The Energy Critical Wave Equation: The Non–radial Case

Carlos Kenig

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We are considering the focusing, energy critical nonlinear wave equation in 3 space dimensions

$$\begin{cases} \partial_t^2 u - \Delta u - u^5 = 0, \quad x \in \mathbb{R}^3, t \in \mathbb{R} \\ u\big|_{t=0} = u_0 \in \dot{H}^1, \\ \partial_t u\big|_{t=0} = u_1 \in L^2. \end{cases} \tag{NLW}$$

By using the contraction mapping principle one can show that if

$$\|S_L(t)(u_0, u_1)\|_{L^5_I L^{10}_x} \leq \delta_0,$$

then we have a unique solution in the interval I, and hence, by the Strichartz estimate, if (u_0, u_1) is small in $\dot{H}^1 \times L^2$ we can take $I = \mathbb{R}$. Moreover, the solution scatters, i.e. $\exists (w_0^+, w_1^+) \in \dot{H}^1 \times L^2$ such that

$$\lim_{t \to \infty} \|\vec{u}(t) - \vec{S}_L(t)(w_0^+, w_1^+)\|_{\dot{H}^1 \times L^2} = 0.$$

We can also show that, for large data (u_0, u_1), we have a solution u with maximal interval of existence $I = (T_-(u), T_+(u))$, and $T_+ < \infty$ if and only if $\|u\|_{L^5_{[0,T_+)} L^{10}_x} = \infty$. Moreover, if $\|u\|_{L^5_{[0,T_+)} L^{10}_x} < \infty$ then $T_+ = \infty$ and u scatters.

The energy norm $\|\cdot\|_{\dot{H}^1 \times L^2}$ is "critical," since, for $\lambda > 0$,

$$u_\lambda(x,t) = \lambda^{-1/2} u(x/\lambda, t/\lambda)$$

is also a solution and $\|(u_{0,\lambda}, u_{1,\lambda})\|_{\dot{H}^1 \times L^2} = \|(u_0, u_1)\|_{\dot{H}^1 \times L^2}$.

The equation has two conservation laws, energy and momentum

$$E(u_0, u_1) = \frac{1}{2} \int |\nabla u_0|^2 + |u_1|^2 \, dx - \frac{1}{6} \int |u_0|^6 dx,$$

$$P(u_0, u_1) = \int u_1 \nabla u_0 \, dx.$$

The negative sign in the energy indicates that the equation is focusing.

Moreover, there are solutions which blow–up in finite time (i.e. $T_+ < \infty$) even with $\sup_{0<t<T_+} \|\vec{u}(t)\|_{\dot{H}^1 \times L^2} < \infty$ (bounded solutions).

There are also bounded solutions with $T_+ = \infty$, which do not scatter, for instance nontrivial solutions of the elliptic equation, $Q \in \dot{H}^1 \backslash \{0\}$, $\Delta Q + Q^5 = 0$ $(Q \in \Sigma)$. For example,

$$W(x) = \left(1 + \frac{|x|^2}{3}\right)^{-1/2},$$

which is the one of smallest energy (the "ground state"). $\pm W_\lambda$ are the only radial elements of Σ.

Other non–scattering solutions are the traveling wave solutions, obtained by the Lorentz transformations of $Q \in \Sigma$. For $\vec{\ell} \in \mathbb{R}^3$, $|\vec{\ell}| < 1$, we have $Q_{\vec{\ell}}(x,t) = Q_{\vec{\ell}}(x - \vec{\ell} t, 0)$, where

$$Q_{\vec{\ell}}(x,0) = Q\left(\left[\frac{1}{\sqrt{1-|\vec{\ell}|^2}} - 1\right]\frac{\vec{\ell}\cdot x}{|\vec{\ell}|^2}\vec{\ell} + x\right),$$

$Q \in \Sigma$. These are are all of the traveling wave solutions (DKM 14).

Theorem ("Ground–state conjecture," K–Merle 08)

If $E(u_0, u_1) < E(W, 0)$ *then:*

a) *If* $\|\nabla u_0\| < \|\nabla W\|$, *then* $T_+ = \infty$, $T_- = -\infty$ *and* u *scatters.*

b) *If* $\|\nabla u_0\| > \|\nabla W\|$, *then* $T_+ < \infty$, $T_- > -\infty$.

c) *The case* $\|\nabla u_0\| = \|\nabla W\|$ *is impossible.*

What happens beyond this? This is what the soliton resolution conjecture addresses, and this only can hold for bounded solutions.

For a bounded solution with $T_+ < \infty$, we define the singular set S as follows: x_0 is regular if $\forall \varepsilon > 0$, $\exists R > 0$ such that $\forall 0 < t < T_+$, we have

$$\int_{|x-x_0|<R} |\nabla u_{x,t}(x,t)|^2 \, dx < \varepsilon.$$

S is the complement of the set of regular points.

We have (DKM 11):

Theorem

S is a non–empty finite set. Moreover,

$$\vec{u}(t) \rightharpoonup_{t\to T_+} (v_0, v_1)$$

in $\dot{H}^1 \times L^2$ and if v is the solution of (NLW) with $\vec{v}(T_+) = (v_0, v_1)$, we have

$$supp\, \vec{a}(t) \subset \bigcup_{k=1}^{N} \{(x,t) : |x - x_k| < |T_+ - t|\},$$

where $S = \{x_1, x_2, \ldots, x_N\}$ and $a(t) = u(t) - v(t)$.

In the radial case, $S = \{0\}$.

Moreover, soliton resolution holds, for a well chosen sequence of times, (DKM 12), and then for all times (DKM 13).

The key idea in the proof of (DKM 13) was to use the “channel of energy” method to quantify the ejection of energy that occurs as we approach T_+ in an unbounded spatial domain.

This was accomplished by using a strong "outer energy lower bound" for radial solutions of the linear wave equation, which was then used to give a new dynamical characterization of W.

This "outer energy lower bound" fails in the non–radial setting and in even dimensions in the radial setting. In higher odd dimensions, a weaker form does hold.

The proof of DKM 12, for a well–chosen sequence of times relied first on the fact that there is no self–similar behavior at the blow–up time (or at infinity for non–scattering solutions).

This can be proved for radial solutions in all dimensions through an analogy with wave maps (CKLS 2014).

One then uses virial identities to prove that (say when $T_+ = 1$)

$$\lim_{t\uparrow 1} \frac{1}{1-t} \int_t^1 \int_{|x|<1-s} (\partial_t u)^2 \, dxds = 0,$$
$$\lim_{t\uparrow 1} \frac{1}{1-t} \int_t^1 \int_{|x|<1-s} \left[|\nabla u|^2 - u^6 \right] \, dxds = 0.$$

From these facts and a Tauberian argument to select a well–chosen sequence of times, the result can be proved in all dimensions (Jia–K 2015). Unfortunately, these facts don't hold in the non–radial case.

Thus, some of the new difficulties in the non–radial case are that the set of traveling waves $Q_{\vec{\ell}}$ is very large and far from being understood, and that the analogs of the "outer energy lower bounds" for the linear equation fail in the non–radial case. Thus, an approach based on a dynamical characterization of traveling waves seems doomed to failure.

We now turn to the result in the non–radial case, treated in works of Jia 2015 and DKM 16, DJKM 16. We have:

Theorem

Let u be a solution to (NLW) *such that*

$$\sup_{0\leq t<T_+} \|(u(t),\partial_t u(t))\|_{\dot H^1\times L^2} < \infty.$$

i) $T_+ < \infty$. *Let S be the set of singular points. Fix $x_* \in S$. Then* $\exists J_* \in \mathbb{N}, J_* \geq 1, r_* > 0, (v_0, v_1) \in \dot H^1 \times L^2$, *a time sequence $t_n \uparrow T_+$ (well chosen), scales $\lambda_n^j, 0 < \lambda_n^1 \ll \lambda_n^2 \ll \cdots \ll \lambda_n^j \ll (T_+ - t_n)$, positions $c_n^j \in \mathbb{R}^3$ such that $c_n^j \in B_{\beta(T_+-t_n)}(x_*)$, $\beta \in [0,1)$, with $\vec\ell_j = \lim_n \frac{c_n^j - x_*}{T_+ - t_n}$ well defined, $|\vec\ell_j| \leq \beta$, and traveling waves $Q^j_{\vec\ell_j}$ for $1 \leq j \leq J_*$ such that in the ball $B_{r_*}(x_*)$ we have*

$$\begin{aligned}\vec u(t_n) &= (v_0, v_1)\\ &+ \sum_{j=1}^{J^*}\left((\lambda_n^j)^{-\frac12} Q^j_{\vec\ell_j}\left(\frac{x-c_n^j}{\lambda_n^j},0\right),(\lambda_n^j)^{-\frac32}\partial_t Q^j_{\vec\ell_j}\left(\frac{x-c_n^j}{\lambda_n^j},0\right)\right)\\ &+ o_{\dot H^1\times L^2}(1).\end{aligned}$$

Moreover, $\lambda_n^j/\lambda_n^{j'} + \lambda_n^{j'}/\lambda_n^j + |c_n^j - c_n^{j'}|/\lambda_n^j \to_n \infty$, $j \neq j'$.

Theorem (Non–radial result cont.)

ii) $T_+ = \infty$. $\exists!\ v_L$ *solving the linear wave equation, such that,* $\forall A \geq 0$

$$\lim_{t\to\infty} \int_{|x|\geq t-A} |\nabla_{x,t}(u - v_L)(x,t)|^2 = 0.$$

Moreover, there exist $J_* \in \mathbb{N}, J_* \geq 0$, *a time sequence* $t_n \uparrow \infty$ *(well chosen), scales* $\lambda_n^j, 0 < \lambda_n^1 \ll \lambda_n^2 \ll \cdots \ll \lambda_n^j \ll t_n$, *positions* $c_n^j \in B_{\beta t_n}(0)$, $\beta \in [0,1)$, *with* $\vec{\ell}_j = \lim_n \frac{c_n^j}{t_n}$ *well defined,* $|\vec{\ell}_j| \leq \beta$, *and traveling waves* $Q^j_{\vec{\ell}_j}$ *for* $1 \leq j \leq J_*$ *such that*

$$\begin{aligned}\vec{u}(t_n) &= \vec{v}_L(t_n) \\ &+ \sum_{j=1}^{J^*} \Big((\lambda_n^j)^{-\frac{1}{2}} Q^j_{\vec{\ell}_j}\Big(\frac{x - c_n^j}{\lambda_n^j}, 0\Big), (\lambda_n^j)^{-\frac{3}{2}} \partial_t Q^j_{\vec{\ell}_j}\Big(\frac{x - c_n^j}{\lambda_n^j}, 0\Big)\Big) \\ &+ o_{\dot{H}^1\times L^2}(1),\end{aligned}$$

and $\lambda_n^j/\lambda_n^{j'} + \lambda_n^{j'}/\lambda_n^j + |c_n^j - c_n^{j'}|/\lambda_n^j \to_n \infty$, $j \neq j'$.

Remark 1. On the radiation term. When $T_+ < \infty$, as we saw, the singular set is finite and $(v_0, v_1) =$ weak limit as $t \to T_+$ of $\vec{u}(t)$ exists, and if v is the solution of (NLW) with data (v_0, v_1) at $t = T_+$, then supp $\vec{a}(t) \subset \cup_{x_* \in S}\{|x - x_*| < |T_+ - t\}$, $0 < t < T_+$. The number r_* in *i)* is chosen so that $|x - x_*| < r_*$ stays away from the other cones.

The case $T_+ = \infty$ is much more delicate. In fact, the existence of v_L as in *ii)* is proved in DKM 16, where it is called the "scattering profile." It was shown there that $\vec{S}_L(-t)(\vec{u}(t)) \rightharpoonup_{t\to\infty} (v_0, v_1)$ and that $v_L(x, t)$ verifies the first property in *ii)*.

The key idea in the proof of this is the use of a combination of virial identities to show that there are no "blocks" in u (non–linear profiles) which remain close to the light cone $\{|x| = t\}$, which is where linear solutions live. This is coherent with the fact that traveling waves with bounded energy travel at a speed strictly smaller than 1. The argument is tricky.

Remark 2. *i)* was proved by Jia (2015), but with the weaker vanishing property of the error $(\varepsilon_{0,n}, \varepsilon_{1,n})$, that $\|S_L(t)(\varepsilon_{0,n}, \varepsilon_{1,n})\|_{L^5_t L^{10}_x} \to_n 0$ (dispersive norm goes to 0).

The rest of the theorem is in DJKM 16, except, as explained before, for the existence of v_L in *ii)* which is in DKM 16.

We will now sketch some ideas for the proof of *i)*. The proof of *ii)* is analogous, once we have constructed v_L.

For simplicity of notation, we will assume that u is defined for $0 < t < \delta$, with blow–up time $T_- = 0$. We choose r_* so that $\vec{u}(x,t) = \vec{v}(x,t)$, for $t < |x| < r_*$, $\delta < r_*$. Our first observation is that, since $v \in L^5_{(0,\delta)} L^{10}_x$, and $\partial_t v \in L^\infty_{(0,\delta)} L^2_x$, $v^6 \in L^1\big(\{(x,t) : |x| = t, 0 < t < \delta\}\big)$. Thus, the same property holds for u. This allows one to control the energy flux, namely to show that

$$\lim_{\tau \to 0} \int_\tau^t \int_{|x|=s} \frac{|\not\partial u|^2}{2} + \left| \partial_t u + \frac{x}{|x|} \cdot \nabla u \right|^2 d\sigma < \infty,$$

where $|\not\partial u|^2 = |\nabla u|^2 - \left| \frac{x}{|x|} \cdot \nabla u \right|^2$. (This is by differentiating the truncated energy $\int_{|x|<t} \frac{1}{2}(|\nabla u|^2 + |\partial_t u|^2) - \frac{1}{6}|u|^6 \, dx$.)

Then, one uses this information, inspired by the wave map case, to obtain a crucial Morawetz identity: For $0 < 10t_1 < t_2 < \delta$, we have

$$\int_{t_1}^{t_2} \int_{|x|<t} \left[\partial_t u + \frac{x}{t} \cdot \nabla u + \frac{u}{2t} \right]^2 dx \frac{dt}{t} \leq C \log \left(\frac{t_2}{t_1} \right)^{1/2} .$$

This is very useful because the right hand side has the power $1/2 < 1$, which forces $\int_{|x|<t} \left[\partial_t u + \frac{x}{t} \cdot \nabla u + \frac{u}{2t} \right]^2 dx$ to vanish on average.

A Tauberian type real variable argument then shows that we can choose $\mu_n \downarrow 0$, $t_{1,n} \in \left(\frac{4}{3}\mu_n, \frac{13}{9}\mu_n\right)$, $t_{2,n} \in \left(\frac{14}{9}\mu_n, \frac{5}{3}\mu_n\right)$ such that

$$\sup_{0<\tau<\frac{t_{i,n}}{16}} \frac{1}{\tau} \int_{|t_{i,n}-t|<\tau} \int_{|x|<t} \left[\partial_t u + \frac{x}{t} \cdot \nabla u + \frac{u}{2t}\right]^2 dx\, dt \to_n 0, \qquad (\ddagger)$$

for $i = 1, 2$.

From this, at each $t_{i,n}$, an analysis of different "blocks" (non–linear profiles) in $\vec{u}(t_{i,n}) - \vec{v}(t_{i,n})$, gives a preliminary decomposition: each "block" has a space–time "center" $(c^j_{i,n}, t^j_{i,n})$ and a "scale" $\lambda^j_{i,n}$. For convenience, let's drop the indices i, and call $t_n = t_{i,n}$. Fix a "block" U^{j_0}.

Case 1: $t_n^{j_0} \equiv 0$, $\lambda_n^{j_0} \simeq t_n$. Then (‡) gives that the "block" U^{j_0} verifies the first order equation

$$\partial_t U^{j_0} + \frac{x}{t+1} \cdot \nabla U^{j_0} + \frac{1}{2(t+1)} U^{j_0} = 0,$$

which forces $U^{j_0}(x,t) = (t+1)^{-1/2}\Psi\left(\frac{x}{t+1}\right)$ and U^{j_0} solves (NLW). This is a self–similar solution, ruled out by K–Merle 08.

Case 2: $t_n^{j_0} \equiv 0$, $\lambda_n^{j_0} \ll t_n$. In this case, (‡) gives that the "block" U^{j_0} verifies the first order equation

$$\partial_t U^{j_0} + \vec{\ell}_{j_0} \cdot \nabla U^{j_0} = 0,$$

where $\vec{\ell}_{j_0} = \lim_n c_n^{j_0}/t_n$. But then, $U^{j_0}(x,t) = \Psi(x - \vec{\ell}_{j_0} t)$ and solves (NLW), so that DKM 14 gives that $U^{j_0}(x,t) = Q^{j_0}_{\vec{\ell}_{j_0}}(x - \vec{\ell}_{j_0} t, 0)$.

Case 3: $\left| t_n^{j_0}/\lambda_n^{j_0} \right| \to_n \infty$. Here we just add these "blocks" to the residue term, and use that they scatter, and for a linear solution h, $\|h(t)\|_{L^6_x} \to_{|t|\to\infty} 0$.

Since we have only finitely many "blocks" by the boundedness of u and the Pythagorean property of blocks, together with the lower bound $\|Q\|_{L^6} \geq \|W\|_{L^6}$, $Q \in \Sigma$, and by the localization of the support of $\vec{u}-\vec{v}$, we have $|c_n^j| + |t_n^j| + \lambda_n^j \lesssim |t_n|$, we have covered all cases and we have obtained a preliminary decomposition at $t_{1,n}$ and at $t_{2,n}$, with the error going to 0 in L^6.

We next improve this, by using an argument that originated in the work of Jia–K 2015, using virial identities. It is in this argument that we need the two sequences $\{t_{1,n}\}, \{t_{2,n}\}$.

By multiplying the equation by u and integrating by parts, we have

$$
\begin{aligned}
0 = &\frac{1}{\sqrt{2}} \int_{t_{1,n}}^{t_{2,n}} \int_{|x|=t} \left(-\partial_t u - \frac{x}{|x|} \cdot \nabla u \right) u \, d\sigma \\
&+ \int_{|x|<t_{2,n}} (u \partial_t u)(x, t_{2,n}) dx - \int_{|x|<t_{1,n}} (u \partial_t u)(x, t_{1,n}) dx \\
&+ \int_{t_{1,n}}^{t_{2,n}} \int_{|x|<t} \left[|\nabla u|^2 - |\partial_t u|^2 - |u|^6 \right] dx \, dt.
\end{aligned}
$$

Using the preliminary decomposition at $t_{1,n}$, $t_{2,n}$, we can show that $\|u(\cdot, t_{i,n})\|_{L^2(B_{i,n})} = o(t_{i,n})$. This, combined with the above identity and the control of the energy flux gives us:

$$\frac{1}{t_{2,n} - t_{1,n}} \int_{t_{1,n}}^{t_{2,n}} \int_{|x|<t} \left[|\nabla u|^2 - |\partial_t u|^2 - |u|^6 \right] \, dx \, dt \to_n 0.$$

From this, a Tauberian real variable argument shows that we can find $t_n \in [t_{1,n}, t_{2,n}]$ such that

$$\lim_{n\to\infty} \sup_{0<\tau<\frac{t_n}{16}} \frac{1}{\tau} \int_{|t_n-t|<\tau} \int_{|x|<t} \left[\partial_t u + \frac{x}{t} \cdot \nabla u + \frac{u}{2t}\right]^2 dxdt = 0,$$

and

$$\overline{\lim_n} \int_{|x|<t_n} [|\nabla u|^2 - |\partial_t u|^2 - |u|^6](x, t_n)dx \leq 0.$$

From the first information, we obtain a decomposition at t_n, with $\|\varepsilon_{0,n}\|_{L^6} \to_n 0$. Since

$$\int [|\nabla Q_{\vec{\ell}}|^2 - |\partial_t Q_{\vec{\ell}}|^2 - |Q_{\vec{\ell}}|^6] dx = 0, \ \forall t,$$

$$\int [\partial_t Q_{\vec{\ell}} + \vec{\ell} \cdot \nabla Q_{\vec{\ell}}]^2 dx = 0,$$

we deduce that

$$\lim_{n\to\infty} \int_{|x|<t_n} \left[\varepsilon_{1,n} + \frac{x}{t_n} \cdot \nabla \varepsilon_{0,n} \right]^2 dx = 0,$$

$$\overline{\lim_{n\to\infty}} \int_{|x|<t_n} \int_{|x|<t_n} \left[|\nabla \varepsilon_{0,n}|^2 - \varepsilon_{1,n}^2 \right] dx \leq 0.$$

These two facts, combined with the localization property

$$\lim_{n\to\infty}\int_{|x|>t_n}[|\nabla \varepsilon_{0,n}|^2+\varepsilon_{1,n}^2]dx=0,$$

easily yield

$$\begin{aligned}&\|\varepsilon_{0,n}\|_{L^6}+\|\not\partial\varepsilon_{0,n}\|_{L^2}+\|\nabla\varepsilon_{0,n}\|_{L^2(B_{\lambda t_n}\cup B^c_{t_n})}\\ &\quad+\|\varepsilon_{1,n}\|_{L^2(B_{\lambda t_n}\cup B^c_{t_n})}+\left\|\varepsilon_{1,n}+\frac{x}{t_n}\cdot\nabla\varepsilon_{0,n}\right\|_{L^2(B_{t_n})}\\ &\to_n 0\end{aligned}$$

for any $0<\lambda<1$.

The next step is to show, from these properties, that

$$\|S_L(t)(\varepsilon_{0,n}, \varepsilon_{1,n})\|_{L^5_t L^{10}_x} \to_n 0.$$

The arguments here are similar to some contained in the proof of the extraction of the "scattering profile."

The final step in the proof is provided by a new "channel of energy" argument which is very robust, valid in all dimensions, non–radial, and a key new ingredient in the theory.

Lemma

Let $(\varepsilon_{0,n}, \varepsilon_{1,n})$ *be a bounded sequence in* $\dot{H}^1 \times L^2$ *such that* $\forall \lambda \in (0,1)$ *we have*

$$\begin{aligned} &\|\varepsilon_{0,n}\|_{L^6} + \|(\varepsilon_{0,n}, \varepsilon_{1,n})\|_{\dot{H}^1 \times L^2(B_\lambda \cup B_1^c)} \\ &\quad + \|\not{\partial} \varepsilon_{0,n}\|_{L^2} + \|\varepsilon_{1,n} + \partial_r \varepsilon_{0,n}\|_{L^2} \\ &\to_n 0. \end{aligned}$$

Then, if $\inf_n \|(\varepsilon_{0,n}, \varepsilon_{1,n})\|_{\dot{H}^1 \times L^2} > 0$*, and* $w_n(x,t) = S_L(t)(\varepsilon_{0,n}, \varepsilon_{1,n})$*, we have,* $\forall \eta_0 \in (0,1)$*, n large that*

$$\int_{|x| \geq t + \eta_0} |\nabla_{x,t} w_n(t)|^2 dx \geq \frac{\eta_0}{2} \|(\varepsilon_{0,n}, \varepsilon_{1,n})\|^2_{\dot{H}^1 \times L^2},$$

for all $t > 0$.

Next, in our case, the fact that $\|w_n\|_{L^5_t L^{10}_x} \to_n 0$, allows us to pass to the solution of (NLW). To obtain a contradiction, we bypass the traveling waves by choosing $\eta_0 > \beta$, where $|\vec{\ell}_j| \leq \beta$, and use finite speed:

Write $(h_{0,n}, h_{1,n}) = \vec{v}(t_n) + (\varepsilon_{0,n}, \varepsilon_{1,n})$, and use that, if $\eta_0 > \beta$,

$$\|(h_{0,n}, h_{1,n}) - \vec{u}(t_n)\|_{\dot{H}^1 \times L^2(|x| \geq \eta_0 t_n)} \to 0,$$

and let h_n be the corresponding solution of (NLW).

Then $\vec{h}_n(t) = \vec{v}(t) + \vec{w}_n(t) + \vec{r}_n(t)$ by the Approximation Theorem and

$$\sup_{t\in(t_n,\delta)} \|\vec{u}(t) - \vec{h}_n(t)\|_{\dot{H}^1\times L^2(\delta\geq|x|\geq t-t_n+\eta_0 t_n)} \to_n 0,$$

by finite speed.

We obtain a contradiction by using the channel of energy Lemma rescaled to scale t_n to get, if $\|(\varepsilon_{0,n},\varepsilon_{1,n})\|_{\dot{H}^1\times L^2} \geq \mu_0$, that for $0 < t < \delta$,

$$\int_{t\geq|x|\geq t-t_n+\eta_0 t_n} |\nabla_{x,t} u(t)|^2 \geq \frac{\eta_0\mu_0}{100},$$

and letting $t = \delta/2$, we get a contradiction for n large.

Thank you for your attention.