particular $\hat{u}$ is $2\pi-$periodic.

Imposing $\{\alpha_k\} \in l^{2,s}$ translates into $\hat{u} \in H^s(0,2\pi)$.

We denote

$$H^s_{pF} := \{u \in \mathcal{S}'(\mathbb{R}),\ \hat{u}(x+2\pi) = \hat{u}(x), \hat{u} \in H^s(0,2\pi)\}.$$

Result for 1-D cubic NLS with several Dirac data

Theorem (B.-Vega '17)

Let $T > 0$, $s > \frac{1}{2}$, $s - \frac{1}{2} < \tilde{s} \leq s$, $0 < \gamma < 1$ and $\{\alpha_k\} \in l^{2,s}$.
We consider the 1-D cubic NLS equation:

$$\begin{cases} i\partial_t u + \Delta u \pm (|u|^2 - \frac{M}{2\pi t})u = 0, \\ u_{|t=0} = \sum_{k\in\mathbb{Z}} \alpha_k \delta_k, \end{cases}$$

with $M = \sum_{k\in\mathbb{Z}} |\alpha_k|^2$.
There exists $\epsilon_0 = \epsilon_0(T) > 0$ such that if $\|\{\alpha_k\}\|_{l^{2,s}} \leq \epsilon_0$ then we have a local solution $u \in \mathcal{C}([0,T]; H^s_{pF})$.
Moreover, this solution is unique of the form

$$u(t,x) = \sum_{k\in\mathbb{Z}} (\alpha_k + R_k(t)) e^{it\Delta} \delta_k(x),$$

with $\{R_k\}$ satisfying the decay

$$t^{-\gamma} \|\{R_k(t)\}\|_{l^{2,s}} + t\,\|\{\partial_t R_k(t)\}\|_{l^{\infty,\tilde{s}}} < C.$$

Result for 1-D cubic NLS with several Dirac data

Remarks:

- the theorem is a generalization of a result of Kita 2006 valid for subcubic nonlinearities.
- the proof goes as follows:
  - plugging the ansatz $u(t,x) = \sum_{k\in\mathbb{Z}} A_k(t)e^{it\Delta}\delta_k(x)$ into the equation leads to a discrete system on $\{A_k(t)\}$,
  - we solve this discrete system by a fixed point argument, in which we treat separately the nonresonant and the resonant part.
- the resonant part is related to the system

$$i\partial a_k(t) = \frac{1}{2\pi t} a_k(t)(\sum_j |a_j(t)|^2 - M),$$

which has only the constant in time solutions for $M = \sum_j |a_j(0)|^2$. (without the extra-factor M in the initial equation, that is treating directly the cubic equation, one ends up with the resonant system above without M, so we get $a_k(t) = e^{i\frac{\sum_j |a_j(0)|^2}{2\pi}\log t} a_k(0)$ which has no limit at $t = 0$).

The proof: the nonlinearity action on the ansatz

We denote $\mathcal{N}(u) = |u|^2 u$. Plugging $u(t) = \sum_{k\in\mathbb{Z}} A_k(t) e^{it\Delta}\delta_k$ into the equation we get

$$\sum_{k\in\mathbb{Z}} i\partial_t A_k(t) e^{it\Delta}\delta_k = \pm\mathcal{N}(\sum_{j\in\mathbb{Z}} A_j(t) e^{it\Delta}\delta_j) \mp \frac{M}{2\pi t}(\sum_{k\in\mathbb{Z}} A_k(t) e^{it\Delta}\delta_k).$$

As Kita we can rewrite the nonlinear term:

$$\mathcal{N}(\sum_{j\in\mathbb{Z}} A_j(t) e^{it\Delta}\delta_j)(x) = \mathcal{N}(\sum_{j\in\mathbb{Z}} A_j(t) \frac{e^{i\frac{(x-j)^2}{4t}}}{\sqrt{t}}) = \frac{e^{i\frac{x^2}{4t}}}{t\sqrt{t}}\mathcal{N}(\sum_{j\in\mathbb{Z}} A_j(t) e^{ij\cdot} e^{i\frac{j^2}{4t}})(-\frac{x}{2t})$$

and use the $2\pi-$periodicity of $\sum_{j\in\mathbb{Z}} A_j(t) e^{ij\cdot} e^{i\frac{j^2}{4t}}$:

$$= \frac{e^{i\frac{x^2}{4t}}}{t\sqrt{t}} \sum_{k\in\mathbb{Z}} e^{-ik\frac{x}{2t}} \frac{1}{2\pi}\int_0^{2\pi} e^{-ik\theta}\mathcal{N}(\sum_{j\in\mathbb{Z}} A_j(t) e^{ij\theta} e^{i\frac{j^2}{4t}})\, d\theta$$

$$= \sum_{k\in\mathbb{Z}} \left(\frac{e^{-i\frac{k^2}{4t}}}{2\pi t} \int_0^{2\pi} e^{-ik\theta}\mathcal{N}(\sum_{j\in\mathbb{Z}} A_k(t) e^{ij\theta} e^{i\frac{j^2}{4t}})\, d\theta \right) (e^{it\Delta}\delta_k)(x).$$

The proof: the discrete system

The family $e^{it\Delta}\delta_k(x) = \frac{e^{i\frac{(x-k)^2}{4t}}}{\sqrt{t}}$ is an orthogonal basis of $L^2(0, 2\pi t)$, so

$$i\partial_t A_k(t) = \pm\frac{e^{-i\frac{k^2}{4t}}}{2\pi t}\int_0^{2\pi} e^{-ik\theta}\mathcal{N}\Big(\sum_{j\in\mathbb{Z}} A_k(t)e^{ij\theta}e^{i\frac{j^2}{4t}}\Big)\,d\theta \mp \frac{M}{2\pi t}A_k(t).$$

Now we develop the cubic power and get

$$i\partial_t A_k(t) = \pm\frac{1}{2\pi t}\sum_{k-j_1+j_2-j_3=0} e^{-i\frac{k^2-j_1^2+j_2^2-j_3^3}{4t}} A_{j_1}(t)\overline{A_{j_2}(t)}A_{j_3}(t) \mp \frac{M}{2\pi t}A_k(t).$$

We split the summation indices into the following two sets:

$$NR_k = \{(j_1, j_2, j_3) \in \mathbb{Z}^3, k - j_1 + j_2 - j_3 = 0, k^2 - j_1^2 + j_2^2 - j_3^2 \neq 0\},$$
$$Res_k = \{(j_1, j_2, j_3) \in \mathbb{Z}^3, k - j_1 + j_2 - j_3 = 0, k^2 - j_1^2 + j_2^2 - j_3^2 = 0\}.$$

As we are in one dimension, the second set is simply

$$Res_k = \{(k, j, j), (j, j, k), j \in \mathbb{Z}\}.$$

The proof: the fixed point framework

Finally the system writes

$$i\partial_t A_k(t) = \frac{A_k(t)}{2\pi t}\Big(\sum_j |A_j(t)|^2 - M\Big) + \sum_{(j_1,j_2,j_3)\in NR_k} \frac{e^{-i\frac{k^2-j_1^2+j_2^2-j_3^3}{4t}}}{2\pi t} A_{j_1}(t)\overline{A_{j_2}(t)}A_{j_3}(t).$$

We want to obtain the existence of $A_k(t) = \alpha_k + R_k(t)$, with

$$\{R_k\} \in X^\gamma := \{\{f_k\} \in \mathcal{C}([0,T]; l^{2,s}) \cap C^1(]0,T]; l^{\infty,\tilde{s}}), \|\{f_k\}\|_{X^\gamma} < \infty\},$$

where $\|\{f_k\}\|_{X^\gamma} := \sup_{0\le t<T} t^{-\gamma}\|\{f_k(t)\}\|_{l^{2,s}} + t\,\|\{\partial_t f_k(t)\}\|_{l^{\infty,\tilde{s}}}$.

We shall prove that the operator $\Phi : \{R_k\} \to \{\Phi_k(\{R_j\})\}$ defined as

$$\Phi_k(\{R_j\})(t) := i\int_0^t \frac{\alpha_k + R_k(\tau)}{2\pi\tau}\Big(\sum_j |(\alpha_j + R_j(\tau))|^2 - |\alpha_j|^2\Big)\, d\tau$$

$$+i\int_0^t \sum_{(j_1,j_2,j_3)\in NR_k} \frac{e^{-i\frac{k^2-j_1^2+j_2^2-j_3^2}{4\tau}}}{2\pi\tau}(\alpha_{j_1}+R_{j_1}(\tau))\overline{(\alpha_{j_2}+R_{j_2}(\tau))}(\alpha_{j_3}+R_{j_3}(\tau))\, d\tau$$

is a contraction in X^γ on a small ball of radius δ, to be chosen later.

The resonant part Φ_k^R we perform Cauchy-Schwarz in the summation in j, and then in time:

$$|\Phi_k^R(\{R_j\})(t)| \le C\int_0^t \frac{|\alpha_k| + |R_k(\tau)|}{\tau}(\sum_j |\alpha_j||R_j(\tau)| + \sum_j |R_j(\tau))|^2)\, d\tau$$

$$\le C\int_0^t \frac{|\alpha_k| + |R_k(\tau)|}{\tau}(\epsilon_0\delta\tau^\gamma + \delta^2\tau^{2\gamma})\, d\tau$$

$$\le C|\alpha_k|(\epsilon_0\delta t^\gamma + \delta^2 t^{2\gamma}) + C(\int_0^t \frac{|R_k(\tau)|^2}{\tau^{1+\gamma}}(\epsilon_0^2\delta^2\tau^{2\gamma} + \delta^4\tau^{4\gamma})\, d\tau)^{\frac{1}{2}} t^{\frac{\gamma}{2}}.$$

Then

$$\|\Phi^R(\{R_k\})(t)\|_{l^{2,s}}^2 := \sum_k (1+|k|)^{2s}|\Phi_k^R(\{R_j\})|^2$$

$$\le C\epsilon_0^2(\epsilon_0^2\delta^2 t^{2\gamma} + \delta^4 t^{4\gamma}) + C\int_0^t \frac{\sum_k (1+|k|)^{2s}|R_k(\tau)|^2}{\tau^{1+\gamma}}(\epsilon_0^2\delta^2\tau^{2\gamma} + \delta^4\tau^{4\gamma})\, d\tau\, t^\gamma$$

$$\le C\delta^2 t^{2\gamma}(\epsilon_0^4 + \epsilon_0^2\delta^2 t^{2\gamma} + \delta^4 t^{4\gamma}) \le C\delta^2 t^{2\gamma}(\epsilon_0^2 + \delta^2 t^{2\gamma})^2 \le \frac{\delta^2 t^{2\gamma}}{10}.$$

The $\|\partial_t \Phi^R(\{R_k\}(t))\|_{l^{\infty,\tilde{s}}}^2$ is treated similarly, by using $l^{2,s} \subset l^{2,\tilde{s}} \subset l^{\infty,\tilde{s}}$.

The proof: the non-resonant part

On the non-resonant part operator Φ^{NR} we shall perform an integration by parts to get advantage of the non-resonant phase (without the phase gain we have an issue for the discrete summations ; this is the only reason of adding the derivative in time in the definition of the space X^γ)

For instance the boundary term $\Phi_k^{NR,B}(\{R_j\})(t)$ for the IBP is

$$t \sum_{(j_1,j_2,j_3)\in NR_k} \frac{e^{-i\frac{k^2-j_1^2+j_2^2-j_3^2}{4t}}}{\pi(k^2-j_1^2+j_2^2-j_3^2)} (\alpha_{j_1}+R_{j_1}(t))\overline{(\alpha_{j_2}+R_{j_2}(t))}(\alpha_{j_3}+R_{j_3}(t)).$$

On the non-resonant set $1 \leq |k^2-j_1^2+j_2^2-j_3^2| = 2|j_1-j_2||k-j_1|$, so

$$|\Phi_k^{NR,B}(\{R_j\})(t)| \leq Ct \sum_{j_1,j_2\in\mathbb{Z}} \frac{|(\alpha_{j_1}+R_{j_1}(t))(\alpha_{j_2}+R_{j_2}(t))(\alpha_{j_3}+R_{j_3}(t))|}{(1+|j_1-j_2|)(1+|k-j_1|)}.$$

The proof: example of term in the non-resonant part

By Cauchy-Schwarz in the summation in j_1, j_2 we get

$$|\Phi^{NR,B}(\{R_k\})(t)|^2 \leq Ct^2 \sum_{j_1,j_2\in\mathbb{Z}} (1+|j_1|)^{2s}(1+|j_2|)^{2s}|(\alpha_{j_1}+R_{j_1}(t))(\alpha_{j_2}+R_{j_2}(t))|^2$$

$$\times \sum_{j_1,j_2\in\mathbb{Z}} \frac{|\alpha_{k-j_1+j_2}+R_{k-j_1+j_2}(t)|^2}{(1+|j_1|)^{2s}(1+|j_2|)^{2s}(1+|j_1-j_2|)^2(1+|k-j_1|)^2}.$$

Therefore we control

$$\|\Phi^{NR,B}(\{R_k\})(t)\|^2_{l^{2,s}} \leq Ct^2(\epsilon_0^2+\delta^2 t^{2\gamma})^3,$$

provided that the following sum is finite

$$\sum_{k,j_1,j_2\in\mathbb{Z}} \frac{(1+|k|)^{2s}}{(1+|j_1|)^{2s}(1+|j_2|)^{2s}(1+|k-j_1+j_2|)^{2s}(1+|j_1-j_2|)^2(1+|k-j_1|)^2}.$$

Example of summation estimation

Lemma

For $0 < s - \frac{1}{2} < \tilde{s}$ the following sum is finite:

$$\sum_{k,j_1,j_2\in\mathbb{Z}} \frac{(1+|k|)^{2s}}{(1+|j_1|)^{2\tilde{s}}(1+|j_2|)^{2s}(1+|j_1-j_2|)^2(1+|k-j_1|)^2(1+|k-j_1+j_2|)^{2s}}.$$

We split the summation in k into nine regions, in terms of the comparison of k with j_1 and with $j_1 - j_2$. We denote, for j_1, j_2 fixed, the two series of three exhaustive regions on $\mathbb{Z}$:

$$B_1 = \{|k| \le \frac{1}{2}|j_1-j_2|\},\ B_2 = \{\frac{1}{2}|j_1-j_2| < |k| \le \frac{3}{2}|j_1-j_2|\},\ B_3 = \{\frac{3}{2}|j_1-j_2| < |k|\},$$

$$C_1 = \{|k| \le \frac{1}{2}|j_1|\},\quad C_2 = \{\frac{1}{2}|j_1| < |k| \le \frac{3}{2}|j_1|\},\quad C_3 = \{\frac{3}{2}|j_1| < |k|\}.$$

We split the sum as follows:
$\sum_{k,j_1,j_2\in\mathbb{Z}} = \sum_{j_1,j_2\in\mathbb{Z}}(\sum_{k\in B_1\cup B_3} + \sum_{k\in B_2\cap C_1} + \sum_{k\in B_2\cap C_2} + \sum_{k\in\cup B_2\cap C_3}).$

Example of summation estimation

On $B_1 \cup B_3$ we have $\frac{(1+|k|)^{2s}}{(1+|k-j_1+j_2|)^{2s}} < C$ so, as $2s > 1$,

$$\sum_{j_1,j_2\in\mathbb{Z},k\in B_1\cup B_3} \frac{(1+|k|)^{2s}}{(1+|j_1|)^{2\tilde{s}}(1+|j_2|)^{2s}(1+|j_1-j_2|)^2(1+|k-j_1|)^2(1+|k-j_1+j_2|)^{2s}}$$

$$\leq C \sum_{j_1,j_2\in\mathbb{Z}} \frac{1}{(1+|j_2|)^{2s}(1+|j_1-j_2|)^2} \sum_{k\in B_1\cup B_3} \frac{1}{(1+|k-j_1|)^2} < \infty.$$

On C_1 we have $|k| \leq \frac{1}{2}|j_1|$ so, as $2s > 1$ and $2\tilde{s}+2-2s > 1$,

$$\sum_{j_1,j_2\in\mathbb{Z},k\in B_2\cap C_1} \frac{(1+|k|)^{2s}}{(1+|j_1|)^{2\tilde{s}}(1+|j_2|)^{2s}(1+|j_1-j_2|)^2(1+|k-j_1|)^2(1+|k-j_1+j_2|)^{2s}}$$

$$\leq C \sum_{j_1,j_2\in\mathbb{Z}} \frac{(1+|j_1|)^{2s}}{(1+|j_1|)^{2\tilde{s}+2}(1+|j_2|)^{2s}(1+|j_1-j_2|)^2} \sum_{k\in B_2\cap C_1} \frac{1}{(1+|k-j_1+j_2|)^{2s}}$$

$$\leq C \sum_{j_1\in\mathbb{Z}} \frac{1}{(1+|j_1|)^{2\tilde{s}+2-2s}} \sum_{j_2\in\mathbb{Z}} \frac{1}{(1+|j_1-j_2|)^2} < \infty.$$

Example of summation estimation

On $B_2 \cap C_2$ we have $|k| \leq \frac{3}{2}|j_1|$ and $|j_1| < 3|j_1 - j_2|$ so

$$\sum_{j_1,j_2\in\mathbb{Z},k\in B_2\cap C_2} \frac{(1+|k|)^{2s}}{(1+|j_1|)^{2\tilde{s}}(1+|j_2|)^{2s}(1+|j_1-j_2|)^2(1+|k-j_1|)^2(1+|k-j_1+j_2|)^{2s}}$$

$$\leq C\sum_{j_1,j_2\in\mathbb{Z}} \frac{(1+|j_1|)^{2s}}{(1+|j_1|)^{2\tilde{s}}(1+|j_2|)^{2s}(1+|j_1-j_2|)^2} \sum_{k\in B_2\cap C_2} \frac{1}{(1+|k-j_1|)^2}$$

$$\leq C\sum_{j_1,j_2\in\mathbb{Z}} \frac{1}{(1+|j_1|)^{2\tilde{s}+2-2s}(1+|j_2|)^{2s}} < \infty.$$

On $B_2 \cap C_3$ we have $\frac{3}{2}|j_1| < |k| \leq \frac{3}{2}|j_1 - j_2| \leq \frac{3}{2}(|j_1| + |j_2|)$:

$$\sum_{j_1,j_2\in\mathbb{Z},k\in B_2\cap C_3} \frac{(1+|k|)^{2s}}{(1+|j_1|)^{2\tilde{s}}(1+|j_2|)^{2s}(1+|j_1-j_2|)^2(1+|k-j_1|)^2(1+|k-j_1+j_2|)^{2s}}$$

$$\leq C\sum_{j_1,j_2\in\mathbb{Z}} \frac{(1+|j_1|)^{2s}+(1+|j_2|)^{2s}}{(1+|j_1|)^{2\tilde{s}+2}(1+|j_2|)^{2s}(1+|j_1-j_2|)^2} \sum_{k\in B_2\cap C_3} \frac{1}{(1+|k-j_1+j_2|)^{2s}}$$

$$\leq C\sum_{j_1,j_2\in\mathbb{Z}} \frac{1}{(1+|j_1|)^{2\tilde{s}+2-2s}(1+|j_2|)^{2s}} + C\sum_{j_1,j_2\in\mathbb{Z}} \frac{1}{(1+|j_1|)^{2\tilde{s}+2}(1+|j_1-j_2|)^2} < \infty.$$

A Talbot effect

The linear and nonlinear Schrödinger evolution on the torus of functions with bounded variation was proved to present Talbot effect features (Berry, Klein; Oskolkov; Kapitanski, Rodnianski; Taylor ; Erdogan, Tzirakis '96-'13).

Here we place ourselves in a more singular setting on $\mathbb{R}$.

A consequence of the Theorem is that the solution $u(t)$ of the modified cubic NLS with initial data $u_0 = \sum_{k\in\mathbb{Z}} \alpha_k \delta_k$ such that $\|\{\alpha_k\}\|_{l^{2,s}} \leq \epsilon_0$ behaves for small times like the linear evolution $e^{it\Delta}u_0$.

We compute first the linear evolution $e^{it\Delta}u_0$ which display a Talbot effect.

A Talbot effect for linear evolutions of several Diracs

Proposition

Let $t = \frac{1}{2\pi}\frac{p}{q}$ with q odd. Let u_0 be such that $\hat{u}_0$ is $2\pi-$periodic and $\hat{u}_0$ located modulo 2π only in a neighborhood of zero of radius less than $\frac{\pi}{p}$. For a given $x \in \mathbb{R}$ we denote $l_x \in \mathbb{Z}$ and $0 \leq m_x < q$ the unique numbers such that

$$x - l_x - \frac{m_x}{q} \in [0, \frac{1}{q}),$$

and we define

$$\xi_x := \frac{\pi q}{p}(x - l_x - \frac{m_x}{q}) \in [0, \frac{\pi}{p}).$$

Then for some $\theta_{m_x} \in \mathbb{R}$

$$e^{it\Delta} u_0(x) = \frac{1}{\sqrt{q}}\, \hat{u}_0(\xi_x)\, e^{-it\,\xi_x^2 + ix\,\xi_x + i\theta_{m_x}}.$$

The data $u_0 = \sum_{k\in\mathbb{Z}} \delta_k$ enters the above setting of the $2\pi-$periodicity in Fourier and localization in Fourier, as $\hat{u}_0 = u_0 = \sum_{k\in\mathbb{Z}} \delta_k$. Therefore $e^{it\Delta}u_0(x) = 0$ for $x \notin \mathbb{Z} + \frac{\mathbb{Z}}{q}$, and is a Dirac mass otherwise, which is a Talbot effect. This kind of data does not enter our nonlinear framework.

A Talbot effect for nonlinear evolutions of several Diracs

If moreover $\hat{u}_0$ is located modulo 2π only in a neighborhood of zero of radius less than $\eta\frac{\pi}{p}$ with $0 < \eta < 1$, then the previous linear evolution vanishes for x at distance larger than $\frac{\eta}{q}$ from $\mathbb{Z} + \frac{\mathbb{Z}}{q}$.

Proposition

Let u_0 such that $\hat{u}_0$ is a $2\pi-$periodic, located modulo 2π only in a neighborhood of zero of radius less than $\eta\frac{\pi}{p}$ with $0 < \eta < 1$ and having Fourier coefficients $\{\alpha_k\}$ satisfying $\|\{\alpha_k\}\|_{l^{2,s}} \leq \epsilon_0$.

Let $u(t,x)$ be the local in time solution obtained in the Theorem.

Then for $t = \frac{1}{2\pi}\frac{p}{q}$ with q odd and for all x at distance larger than $\frac{\eta}{q}$ from $\mathbb{Z} + \frac{\mathbb{Z}}{q}$ the function $u(t,x)$ almost vanish for small times, in the sense:

$$u(t,x) = \sum_{k\in\mathbb{Z}} R_k(t) e^{it\Delta}\delta_k(x),$$

with $\{R_k\}$ satisfying the decay

$$t^{-\gamma}\|\{R_k(t)\}\|_{l^{2,s}} + t\,\|\{\partial_t R_k(t)\}\|_{l^{\infty,\tilde{s}}} < C.$$

A model for one vortex filament dynamics

In a 3D homogeneous incompressible inviscid fluid a vortex filament is a a vortex tube with infinitesimal cross section: the vorticity is a singular measure supported along a curve in $\mathbb{R}^3$ that moves with the flow.

The binormal flow is the oldest, simpler and richer model for one vortex filament dynamics (Da Rios 1906, Arms-Hama 1965 using a truncated Biot-Savart's law). It imposes the evolution in time of a $\mathbb{R}^3$-curve $\chi(t)$ by

$$\chi_t = \chi_s \wedge \chi_{ss} = c\, b.$$

Frenet's system (direct computations)	Hasimoto 72 (...Madelung^{-1})

$\downarrow$

The filament function $u(t,s) = c(t,s)e^{i\int_0^s \tau(t,s)ds}$ satisfies the 1D NLS

$$iu_t + u_{ss} + \frac{1}{2}\left(|u|^2 - A(t)\right) u = 0,$$

with $A(t)$ in terms of curvature and torsion $(c,\tau)(t,0)$.

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Frenet's system (direct computations)	↓	Hasimoto 72 (...Madelung^{-1})

The filament function $u(t,s) = c(t,s)e^{i\int_0^s \tau(t,s)ds}$ satisfies the 1D NLS

$$iu_t + u_{ss} + \frac{1}{2}\left(|u|^2 - A(t)\right) u = 0,$$

with $A(t)$ in terms of curvature and torsion $(c,\tau)(t,0)$.

Binormal flow results

Conversely, for $A(t)$ and u s.t. $iu_t + u_{ss} + \frac{1}{2}\left(|u|^2 - A(t)\right) u = 0$, one can construct a solution of the binormal flow.

Examples:

- Lines: $u(t,s) = 0$, $A(t) = 0$.
- Circles: $u(t,s) = 1$, $A(t) = -1$.
- Helices: $u(t,s) = e^{-itN^2} e^{iNs}$, $A(t) = -1$.
- Travelling waves: $u(t,s) = e^{-itN^2} e^{iNs} \frac{1}{2\sqrt{2}} \frac{1}{\cosh(s-2Nt)}$, $A(t) = -1$, (Hasimoto 72, Hopfinger-Browand 81).
- Self-similar solutions $u(t,s) = a\frac{e^{i\frac{x^2}{4t}}}{\sqrt{t}}$ for $A(t) = \frac{|a|^2}{t}$ and perturbations (physicists 70-80, Guttierez-Rivas-Vega 03, Guttierez-Vega 04, Banica-Vega 08-15).

  $\chi(0)$ admits a corner at $x = 0 \rightsquigarrow u_0$ presents a Dirac mass at $x = 0$.

Local well-posedness for (c, τ) in Sobolev spaces (Hasimoto 72, Nishiyama-Tani 94-97, Koiso 97-08), for currents for a weak formulation, with analysis at the level of the frame (Jerrard-Smets 11), for curves with a corner and curvature in weighted space (B.-Vega 15).

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Corners interaction through the binormal flow

• A non-closed curve with one corner and curvature in weighted space smoothens instantaneously is an oscillating way (B.-Vega 15). This is in link with the Kelvin waves observed in vortex reconnections.

• A planar regular polygon with M sides is expected to evolve through the binormal flow to skew polygons with Mq sides at times of type $\frac{p}{q}$ (numerical simulations Grinstein-De Vore 96, Jerrard-Smets 15 and integration of the Frenet frame at rational times De la Hoz-Vega 15).

At infinitesimal times evidence is given for the evolution to be the superposition of the evolutions of each initial corner (De la Hoz-Vega 17).

• Here the framework is of a broken line, for instance with two corners.

The Theorem says that the curve gets through the binormal flow instantaneously smooth. Moreover, for infinitesimal times the evolution is as a superposition of the evolutions of each initial corner.

The smoothening is in an oscillating way: the Proposition insures that at times of type $\frac{p}{q}$ the curvature of $\chi(t)$ displays concentrations near the locations x such that $xq \in \mathbb{Z}$, and almost straight segments are between.