The Energy Critical Wave Equation: The Radial Case

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2017

We are considering the focusing, energy critical nonlinear wave equation in 3 space dimensions

$$\begin{cases} \partial_t^2 u - \Delta u - u^5 = 0, & x \in \mathbb{R}^3, t \in \mathbb{R} \\ u\big|_{t=0} = u_0 \in \dot{H}^1, \\ \partial_t u\big|_{t=0} = u_1 \in L^2. \end{cases} \tag{NLW}$$

The associated linear equation is

$$\begin{cases} \partial_t^2 w - \Delta w = h, \\ w\big|_{t=0} = w_0, \\ \partial_t w\big|_{t=0} = w_1, \end{cases} \tag{LW}$$

whose solution is given, by the Fourier method, as

$$\begin{aligned} w(t) = \cos(\sqrt{-\Delta}\, t) w_0 + \frac{\sin(\sqrt{-\Delta}\, t)}{\sqrt{-\Delta}} w_1 \\ + \int_0^t \frac{\sin(\sqrt{-\Delta}(t - t'))}{\sqrt{-\Delta}} h(t') dt', \end{aligned}$$

or

$$\begin{aligned} w(t) &= S_L(t)(w_0, w_1) + D_L(t)h \\ &= S_L(t)\vec{w}(0) + D_L(t)h. \end{aligned}$$

An important property of the (LW) is finite speed of propagation: If $\mathrm{supp}(w_0, w_1) \cap B(x_0, a) = \emptyset$, and

$$\mathrm{supp}\ h \cap \left(\cup_{0 \le t \le a} B(x_0, a - t) \times \{t\}\right) = \emptyset,$$

then $\vec{w} = 0$ on $\cup_{0 \le t \le a} B(x_0, a - t) \times \{t\}$.

In odd dimensions we also have the “strong Huygens principle.” If $\vec{h} = 0$, $\mathrm{supp}(w_0, w_1) \subset B(\bar{x}, b)$, then, for $t > 0$

$$\mathrm{supp}\ \vec{w}(t) \subset \{x : t - b < |x - \bar{x}| < t + b\}$$

In even dimensions, only the upper bound holds.

In order to study the local theory of the Cauchy problem for (NLW) we need the Strichartz estimate for (LW):

$$\sup_t \|\vec{w}(t)\|_{\dot{H}^1\times L^2} + \|w\|_{L^5_t L^{10}_x} \leq C\big(\|(w_0, w_1)\|_{\dot{H}^1\times L^2} + \|h\|_{L^1_t L^2_x}\big). \quad \text{(S)}$$

By using the standard contraction mapping theorem, we can show (by a solution will mean $u \in C(\mathring{I}, \dot{H}^1 \times L^2), u \in L^5_J L^{10}_x \ \forall J \Subset I$, and $u(t) = S_L(t)(u_0, u_1) + D_L(t)(u^5)$): $\exists \delta_0$ such that, if

$$\|S_L(t)(u_0, u_1)\|_{L^5_I L^{10}_x} < \delta_0,$$

then we have a unique solution in the interval I. Thus, if $\|(u_0, u_1)\|_{\dot{H}^1 \times L^2}$ is small, we can take $I = \mathbb{R}$. These solutions "scatter" i.e. there exists (w_0^+, w_1^+) such that

$$\lim_{t \to \infty} \|\vec{u}(t) - \vec{S}_L(t)(w_0^+, w_1^+)\|_{\dot{H}^1 \times L^2} = 0.$$

For large data we have solutions u with a maximal interval of existence $I = (T_-(u), T_+(u))$.

The energy norm is "critical" since, for $\lambda > 0$,

$$u_\lambda(x,t) = \lambda^{-1/2} u(x/\lambda, t/\lambda)$$

is also a solution and $\|(u_{0,\lambda}, u_{1,\lambda})\|_{\dot{H}^1 \times L^2} = \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}$.

The equation is focusing and has two conservation laws, energy and momentum:

$$E(u_0,u_1) = \frac{1}{2}\int |\nabla u_0|^2 + |u_1|^2\,dx - \frac{1}{6}\int |u_0|^6 dx,$$

$$P(u_0,u_1) = \int \nabla u_0 u_1 dx \quad \text{(important in the non–radial case).}$$

There are solutions which blow–up in finite time, i.e. $T_+ < \infty$ even with $\sup_{0<t<T_+} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} < \infty$ (type II blow–up solutions) (Krieger–Schlag–Tataru 09, Hillairet–Raphaël 12, Jendrej 15).

There are also solutions with $T_+ = \infty$ which do not scatter, for instance solutions Q of the elliptic equation, $Q \in \dot{H}^1 \setminus \{0\}$, $\Delta Q + Q^5 = 0$ ($Q \in \Sigma$). As an example we have

$$W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-1/2},$$

which is the one of smallest energy, the "ground state." Also, $\pm W_\lambda(x)$ are the only radial elements of Σ.

Other non–scattering solutions are the traveling wave solutions, obtained by the Lorentz transformations of $Q \in \Sigma$. They have the form, for $|\vec{\ell}| < 1$, $Q_{\vec{\ell}}(x,t) = Q_{\vec{\ell}}(x - \vec{\ell}t, 0)$, where

$$Q_{\vec{\ell}}(x,0) = Q\left(\left[\frac{1}{\sqrt{1-|\vec{\ell}|^2}} - 1\right]\frac{\vec{\ell}\cdot x}{|\vec{\ell}|^2}\vec{\ell} + x\right),$$

$Q \in \Sigma$. These are are all of the traveling wave solutions (DKM 14).

We have the finite time blow–up criterion: $T_+ < \infty$ iff

$$\|u\|_{L^5_{[0,T_+)} L^{10}_x} = \infty.$$

If $\|u\|_{L^5_{[0,T_+)} L^{10}_x} < \infty$, then $T_+ = \infty$ and u scatters.

An important role in the analysis (via the "concentration compactness/rigidity theorem" method of K–Merle 05) is played by the "profile decomposition" (Bahouri–Gérard, Merle–Vega). It can be thought of as a way to quantify the lack of compactness in the Strichartz estimate (S).

Let $\{(u_{0,n}, u_{1,n})\}$ be a bounded sequence in $\dot{H}^1 \times L^2$. Then (Bahouri–Gérard 99), after extraction, we can find $\{U_L^j\}_j$ a sequence of solutions of (LW) and parameters $\{(\lambda_n^j, x_n^j, t_n^j)\}$, $\lambda_n^j > 0, x_n^j \in \mathbb{R}^3$, $t_n^j \in \mathbb{R}$ verifying the orthogonality condition:

$$j \neq k \implies \lim_n \frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} + \frac{|t_n^j - t_n^k|}{\lambda_n^j} + \frac{|x_n^j - x_n^k|}{\lambda_n^j} = \infty,$$

such that, with $U_{L,n}^j(x,t) = \frac{1}{(\lambda_n^j)^{1/2}} U_L^j\left(\frac{x-x_n^j}{\lambda_n^j}, \frac{t-t_n^j}{\lambda_n^j}\right)$ and $w_n^J(x,t) = S_L(t)(u_{0,n}, u_{1,n}) - \sum_{j=1}^J U_{L,n}^j(t)$, we have $\overline{\lim}_J \overline{\lim}_n \|(w_{0,n}^J, w_{1,n}^J)\|_{\dot{H}^1 \times L^2} < \infty$ and $\overline{\lim}_J \overline{\lim}_n \|w_n^J\|_{L^5_{\mathbb{R}} L^{10}_x} = 0$, $(w_{0,n}^J, w_{1,n}^J) = \vec{w}_n^J(\cdot, 0)$. In addition, we have $\overline{\lim}_J \overline{\lim}_n \|w_n^J\|_{L^\infty_{\mathbb{R}} L^6_x} = 0$.

The profiles are constructed as weak limits: for each j,

$$\vec{S}_L(t_n^j/\lambda_n^j)\big((\lambda_n^j)^{1/2}u_{0,n}(\lambda_n^j\cdot+x_n^j),(\lambda_n^j)^{3/2}u_{1,n}(\lambda_n^j\cdot+x_n^j)\big)\rightharpoonup_n \vec{U}_L^j(0)$$

in $\dot{H}^1\times L^2$. Moreover, $\forall 1\le j\le J$,

$$\big((\lambda_n^j)^{1/2}w_n^J(\lambda_n^j\cdot+x_n^j),(\lambda_n^j)^{3/2}\partial_t w_n^J(\lambda_n^j\cdot+x_n^j)\big)\rightharpoonup_n (0,0)$$

weakly in $\dot{H}^1\times L^2$.

Pythagorean expansions: $\forall J \geq 1$

$$\lim_n \Big[\|u_{0,n}\|^2_{\dot{H}^1} + \|u_{1,n}\|^2 - \Big(\sum_{j=1}^{J} \|U^j_{L,n}(0)\|^2_{\dot{H}^1} + \|\partial_t U^j_{L,n}(0)\|^2 + \|w^J_{0,n}\|^2_{\dot{H}^1} + \|w^J_{1,n}\|^2 \Big) \Big] = 0,$$

$$\lim_n \Big[\|u_{0,n}\|^6_{L^6} - \Big(\sum_{j=1}^{J} \|U^j_{L,n}(0)\|^6_{L^6} + \|w^J_{0,n}\|^6_{L^6} \Big) \Big] = 0.$$

By extracting subsequence and changing the profiles it is always possible to assume for each j, $t_n^j \equiv 0$, or $t_n^j/\lambda_n^j \to \pm\infty$.

Definition

For $j \geq 1$, a nonlinear profile U^j associated to U_L^j, $\{(\lambda_n^j, x_n^j, t_n^j)\}_n$ is a solution U^j of (NLW) such that for large n, $-t_n^j/\lambda_n^j \in I_{max}(U^j)$ and

$$\lim_{n\to\infty} \left\| \vec{U}_L^j(-t_n^j/\lambda_n^j) - \vec{U}^j(-t_n^j/\lambda_n^j) \right\|_{\dot{H}^1 \times L^2} = 0.$$

The nonlinear profiles can be used as “building blocks” for solutions, through the following theorem.

Theorem (Approximation Theorem)

Let $(u_{0,n}, u_{1,n})$ be a bounded sequence in $\dot{H}^1 \times L^2$ which admits a profile decomposition. Let u_n be the corresponding solutions of (NLW).
i) Assume that for all j, U^j scatters forward in time. Letting

$$r_n^J(t) = u_n(t) - \sum_{j=1}^{J} U_n^j(t) - w_n^J(t),$$

where

$$U_n^j(x,t) = \frac{1}{(\lambda_n^j)^{1/2}} U^j\left(\frac{x - x_n^j}{\lambda_n^j}, \frac{t - t_n^j}{\lambda_n^j}\right),$$

we have

$$\overline{\lim_J}\left[\overline{\lim_n}\, \|r_n^J\|_{L^5_{(0,\infty)} L^{10}_x} + \sup_{t\in[0,\infty)} \|\vec{r}_n^J(t)\|_{\dot{H}^1\times L^2}\right] = 0.$$

Theorem (Approximation Theorem contd.)

ii) *Let* $\theta_n \in (0, \infty)$. *Assume that for all* j, n

$$\frac{\theta_n - t_n^j}{\lambda_n^j} < T_+(U^j), \ \overline{\lim_n} \|U_n^j\|_{L^5_{(0,\theta_n)} L^{10}_x} < \infty.$$

Then for all large n, u_n *is defined in* $[0, \theta_n]$, $\overline{\lim}_n \|u_n\|_{L^5_{(0,\theta_n) L^{10}_x}} < \infty$, *and for all* $t \in [0, \theta_n]$

$$\vec{u}_n(x,t) = \sum_{j=1}^{J} \vec{U}_n^j(x,t) + \vec{w}_n^J(x,t) + \vec{r}_n^J(x,t),$$

where

$$\overline{\lim_J} \left[\overline{\lim_n} \|r_n^J\|_{L^5_{(0,\theta_n)} L^{10}_x} + \sup_{t \in [0,\theta_n]} \|\vec{r}_n^J(t)\|_{\dot{H}^1 \times L^2} \right] = 0.$$

Moreover, in all cases, orthogonal expansions hold, for $0 < t < \theta_n$.

Recall the K–Merle solution to the "ground state conjecture":

Theorem (K–Merle 08)

If $E(u_0, u_1) < E(W, 0)$*, then*

i) *If* $\|\nabla u_0\| < \|\nabla W\|$*, then* $T_+ = \infty$, $T_- = -\infty$*, u scatters.*

ii) *If* $\|\nabla u_0\| > \|\nabla W\|$*, then* $T_+ < \infty$, $T_- > -\infty$*.*

iii) *The case* $\|\nabla u_0\| = \|\nabla W\|$ *is impossible.*

What happens beyond this? This is what the “soliton resolution conjecture” addresses. As we saw earlier, this can only hold for solutions which remain bounded in the energy norm up to T_+.

When $T_+ < \infty$, we define the singular set S as follows: x_0 is regular if $\forall \epsilon > 0$, $\exists R > 0$ such that $\forall 0 < t < T_+$ we have

$$\int_{|x-x_0|<R} |\nabla_{x,t} u(x,t)|^2 \, dx < \epsilon.$$

S is the complement of the set of regular points.

Theorem (DKM 11)

If $T_+ < \infty$, then S is a non–empty finite set. Moreover,

$$\vec{u}(t) \rightharpoonup_{t \to T_+} (v_0, v_1)$$

in $\dot{H}^1 \times L^2$, and if v is the solution of (NLW) with $\vec{v}(T_+) = (v_0, v_1)$ (the regular part of u) we have

$$supp\, \vec{a}(t) \subset \bigcup_{k=1}^{N} \{(x,t) : |x - x_k| \leq |T_+ - t|\},$$

where $S = \{x_1, x_2, \ldots, x_N\}$, $\vec{a}(t) = \vec{u}(t) - \vec{v}(t)$.

Note that if u is radial, $S = \{0\}$.

We now turn to soliton resolution in the radial case. This was proved by DKM 12 for a well–chosen sequence of times converging to T_+, for bounded in energy norm solutions in the radial case, $T_+ < \infty$, $T_+ = \infty$, $N = 3$. This was extended to any sequence by DKM 13.

Theorem (Classification, DKM 13)

Let u be a radial solution of (NLW). *Then one of the following holds:*
i) $T_+ < \infty$ *and* $\lim_{t \to T_+} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} = \infty$ *(type I blow–up),*
ii) $T_+ < \infty$ *and* $\sup_{0<t<T_+} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} < \infty$ *(type II blow–up).*
Moreover, there exist $J \geq 1$, *and for* $1 \leq j \leq J$, $i_j \in \{\pm 1\}$ *and* $\lambda_j(t) > 0$, $0 < \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_J(t) \ll (T_+ - t)$ *with*

$$\vec{u}(t) = \vec{v}(t) + \sum_{j=1}^{J} i_j (W_{\lambda_j(t)}, 0) + o_{\dot{H}^1 \times L^2}(1).$$

iii) $T_+ = \infty$. *Then* $\exists v_L$ *a solution to* (LW) *and* $J \geq 0$ *and for* $1 \leq j \leq J$, $i_j \in \{\pm 1\}$ *and* $\lambda_j(t) > 0$,
$0 < \lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_J(t) \ll t$ *with*

$$\vec{u}(t) = \vec{v}_L(t) + \sum_{j=1}^{J} i_j (W_{\lambda_j(t)}, 0) + o_{\dot{H}^1 \times L^2}(1).$$

The key idea in this proof was to use the "channel of energy" method (DKM 11) to quantify the ejection of energy that occurs as we approach T_+ is an unbounded spatial domain. This was reflected in a fundamental new dynamical characterization of W.

This says that if u is a radial, bounded in energy norm solution of (NLW), when $N = 3$, which exists for all time and is not $0, \pm W_\lambda, \lambda \in (0, \infty)$, then for some $R > 0$, some $\eta > 0$, we have

$$\int_{|x| \geq R + |t|} |\nabla_{x,t} u(x,t)|^2 dx \geq \eta, \qquad (*)$$

for all $t \geq 0$ or for all $t \leq 0$. (The difference in time direction has to do with the incoming or outgoing nature.)

The proof of $(*)$ relied on some "outer energy lower bound" for radial solutions v of the linear wave equation in $N=3$ (DKM 11). This is: $\forall t \geq 0$ or $\forall t \leq 0$ we have, for any $r_0 \geq 0$

$$\int_{|x|\geq r_0+|t|} |\nabla v_{x,t}|^2 dx \geq \frac{1}{2} \int_{|x|\geq r_0} \left[(\partial_r(rv_0))^2 + (rv_1)^2\right] dx. \qquad (**)$$

$(**)$ can easily be seen to give $(*)$ for (LW). The passage to $(*)$ uses this and "elliptic arguments" of iterative nature.

To use $(*)$ for the proof of soliton resolution when $T_+ < \infty$, one takes $t_n \to T_+ = 1$. We then break up $\vec{u}(t_n) - \vec{v}(t_n)$ into a sum of nonlinear "blocks," through the approximation theorem, plus an error w_n, which is small in the weaker dispersive norm $L^5_t L^{10}_x$.

Because of $(*)$, if one of the blocks is not $\pm W_{\lambda_n}$, it will send energy outside the inverted light cone with vertex (0,1) (a contradiction to the localization of $\vec{u} - \vec{v}$) in case $t \geq 0$ in $(*)$ or, arbitrarily close the boundary of the inverted light cone intersected with $t = 0$, in the case $t \leq 0$ in $(*)$, a contradiction to the fact that $\vec{u}(0) - \vec{v}(0)$ is in $\dot{H}^1 \times L^2$. To show that w_n has to be small in energy, one uses $(**)$ this time and a similar argument. The details are quite lengthy and involved.

At this point, it is worth pointing out that $(**)$ and its variants are false, even for $r_0 = 0$, in all even dimensions (Côte–K–Schlag 14). A variant of $(**)$ does hold in all odd dimensions (K–Lawrie–Liu–Schlag 15) but the expression on the right–hand side becomes (necessarily) increasingly complicated as the dimension grows, thus precluding this argument to be extended to higher odd dimensions too.

In light of these last comments, we now turn to the proof of the DKM 12 result, for radial solutions bounded in the energy norm, for a well–chosen sequence of times. This proof was different. The proof proceeded (say when $T_+ = 1$) by first showing that

$$\lim_{t \uparrow 1} \int_{\lambda(1-t)<|x|<1-t} |\nabla_{x,t} u(x,t)|^2 dx = 0, \qquad (***)$$

for each $\lambda \in (0,1)$.

The proof of DKM 12 of this fact relied on $(**)$. One then used virial identities, namely

$$\partial_t \int x \cdot \nabla u \partial_t u \phi = -\frac{3}{2} \int (\partial_t u)^2 \phi + \frac{1}{2} \int \left[|\nabla u|^2 - u^6 \right] \phi + \text{error}, \quad \text{(a)}$$

$$\partial_t \int u \partial_t u \phi = \int (\partial_t u)^2 \phi - \int \left[|\nabla u|^2 - u^6 \right] \phi + \text{error}, \quad \text{(b)}$$

$$\partial_t \int \phi x \left\{ \frac{1}{2} |\nabla u|^2 + \frac{1}{2} (\partial_t u)^2 - \frac{1}{6} u^6 \right\} = - \int \phi \nabla u \partial_t u + \text{error}. \quad \text{(c)}$$

(c) is only useful in non–radial situations.

One uses $(***)$ and chooses ϕ appropriately to show, via, $\frac{1}{2}$(b) + (a), that

$$\lim_{t\uparrow 1} \frac{1}{1-t} \int_t^1 \int_{|x|<1-s} (\partial_t u)^2 dxds = 0.$$

Thus, on average, u is a time independent solution of (NLW), which is radial, i.e. $\pm W_\lambda$! Then, by a Tauberian argument, for a well–chosen $t_n \to 1$, one has $\int_{|x|<1-t_n} (\partial_t u(t_n))^2 dx \to 0$, and hence $\int \left[\partial_t u(t_n) - \partial_t v(t_n)\right]^2 dx \to 0$. From this, one can show that all the nonlinear blocks must be time independent solutions. Finally, one uses $(**)$, with $r_0 = 0$, to show that the dispersive errors in fact tend to 0 in energy norm.

Subsequently, C. Rodriguez (2014) was able to extend this argument to all N odd, showing that the variant of $(**)$ in all odd dimensions is "strong enough" for this.

What about the case N even? When $N = 4$, Côte–K–Lawrie–Schlag (2015) found an analogy with wave maps in the equivariant case (and which applied to radial solutions for all N) to find a direct proof of $(***)$, which follows the classical one for equivariant wave maps due to Christodoulou, Shatah and Tahvildar–Zadeh from the early 90's.

One can then also show that for the error, $\partial_t w_n$ goes to 0 in L^2. One then uses the fact (Côte–K–Schag 2014) that when $N = 4$, $(**)$ for $r_0 = 0$ holds for data of the form $(v_0, 0)$, which can then be applied to finish the proof. When $N = 6$, the relevant "good data" is $(0, v_1)$ and this argument collapses.

But, Jia–K (2015) observed that once $\int \left[\partial_t u(t_n) - \partial_t v(t_n)\right]^2 dx \to 0$, using (a), (b) again

$$\frac{1}{1-t}\int_t^1 \int \left[|\nabla a(s)|^2 - a^6(s)\right] dxds \to 0,$$

where $a(x,t) = u(x,t) - v(x,t)$, and hence again by a Tauberian argument,

$$\int \left[|\nabla a(\tilde{t}_n)|^2 - a^6(\tilde{t}_n)\right] dx \to_n 0.$$

A real variable argument allows one to choose a new sequence t_n' so that both properties hold. We can then use that all blocks are $\pm W_\lambda$, and that, for W_λ, $\int \left[|\nabla W_\lambda|^2 - W_\lambda^6 \right] dx = 0$, by the elliptic equation. Using that for the error w_n, we know $\int w_n^6 \to 0$ (dispersive norm), we finally conclude that $\int |\nabla w_n|^2 \to 0$. This proof applies to all dimensions, for a well–chosen sequence of times, and does not use channels of energy.

Next time, we will discuss the non–radial case.

Thank you for your attention.