Spectral inequalities and their applications

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Introduction

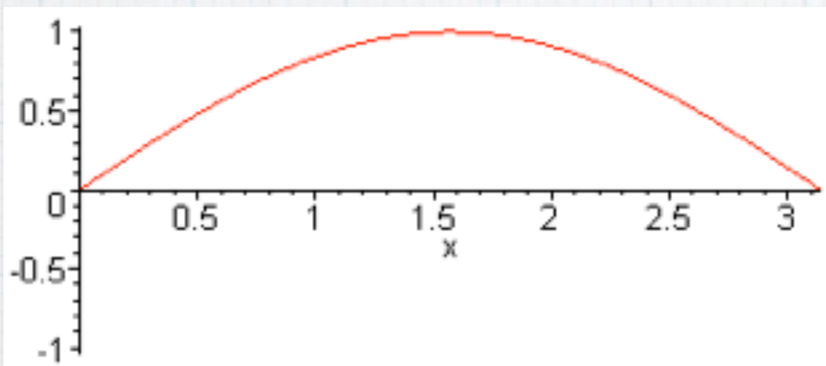
Weyl Asymptotics

Let us consider the problem

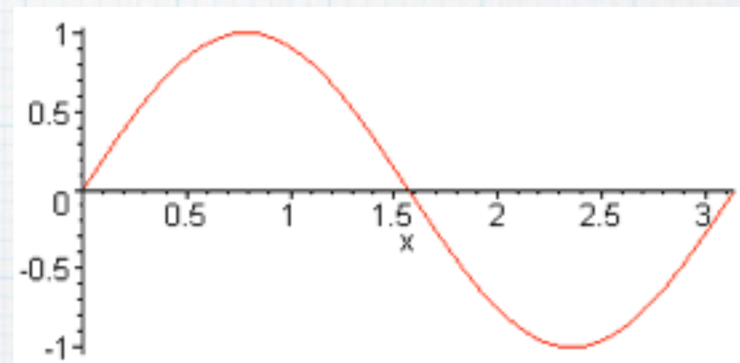
$$-u''(x) = \lambda u(x), \qquad u(0) = u(\pi) = 0.$$

Clearly for any $k = 1, 2, 3, \ldots$ we find

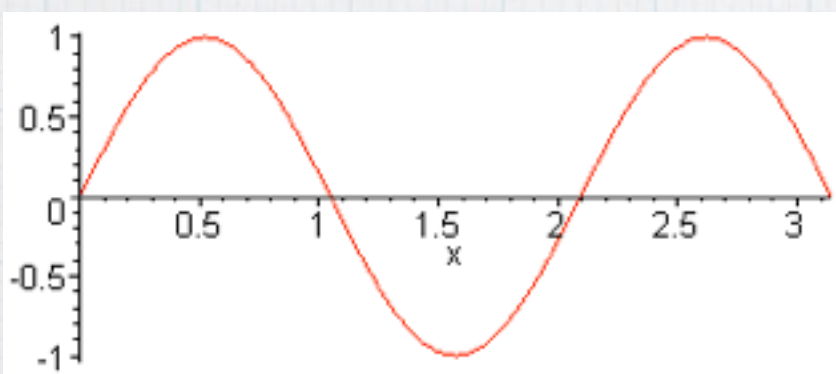
$$u_k(x) = \sin kx, \qquad \lambda_k = k^2.$$



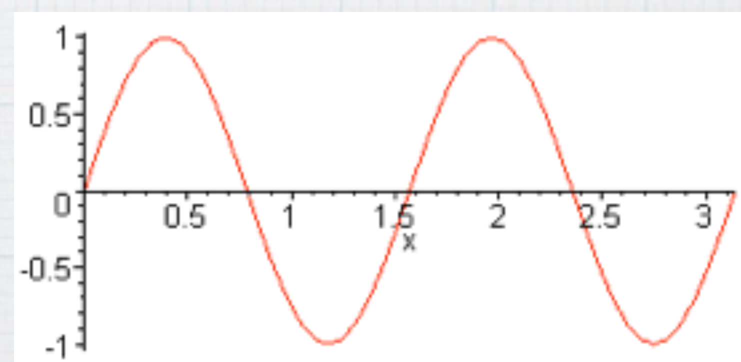
$\sin x, \quad \lambda_1 = 1$



$\sin 2x, \quad \lambda_2 = 4$



$\sin 3x, \quad \lambda_3 = 9$

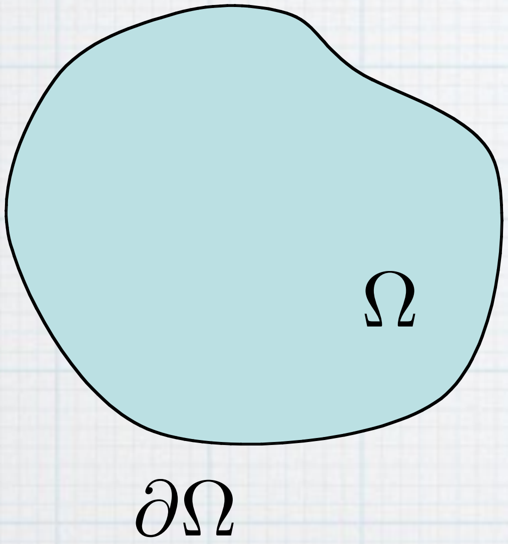


$\sin 4x, \quad \lambda_4 = 16$






Dirichlet boundary value problem.



Consider a bounded domain $\Omega \subset \mathbb{R}^d$ with piecewise smooth boundary $\partial\Omega$.

Dirichlet boundary value problem for the Laplace operator in $L^2(\Omega)$

$$-\Delta u(x) = \lambda u(x), \qquad x \in \Omega,$$

$$u\Big|_{\partial\Omega} = 0.$$

The Dirichlet Laplacian has a discrete spectrum of infinitely many positive eigenvalues with no finite accumulation point (F. Pockels - 1892)

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots,$$

$\lambda_k \to \infty$ as $k \to \infty$.

Weyl's asymptotic formula for eigenvalues of a Dirichlet Laplacian.

Hermann Weyl
1885-1955

My work always tried to unite the Truth with the Beautiful, but when I had to choose one or the other, I usually choose the Beautiful.

H.Weyl: "Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen" Math. Ann. , 71 (1911) pp. 441–479.

Theorem.

$$\lambda_k = \frac{4\pi^2 k^{2/d}}{C_d |\Omega|^{2/d}} + o(k^{2/d}),$$

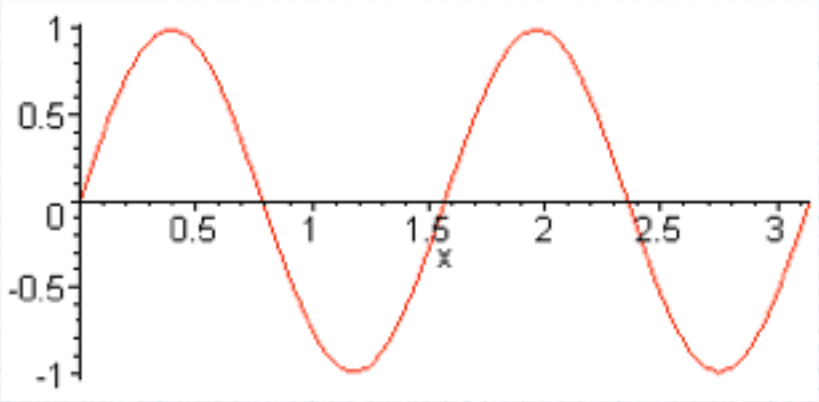
where $|\Omega|$ and $C_d = \pi^{d/2}/\Gamma(d/2+1)$ are respectively the Lebesgue measure of Ω and of the unit ball in $\mathbb{R}^d$.

It is useful to rewrite Weyl's asymptotic formula in term of the counting function of the spectrum as $\lambda \to \infty$

$$N(\lambda) = \#\{k : \lambda_k < \lambda\} = (2\pi)^{-d} \lambda^{d/2} |\Omega| \int_{|\xi|<1} d\xi + o(\lambda^{d/2})$$

$$= (2\pi)^{-d} \int_\Omega \int_{|\xi|^2 \le \lambda} d\xi \, dx + o(\lambda^{d/2}),$$

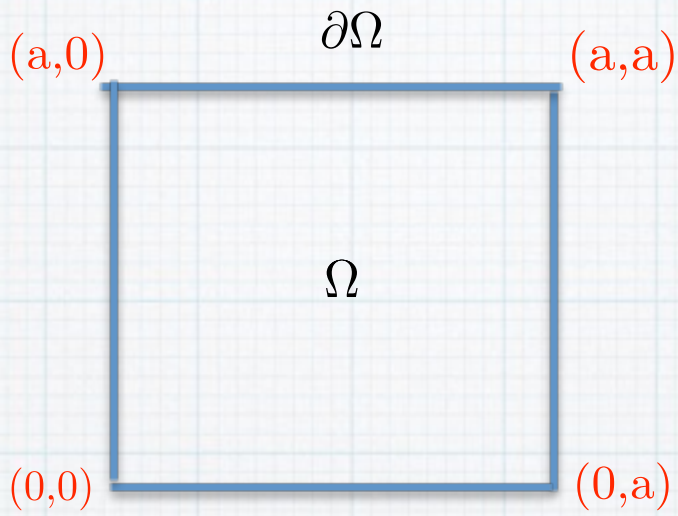
phase volume asymptotics.



$$u_k(x) = \sin kx, \quad \lambda_k = k^2$$

In the case $d = 1$, $\Omega = (0, \pi)$, $|\Omega| = \pi$, Weyl's asymptotic formula in term of the counting function could be written in a more precise way

$$N(\lambda) = \#\{k : \lambda_k = k^2 < \lambda\} = (2\pi)^{-d} \lambda^{d/2} |\Omega| \int_{|\xi|<1} d\xi + o(\lambda^{d/2})$$
$$= (2\pi)^{-1} \sqrt{\lambda}\, \pi\, 2 + O(1) = \sqrt{\lambda} + O(1), \qquad \text{as} \quad \lambda \to \infty.$$



In this case Dirichlet and Neumann boundary value problems

$$-\Delta u(x,y) = \lambda u(x,y), \qquad -\Delta v(x,y) = \mu v(x,y)$$

$$u|_{\partial\Omega} = 0 \qquad \left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0$$

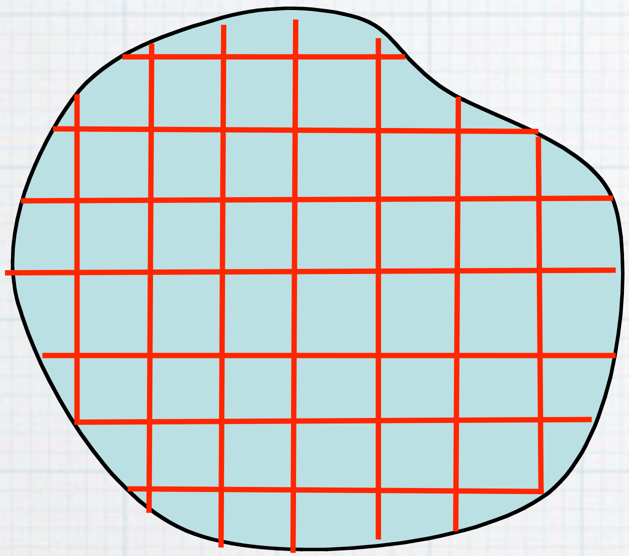
have solutions

$$u_{nm}(x,y) = \sin \pi a^{-1} n x \cdot \sin \pi a^{-1} m y, \qquad v_{nm}(x,y) = \cos \pi a^{-1} n x \cdot \cos \pi a^{-1} m y,$$

$$\lambda_{nm} = \pi^2 a^{-2}(n^2+m^2), \quad n,m = 1,2,\ldots \qquad \mu_{nm} = \pi^2 a^{-2}(n^2+m^2), \quad n,m = 0,1,2,\ldots$$

$$N^{\mathcal{D}}(\lambda) = \#\{n,m = 1,2,\cdots : \ \lambda_{nm} = n^2+m^2 < \lambda\} \sim (4\pi)^{-1} a^2 \lambda$$

$$N^{\mathcal{N}}(\mu) = \#\{n,m = 0,1,2,\cdots : \ \mu_{nm} = n^2+m^2 < \mu\} \sim (4\pi)^{-1} a^2 \mu$$



Proof.

Weyl used a version of the max-min principle.

Dirichlet-Neumann bracketing:

$$N^{\mathcal{D}}(\lambda) \leq N(\lambda) \leq N^{\mathcal{N}}(\lambda).$$

For each square with side a we find that the eigenvalues are equal to

$\{\lambda^{\mathbb{D}}_{nm}(a) = \pi^2 a^{-2}(n^2 + m^2) : n, m = 1, 2, 3, \dots\}$ for the Dirichlet problem

and

$\{\mu^{\mathbb{N}}_{nm}(a) = \pi^2 a^{-2}(n^2+m^2) : n, m = 0, 1, 2, 3, \dots\}$ for the Neumann problem.

Counting

$\#\{(n, m) : \lambda^{\mathbb{D}}_{nm}(a) \leq \lambda\}$

and

$\#\{(n, m) : \mu^{\mathbb{N}}_{nm}(a) \leq \lambda\}$,

summing them up and letting $a \to 0$ we proof the result.

Weyl's conjecture.

In 1911 H.Weyl also conjectured that

$$N(\lambda) = (2\pi)^{-d}\, C_d\, \lambda^{d/2}|\Omega| - c_{d-1}\, \lambda^{(d-1)/2}|\partial\Omega| + o(\lambda^{(d-1)/2}),$$

where $c_{d-1} > 0$ is a standard term depending only on dimension d.

Under certain conditions on classical billiards in $T^*\Omega$ V. Ivrii proved this result in 1980.

Inverse spectral problems

Can one hear the shape of a drum ?

Mark Kac

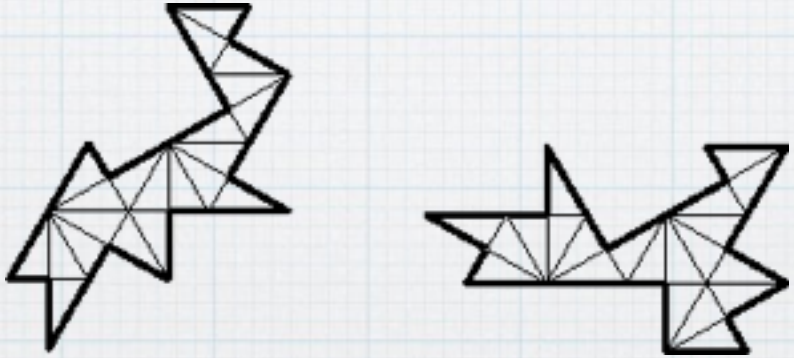
1914-1984

Isospectral domains.

In 1965 Mark Kac asked: 'Can one hear the shape of a drum?' (question goes back to H.Weyl)

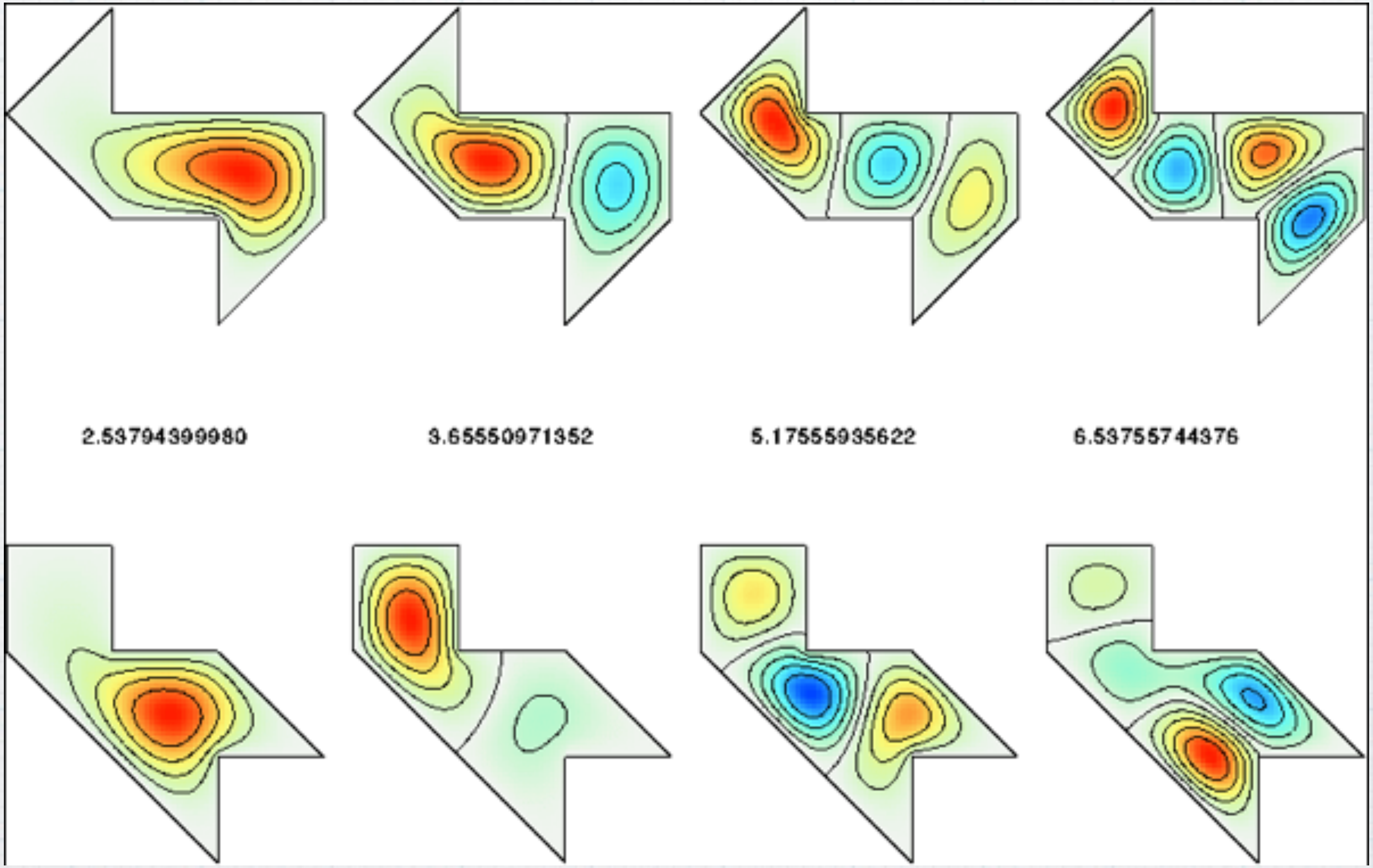
T. Sunada 1985 found two different domains in $\mathbb{R}^{16}$ which have the same "Dirichlet" spectrum.

Gordon, Webb, and Wolpert 1992, found planar isospectral domains.



Peter Buser, John Conway, Peter Doyle, and Klaus-Dieter Semmler, 1994.


Sunada triple:
"almost
conjugate
WASH. U. MATH
WASH. U. MATH



2.53794399980
3.65550971352
5.17555935622
6.53755744376

Spectral inequalities for Dirichlet Laplacians

Weyl's type inequalities. Pólya's conjecture.

George Pólya
1887 - 1985

In 1961 Pólya proved that if $\Omega \subset \mathbb{R}^2$ is a tiling domain then

$$\lambda_k \geq \frac{4\pi k}{|\Omega|}, \qquad k = 1, 2, 3, \dots$$

or equivalently

$$N(\lambda) = \#\{k : \lambda_k \leq \lambda\} \leq (2\pi)^{-2}\, \pi\, \lambda\, |\Omega|$$

$$= (2\pi)^{-2} \int_\Omega \int_{|\xi|^2 \leq \lambda} d\xi\, dx,$$

phase volume inequality.

Pólya's Conjecture

Prove that the latter inequality holds for arbitrary domains.

Pólya's conjecture is still open for $\Omega = \{x \in \mathbb{R}^d : |x| < 1\}$.

Easier question:

Is there a constant $C \geq 1$ such that

$$N(\lambda) \leq C\,(2\pi)^{-2}\,\pi\,\lambda\,|\Omega|.$$

- If Ω is bounded, then it was proved for bounded domains by Birman & Solomyak $'70$ and Ciesielski $'70$ with some constant $C > 1$.

- For domains of finite measure and with some $C > 1$, (3) was proved by Rosenblum $'71$ and Lieb $'80$.

- Some progress was made in my paper JFA $'97$.

Example: Pólya conjecture holds true for
$\Omega = \Omega_1 \times \Omega_2$, where $\Omega_1 \subset \mathbb{R}^{d_1}$, $d_1 \geq 2$, is tiling
and $\Omega_2 \subset \mathbb{R}^{d_2}$ is an arbitrary domain of finite measure.

Another easier question:
Is it true that for some $\gamma \geq 0$ the following phase volume estimate holds?

$$\sum(\lambda - \lambda_k)_+^\gamma \leq (2\pi)^{-d}\, \lambda^{\gamma+d/2}|\Omega| \int_{\mathbb{R}^d} (1 - |\xi|^2)_+^\gamma \, d\xi,$$

where $(x)_+ = (|x| + x)/2$ is the positive part of x.

Theorem. (Berezin, Li-Yau)
If $\gamma \geq 1$ then the above Weyl inequality holds true.

Proof.(see AL, JFA 1997.)
Let $\gamma = 1$ and let φ_k be the orthonormal basis in $L^2(\Omega)$ consisting of eigenfunctions of the Dirichlet Laplacian which is denoted by A. Let $\hat{\varphi}$ be the Fourier transform of φ. Then by using Parceval formula we find

$$\sum_k(\lambda - \lambda_k)_+ = \sum_k \big(\lambda - (A\varphi_k, \varphi_k)\big)_+ = \sum_k \left((2\pi)^{-d} \int_{\mathbb{R}^d} (\lambda - |\xi|^2)\, |\hat{\varphi}_k|^2 \, d\xi\right)_+$$

$$\leq (2\pi)^{-d} \sum_k \int_{\mathbb{R}^d} (\lambda - |\xi|^2)_+ \, |\hat{\varphi}_k|^2 \, d\xi$$

$$= (2\pi)^{-d} \sum_k \int_{\mathbb{R}^d} (\lambda - |\xi|^2)_+ \left| \int_\Omega e^{i(x,\xi)} \varphi_k(x)\, dx\right|^2 d\xi =$$

$$(2\pi)^{-d} \int_{\mathbb{R}^d} (\lambda{-}|\xi|^2)_+ \sum_k \|(e^{i(\cdot,\xi)}, \varphi_k)\|^2 \, d\xi = (2\pi)^{-d} \int_{\mathbb{R}^d} (\lambda{-}|\xi|^2)_+ \, d\xi \, \underbrace{\|e^{i(\cdot,\xi)}\|^2}_{=|\Omega|}.$$

Corollary.

$$N(\lambda) \leq \left(1 + \frac{2}{d}\right)^{d/2} \frac{1}{(2\pi)^d} \int_\Omega \int_{|\xi|^2<\lambda} d\xi dx.$$

Proof.

Remark.

Nobody knows if there is a constant $1 \leq R < \left(1 + \frac{2}{d}\right)^{d/2}$ such that

$$N(\lambda) \leq \frac{R}{(2\pi)^d} \int_\Omega \int_{|\xi|^2 < \lambda} d\xi dx.$$

The Laplace operator in $\mathbb{R}^n$ can be written in polar coordinates

$$-\Delta = -\frac{\partial^2}{\partial r^2} - \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}(-\Delta_{\mathbb{S}^{n-1}}).$$

The spectrum of $-\Delta_{\mathbb{S}^{n-1}}$ consists of eigenvalues

$$l(l+n-2), \qquad l \in \mathbb{N}$$

whose multiplicity is given by

$$\binom{l+n-1}{n-1} + \binom{l+n-3}{n-1}.$$

This leads us to the Dirichlet eigenvalues of the problem

$$-\frac{\partial^2}{\partial r^2}\psi(r) - \frac{n-1}{r}\frac{\partial}{\partial r}\psi(r) + \frac{l(l+n-2)}{r^2}\psi(r) = \lambda\psi(r).$$

Its solution is

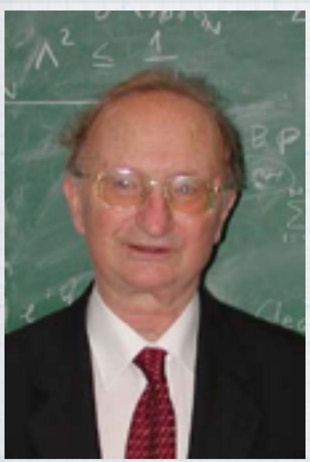
$$\psi(r) = r^{(2-n)/2} J_{l+(n-2)/2}(\sqrt{\lambda}r), \qquad J_{l+(n-2)/2}(\sqrt{\lambda}r)\Big|_{r=1} = 0.$$

Let $j_{\nu,k}$ be the k-zero of the Bessel function $J_\nu(x)$. The following inequalities might be useful (obtained by Ifantis and Siafarikas, see also Elbert)

$$j_{\nu,k} > \nu + k\pi - \frac{\pi}{2} + \frac{1}{2}, \qquad \nu > -\frac{1}{2}, \quad k = 1, 2, \ldots.$$

Spectrum of Schrödinger operators

E.H. Lieb

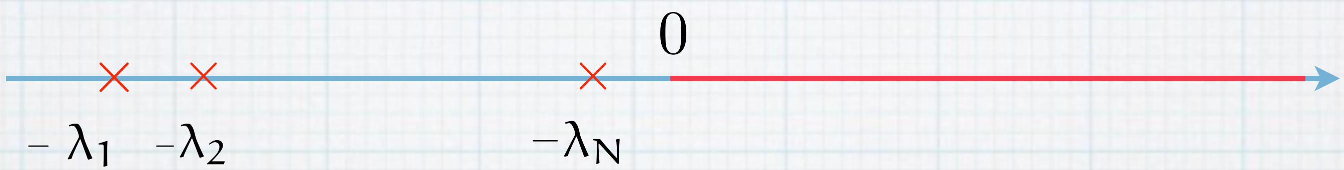


W. Thirring

Lieb-Thirring inequalities.

Let $H = -\Delta + V$ be a Schrödinger operator in $L^2(\mathbb{R}^d)$, $V \to 0$, as $x \to \infty$.

Spectrum:



$$\sum_j \lambda_j^\gamma = \sum_j \lambda_j^\gamma(V) \leq \frac{C_{d,\gamma}}{(2\pi)^d} \iint \left(|\xi|^2 + V(x)\right)_-^\gamma \, dx d\xi = L_{\gamma,d} \int V(x)^{\gamma+d/2} dx.$$

Compare with Weyl's asymptotic formula:

$$\sum_j \lambda_j^\gamma(\alpha V) \sim_{\alpha\to\infty} L_{\gamma,d}^{cl} \int (\alpha V_-)^{\gamma+1/2}\, dx = (2\pi)^{-d} \iint (\xi^2 + \alpha V)_-^\gamma \, d\xi dx.$$

which implies $L_{\gamma,1}^{cl} \leq L_{\gamma,1}$.

Lieb-Thirring inequalities

$$\sum_k \lambda_k^\gamma = \sum_k \lambda_k^\gamma(V) \le \frac{C_{\gamma,d}}{(2\pi)^d} \iint \Big(|\xi|^2 - V(x)\Big)_-^\gamma \, d\xi dx = L_{\gamma,d} \int V_+^{\gamma+d/2}(x)\, dx.$$

Applications.

- Weyl's asymptotics.
- Stability of matter.
- Study of properties of Continuous spectrum of Schrödinger operators.
- Estimate of dimensions of attractors in theory of Navier-Stokes equations.
- Bounds on the maximum ionization of atoms.

Example.

If in $H = -\Delta + V$,

$$V(x) = \begin{cases} -\lambda, & x \in \Omega, \\ +\infty, & x \notin \Omega, \end{cases} \qquad \Omega \in \mathbb{R}^d,$$

then the spectrum of H coincides with the spectrum of the Dirichlet Laplacian in Ω.

Therefore Pólya inequalities are special cases of L-Th inequalities.

$$(-\Delta + V)u = \lambda u.$$

$$\sum_j |\lambda_j|^\gamma \le L_{\gamma,d} \int V(x)_-^{\gamma+d/2}\, dx.$$

Theorem.

The constant $L_{\gamma,d} < \infty$ if $d = 1$, $\gamma \ge 1/2$; $d = 2$, $\gamma > 0$ and $d \ge 3$, $\gamma \ge 0$.

E.Lieb, W.Thirring, T.Weidl, M.Cwikel, G.Rozenblum.

Theorem.

It is known that $L_{1/2,1} = 1/2$ ($L^{cl}_{1/2,1} = 1/4$) and
$L_{\gamma,d} = L^{cl}_{\gamma,d}$ if $\gamma \ge 3/2$, $d \ge 1$.
In other cases the sharp constants are unknown.

E.Lieb, W.Thirring, M.Aizenmann, D.Hundertmark, L.Thomas, AL & T.Weidl.

Buslaev-Faddeev-Zakharov trace formula, $d = 1$.

Let ψ solves the equation

$$-\frac{d^2}{dx^2}\psi + V\psi = k^2\psi, \qquad \psi(x,k) = \begin{cases} e^{ikx}, & \text{as } x \to \infty \\ a(k)e^{ikx} + b(k)e^{-ikx}, & \text{as } x \to -\infty. \end{cases}$$

Fundamental property:

if $k \in \mathbb{R}$ then $W[\psi, \bar{\psi}] = \psi\bar{\psi}' - \psi'\bar{\psi} = \text{const}$.

This implies $1 = |a|^2 - |b|^2 \Leftrightarrow |a| \geq 1$.

Let

$$\lambda_j = (i\kappa_j)^2, \qquad \kappa_j > 0.$$

Theorem. (BFZ trace formula.)
If $V \leq 0$, then

$$\frac{3}{2\pi}\int k^2 \ln|a|^2\, dk + \sum_j \kappa_j^3 = \frac{3}{16}\int V^2\, dx = (2\pi)^{-1}\iint (|\xi|^2 + V)_-^{3/2}\, d\xi\, dx.$$

Corollary. (L-Th inequality.)

$$\sum_j |\lambda_j|^{3/2} = \sum_j \kappa_j^3 \leq \frac{3}{16}\int V^2\, dx.$$

Soliton's approach (Lieb & Thirring, Lax, Kruskal).

Let us consider the KdV equation

$$U_t = 6UU_x - U_{xxx}, \qquad U|_{t=0} = V.$$

Then

$$U_t = \Big[-\frac{d^2}{dx^2} + U, M\Big], \quad \text{where} \quad M = 4\frac{d^3}{dx^3} - 3\Big(U\frac{d}{dx} + \frac{d}{dx}U\Big).$$

- Discrete spectrum is independent of t:

$$\lambda_j\Big(-\frac{d^2}{dx^2} + U\Big) = \lambda_j\Big(-\frac{d^2}{dx^2} + V\Big).$$

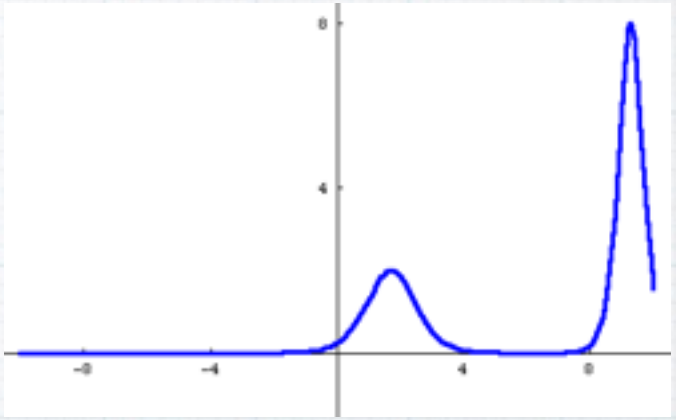
- $a(k,t) = e^{i8k^3t}a(k,0)$.
- $\int U^2(x,t)\,dx = \int V^2(x)\,dx$.

Therefore terms in the trace formula

$$\frac{3}{2\pi}\int k^2 \ln|a|^2\,dk + \sum_j |\lambda_j|^{3/2} = \frac{3}{16}\int U^2\,dx$$

are independent of time.

$$U(x,t) \sim_{t\to\infty} \sum_{j=1}^{N} U_j(x - 4\lambda_j t) + U_\infty,$$



- $\|U_\infty\|_\infty \le \varepsilon(t) \to_{t\to\infty} 0$ and U_j are solitons

$$U_j(x) = -2\lambda_j \cosh^{-2}(\sqrt{\lambda_j} x).$$

- $\left(-\frac{d^2}{dx^2} + U_j\right) \cosh^{-1}(\sqrt{\lambda_j} x) = -\lambda_j \cosh^{-1}(\sqrt{\lambda_j} x).$

Finally, since $4\int \cosh^{-4} x\, dx = 16/3$, we obtain

$$\int V^2\, dx \ge \sum_{j=1}^{N} \int U_j^2\, dx = \frac{16}{3} \sum_{j=1}^{N} \lambda_j^{3/2}.$$

Theorem. Assume that $V \in L^2(\mathbb{R})$, $V \geq 0$. Then for the negative eigenvalues $\{-\lambda_j\}$ of the Schrödinger operator H

$$H\psi = \frac{d^2}{dx^2}\,\psi(x) - V(x)\psi(x)$$

we have

$$\sum_{j=1}^{N} \lambda_j^{3/2} \leq \frac{3}{16} \int V^2\,dx.$$

Proof by using the Darboux transform (Benguria & Loss)

Let us consider the equation

$$H\psi(x) = -\psi''(x) - V(x)\psi(x) = -\lambda\psi(x),$$

with $V \geq 0$ and decaying rapidly, so that the negative spectrum of H is finite and equals $\{-\lambda_1, -\lambda_2, \ldots, -\lambda_N\}$.
Let $(-\lambda_1, \psi_1(x))$ be the lowest negative eigenvalue and eigenfunction respectively.
It is well know that the function ψ_1 could be chosen such that $\psi_1 > 0$ (prove it).

Consider now

$$f_1(x) = \frac{\psi_1'(x)}{\psi_1(x)} \quad \Longrightarrow \quad f_1'(x) = \frac{\psi_1''(x)}{\psi_1(x)} - \left(\frac{\psi_1'(x)}{\psi_1(x)}\right)^2.$$

Therefore

$$f_1' + f_1^2 = \frac{\psi_1''}{\psi_1} = \lambda_1 - V.$$

Denote by Q_1 the differential operator

$$Q_1 = \frac{d}{dx} - f_1 \quad \& \quad Q_1^* = -\frac{d}{dx} - f_1.$$

Then

$$Q_1^* Q_1 = \left(-\frac{d}{dx} - f_1\right)\left(\frac{d}{dx} - f_1\right) = -\frac{d^2}{dx^2} + f_1' + f_1^2$$

$$= -\frac{d^2}{dx^2} - V + \lambda_1.$$

The negative spectrum of the operator $Q_1^* Q_1$ consists of points

$$\{0, -\lambda_2 + \lambda_1,\ -\lambda_3 + \lambda_1,\ -\lambda_4 + \lambda_1, \ldots, -\lambda_N + \lambda_1\}.$$

Clearly

$$Q_1^* Q_1 \psi_1 = 0, \quad \text{and} \quad \psi_1(x) = \begin{cases} e^{-\sqrt{\lambda_1} x}, & x \to \infty, \\ e^{\sqrt{\lambda_1} x}, & x \to -\infty. \end{cases}$$

Thus

$$f_1(x) = \frac{\psi_1'(x)}{\psi_1(x)} = \begin{cases} -\sqrt{\lambda_1}, & x \to \infty, \\ \sqrt{\lambda_1}, & x \to -\infty. \end{cases}$$

Consider the operator

$$Q_1 Q_1^* = \left(\frac{d}{dx} - f_1\right)\left(-\frac{d}{dx} - f_1\right) = -\frac{d^2}{dx^2} - f_1' + f_1^2$$

$$= -\frac{d^2}{dx^2} - 2f_1' - V + \lambda_1.$$

The operators $Q_1^*Q_1$ and $Q_1Q_1^*$ Have the same eigenvalues apart from the point 0. Indeed,

$$Q_1^*Q_1\psi = -\lambda\psi \quad \Longrightarrow \quad Q_1Q_1^*Q_1\psi = -\lambda Q_1\psi = Q_1Q_1^*\varphi,$$

$$\text{with} \quad \varphi = \mathrm{Q}_1\psi.$$

Besides, assume that there is $\psi \in L^2(\mathbb{R})$ such that $Q_1Q_1^*\psi = 0$. Then

$$0 = (Q_1Q_1^*\psi, \psi) = \|Q_1^*\psi\| \quad \Longrightarrow \quad -\psi' - f_1\psi = 0.$$

Since $f_1(x) \sim -\sqrt{\lambda_1}$, as $x \to \infty$, we conclude that $\psi(x) \sim e^{\sqrt{\lambda_1}x}$, as $x \to \infty$, and thus $\psi \notin L^2(\mathbb{R})$.
This implies that the discrete spectrum of operators H and $H_1 = H - 2f_1'$ equal

$$\sigma_d(H) = \sigma_d\left(\frac{-d^2}{dx^2} - V\right) = \{-\lambda_1, -\lambda_2, -\lambda_3, \ldots, -\lambda_N\}$$

and

$$\sigma_d(H_1) = \sigma_d\left(\frac{-d^2}{dx^2} - V - 2f_1'\right) = \{-\lambda_2, -\lambda_3, \ldots, -\lambda_N\}.$$

The operator H_1 is the Schrödinger operator whose potential equals

$$V_1 = V + f_1'.$$

The lowest eigenvalue of H_1 is now $-\lambda_2$ and its respective eigenfunction (that is equal $Q_1\psi_2$) is positive.

Repeating this process we eliminate all negative eigenvalues after N-steps and obtain the Schrödinger operator whose potential equals $-V_N$, where

$$V_N = V_{N-1} + f_N' = V + f_1' + f_2' \cdots + f_N'$$

and where

$$f_N' + f_N^2 = \lambda_N - V_{N-1}.$$

Finally we find

$$\begin{aligned}
0 \le \int V_N^2\,dx &= \int (V_{N-1} + 2f_N')^2\,dx \\
&= \int \left(V_{N-1}^2 + 4f_N'(V_{N-1} + f_N')\right)\,dx = \int \left(V_{N-1}^2 + 4f_N'(\lambda_N - f_N^2)\right)\,dx \\
&= \int V_{N-1}^2\,dx + 4\lambda_N f_N\Big|_{-\infty}^{\infty} - \frac{4}{3}\,f_N^3\Big|_{-\infty}^{\infty} \\
&= \int V_{N-1}^2\,dx - 8\lambda_N^{3/2} + \frac{4}{3}\,\lambda_N^{3/2} = \int V_{N-1}^2\,dx - \frac{16}{3}\lambda_N^{3/2} \\
&= \cdots = \int V^2\,dx - \frac{16}{3}\sum_{j=1}^{N}\lambda_j^{3/2}.
\end{aligned}$$

Multidimensional Lieb-Thirring inequalities.

The main argument is based on 1D matrix Lieb-Thirring inequality.

Theorem. (AL & T.Weidl)
Let $Q \geq 0$ be a Hermitian $m \times m$ matrix-function and let $H = -\Delta - Q$.
Then

$$\sum_j \lambda_j^{3/2}(H) \leq \frac{3}{16} \int \operatorname{Tr} Q^2(x)\, dx.$$

Lifting argument with respect to dimension.

Let for simplicity $d = 2$, $V \in C_0^\infty(\mathbb{R}^2)$, $V \geq 0$, $x = (x_1, x_2)$. Then

$$H = -\Delta - V = -\partial^2_{x_1 x_1} - \underbrace{(\partial^2_{x_2 x_2} + V)}_{\tilde{H}(x_1)}.$$

Spectrum $\sigma(\tilde{H})$ of $\tilde{H}(x_1)$ has a finite number of positive eigenvalues $\mu_l(x_1)$.
Thus $\tilde{H}_+(x_1)$ has a finite rank. Let, for instance, $\gamma = 3/2$

$$\sum_j \lambda_j^{3/2}(H) \leq \sum_j \lambda_j^{3/2}(-\partial^2_{x_1 x_1} - \tilde{H}_+)$$

$$\leq \frac{3}{16} \int \operatorname{Tr} \tilde{H}_+^2(x_1)\, dx_1 \leq \underbrace{\frac{3}{16} L_{2,1}}_{L^{cl}_{3/2,2}} \iint V^{3/2+1}(x)\, dx.$$

1/2 moments of the eigenvalues ($\gamma = 1/2$)

Theorem.

Let $Q \geq 0$ be a Hermitian $M \times M$ matrix-function defined on $\mathbb{R}$, such that $\operatorname{Tr} Q \in L^1(\mathbb{R})$, $Q \geq 0$. Then for the negative eigenvalues $\{-\lambda_j\}$ of the Schrödinger operator H

$$H\psi(x) = \frac{d^2}{dx^2}\,\psi(x) - Q(x)\psi(x)$$

we have

$$\sum_j \lambda_j^{1/2}(H) \leq \frac{1}{2}\int \operatorname{Tr} Q\, dx. \qquad \text{(Hundertmark, Lapt \& Weidl).}$$

The scalar version of this result was first proved by D. Hundertmark, E. Lieb and L.Thomas.

Proof - scalar version (D. Hundertmark, E. Lieb and L.Thomas)

Let $A \geq 0$ be a compact operator in a Hilbert space $\mathcal{H}$ whose eigenvalues $\mu_j(A) \to 0$, as $j \to 0$ such that

$$\sqrt{\mu_1(A^*)\mu_1(A)} \geq \sqrt{\mu_2(A^*)\mu_2(A)} \geq \sqrt{\mu_3(A^*)\mu_3(A)} \geq \dots .$$

It is well known that the functionals

$$\|A\|_n = \sum_{j=1}^{n} \sqrt{\mu_j(A^*)\mu_1(A)}$$

are norms (see, for example, Gohberg I.C. and Krein M.G.: Introduction to the theory of linear non-self-adjoint operators. Trans. Math. Monographs vol 18. AMS 1969, Lemma 4.2).
Thus for any unitary operator U we have

$$\|U^*AU\|_n = \|A\|_n$$

Definition. Let A and B be two compact operators in $\mathcal{H}$. We say that A majorizes B or $B \prec A$, iff

$$\|B\|_n \leq \|A\|_n, \quad \text{for all} \quad n \in \mathbb{N}.$$

Proposition. Let A be a nonnegative compact operator in $\mathcal{H}$, $\{U(\omega)\}_{\omega\in\Omega}$ be a family of unitary operators in on $\mathcal{H}$ and let g be a probability measure on Ω. Then the operator

$$B := \int_\Omega U^*(\omega) A U(\omega)\, g(d\omega)$$

is majorized by A.

Proof. This is a simple consequence of the triangle inequality

$$\|B\|_n \leq \Big\| \int_\Omega U^*(\omega) A U(\omega)\, g(d\omega) \Big\|_n$$

$$\leq \int_\Omega \|U^*(\omega) A U(\omega)\|_n \, g(d\omega) = \|A\|_n \, g(\Omega) = \|A\|_n.$$

The proof is complete.

We now consider the spectral problem

$$-\frac{d^2}{dx^2}u(x) - V(x) = -\lambda u(x), \quad \text{where} \quad V \geq 0,$$

and introduce the operator

$$\mathcal{L}_\varepsilon = W \frac{2\varepsilon}{-\frac{d^2}{dx^2} + \varepsilon^2} W,$$

where $W = \sqrt{V}$. The kernel $M(x-y)$ of the operator $-\frac{d^2}{dx^2} + \varepsilon^2$ equals

$$M(x-y) = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{e^{i(x-y)\xi}}{\xi^2 + \varepsilon^2}\, d\xi = \frac{1}{2\varepsilon}\, e^{-\varepsilon|x-y|}.$$

Therefore, if $\varepsilon \to 0$ we have

$$\mathcal{L}_\varepsilon \psi(x) = \int_{\mathbb{R}} W(x) e^{-\varepsilon|x-y|} W(y)\psi(y)\, dy \to$$

$$\to \mathcal{L}_0 \psi(x) = W(x) \int_{\mathbb{R}} W(y)\psi(y)\, dy,$$

where the latter is the operator of rank one.

One of the main technical results is

Lemma. (Monotonicity)
The operator $\mathcal{L}_\varepsilon$ is majorized by $\mathcal{L}_{\varepsilon'}$. Namely

$$\mathcal{L}_\varepsilon \prec \mathcal{L}_{\varepsilon'} \quad \text{for all} \quad 0 < \varepsilon' < \varepsilon.$$

Proof.

Let us introduce

$$g_\varepsilon(d\xi) = \begin{cases} \frac{\varepsilon}{\pi(\xi^2+\varepsilon^2)}\,d\xi, & \text{if } \varepsilon > 0, \\ \delta(\xi)\,d\xi, & \text{if } \varepsilon = 0, \end{cases}$$

and difine

$$U(\xi)\psi(x) = e^{i\xi x}\psi(x), \qquad \xi \in \mathbb{R}.$$

Then

$$\mathcal{L}_\varepsilon = \int_{\mathbb{R}} U^* \mathcal{L}_0 U \, g_\varepsilon(d\xi).$$

In particular, applying Proposition we find that $L_\varepsilon \prec L_0$. Moreover, Note that

$$g_\varepsilon = g_{\varepsilon'} * g_{\varepsilon-\varepsilon'}$$

which follows from the fact $e^{-\varepsilon|x|} = e^{-\varepsilon'|x|}e^{-(\varepsilon-\varepsilon')|x|}$.

Then

$$\mathcal{L}_\varepsilon\psi(x) = \iint_{\mathbb{R}^2} e^{ix\xi} W(x)W(y)e^{-iy\xi}\,\frac{\varepsilon}{\pi(\xi^2+\varepsilon^2)}\psi(y)\,dyd\xi$$

$$= \iiint_{\mathbb{R}^3} e^{ix\xi} W(x)W(y)e^{-iy\xi}\,\frac{\varepsilon'}{\pi((\xi-\eta)^2+(\varepsilon')^2)}\;\frac{\varepsilon-\varepsilon'}{\pi((\eta)^2+(\varepsilon-\varepsilon')^2)}\,d\eta\,\psi(y)\,dyd\xi$$

$$= \iint_{\mathbb{R}^2} e^{ix\eta}\left(\int_{\mathbb{R}} e^{ix\rho} W(x)W(y)e^{-iy\rho}\frac{\varepsilon'}{\pi(\rho^2+(\varepsilon')^2)}\,d\rho\right)\,e^{-iy\eta}\;\frac{\varepsilon-\varepsilon'}{\pi((\eta)^2+(\varepsilon-\varepsilon')^2)}\,d\eta\,\psi(y)\,dy$$

$$= \iint_{\mathbb{R}^2} e^{ix\eta}\,\mathcal{L}_{\varepsilon'}\,e^{-iy\eta}\;\frac{\varepsilon-\varepsilon'}{\pi((\eta)^2+(\varepsilon-\varepsilon')^2)}\,d\eta\,\psi(y)\,dy.$$

The statement $\mathcal{L}_\varepsilon \prec \mathcal{L}_{\varepsilon'}$ follows now from the Proposition.

Proof of Theorem.

Let us introduce the operator

$$\mathcal{K}_\lambda = \frac{1}{2\sqrt{\lambda}}\mathcal{L}_{\sqrt{\lambda}} = W\left(\frac{-d^2}{dx^2}+\lambda\right)^{-1}W$$

The operator K_λ is compact that due to Birman Schwinger principle we have

$$1 = \mu_j(K_{\lambda_j}),$$

where $-\lambda_j$ is the eigenvalue of the Schrödinger operator H. Therefore by using the Monotonicity Lemma and since $\lambda_1 > \lambda_2$, we have

$$2\sqrt{\lambda_1} = \|\mathcal{L}_{\sqrt{\lambda_1}}\|_1 \le \|\mathcal{L}_{\sqrt{\lambda_2}}\|_1.$$

The inequality $\lambda_2 > \lambda_3$ also implies

$$2\sqrt{\lambda_1} + 2\sqrt{\lambda_2} \le \|\mathcal{L}_{\sqrt{\lambda_2}}\|_2 \le \|\mathcal{L}_{\sqrt{\lambda_3}}\|_2.$$

Repeating this n times we finally obtain

$$\begin{aligned} 2\sqrt{\lambda_1} + 2\sqrt{\lambda_2} + \cdots + 2\sqrt{\lambda_n} &\le \|\mathcal{L}_{\sqrt{\lambda_n}}\|_n \\ &\le \operatorname{Tr} L_{\sqrt{\lambda_n}} = \int_{\mathbb{R}} W^2(x)\,dx = \int_{\mathbb{R}} V(x)\,dx. \end{aligned}$$

The proof is complete.

1-moments estimates ($\gamma = 1$).

Let H be a Schrödinger operator in $L^2(\mathbb{R}^d)$

$$H = -\Delta - Q,$$

Theorem. ((jointly with J.Dolbeault & M.Loss)
Let $Q \geq 0$ be a Hermitian $M \times M$ matrix-function defined on $\mathbb{R}$ and let $\{-\lambda_j\}$ be negative eigenvalues of the operator H. Then

$$\sum \lambda_j \leq \frac{2}{3\sqrt{3}} \int_{\mathbb{R}} \mathrm{Tr}\left[Q^{3/2}(x)\right] dx\,.$$

Corollary. For any dimension $d \geq 1$, the negative eigenvalues of the operator H satisfy inequalities

$$\sum \lambda_j \leq L_{d,1} \int_{\mathbb{R}^d} \mathrm{Tr}\left[Q^{d/2+1}(x)\right] dx\,,$$

where

$$L_{d,1} \leq R \times L^{cl}_{d,1} = R \times \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} (1-|\xi|)_+ \, d\xi,$$

and $R = \frac{\pi}{\sqrt{3}} = 1.8138\ldots$.

Scalar case

For simplicity we consider a scalar version of this theorem based on a 1D generalised Sobolev inequality due to Eden and Foias.

Let $\{\psi_j\}_{j=1}^n$ be in orthonormal system of function in $L^2(\mathbb{R})$ and let

$$\rho(x) = \sum_{j=1}^{n} \psi_j^2(x).$$

Generalised Sobolev inequality in 1D case:

Theorem.

$$\int_{\mathbb{R}} \rho^3(x)\, dx = \int \Big(\sum_{j=1}^{n} |\psi_j(x)|^2\Big)^3 dx \leq \sum_{j=1}^{n} \int_{\mathbb{R}} |\psi'{}_j(x)|^2\, dx.$$

Proof.

We first derive a so-called Agmon inequality

$$\|\psi\|_{L^\infty} \le \|\psi\|_{L^2}^{1/2}\|\psi'\|_{L^2}^{1/2}.$$

Indeed

$$|\psi(x)|^2 = \frac{1}{2}\Big|\int_{-\infty}^{x} |\psi^2|'\,dt - \int_x^\infty |\psi^2|'\,dt\Big| \le \int |\psi||\psi'|\,dt \le \|\psi\|_{L^2}\|\psi'\|_{L^2}.$$

Let now $\xi = (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n$. Then by Agmon inequality

$$\Big|\sum_{j=1}^n \xi_j\psi_j(x)\Big| \le \Big(\sum_{j,k=1}^n \xi_j\bar{\xi}_k(\psi_j,\psi_k)\Big)^{1/4}\Big(\sum_{j,k=1}^n \xi_j\bar{\xi}_k(\psi_j',\psi_k')\Big)^{1/4}$$

$$\le \Big(\sum_{j=1}^n \xi_j^2\Big)^{1/4}\Big(\sum_{j,k=1}^n \xi_j\bar{\xi}_k(\psi_j',\psi_k')\Big)^{1/4}.$$

If we set $\xi_j = \psi_j(x)$ then the latter inequality becomes

$$\rho(x) = \sum_{j=1}^n |\psi_j(x)|^2 \le \rho^{1/4}(x)\Big(\sum_{j,k=1}^n \psi_j(x)\overline{\psi_k(x)}(\psi_j',\psi_k')\Big)^{1/4}.$$

Thus

$$\rho^3(x) \le \sum_{j,k=1}^n \psi_j(x)\overline{\psi_k(x)}(\psi_j',\psi_k').$$

Integrating both sides we arrive at

$$\int\Big(\sum_{j=1}^n |\psi_j(x)|^2\Big)^3\,dx \le \sum_{j=1}^n \int |\psi_j'|^2\,dx.$$

Spectrum of Schrödinger operators

Let $\{\psi_j\}_{j=1}^{\infty}$ be the orthonormal system of eigenfunctions corresponding to the negative eigenvalues of the Schrödinger operator

$$-\frac{d^2}{dx^2}\psi_j - Q\psi_j = -\lambda_j \psi_j,$$

where we assume that $Q \geq 0$. Then by using the latter result and Hölder's inequality we obtain

$$\int \Big(\sum_{j=1} |\psi_j(x)|^2\Big)^3 dx - \Big(\int Q^{3/2}\, dx\Big)^{2/3} \int \Big(\sum_{j=1} |\psi_j(x)|^2\Big)^3 dx\Big)^{1/3}$$

$$\leq \sum_j \int \Big(|\psi_j'|^2 - Q|\psi_j|^2\Big)\, dx = -\sum_j \lambda_j.$$

$$\int \Big(\sum_{j=1}^{n} |\psi_j(x)|^2\Big)^3 dx - \Big(\int Q^{3/2}\,dx\Big)^{2/3} \int \Big(\sum_{j=1}^{n} |\psi_j(x)|^2\Big)^3 dx\Big)^{1/3} \le -\sum_j \lambda_j.$$

Denote

$$X = \Big(\int \Big(\sum_{j=1}^{n} |\psi_j(x)|^2\Big)^3 dx\Big)^{1/3},$$

then the latter inequality can be written as

$$X^3 - \Big(\int V^{3/2}\,dx\Big)^{2/3} X \le -\sum_j \lambda_j.$$

Maximizing the left hand side we find $X = \frac{1}{\sqrt{3}}\Big(\int Q^{3/2}\,dx\Big)^{1/3}$. This implies

$$\frac{1}{3\sqrt{3}} \int Q^{3/2}\,dx - \frac{1}{\sqrt{3}} \int Q^{3/2}\,dx = -\frac{2}{3\sqrt{3}} \int Q^{3/2}\,dx \le -\sum_j \lambda_j$$

and we finally obtain $\sum_j \lambda_j \le \frac{2}{3\sqrt{3}} \int Q^{3/2}\,dx$.

This is the best known constant in the $\gamma = 1$ L-Th inequality.

Weyl Operators

Harmonic Oscillator

Let us begin with the operator of Harmonic Oscillator

$$H = -\frac{d^2}{dx^2} + x^2, \quad x \in \mathbb{R}.$$

The spectrum of this operator is discrete and equals $\{2k+1\}$, $k = 0, 1, 2, \ldots$. In particular,

$$H - 1 := A^* A = \left(-\frac{d}{dx} + x\right)\left(\frac{d}{dx} + x\right) \geq 0$$

which implies $H \geq 1$.

Note that the values of the symbol of the Harmonic oscillator $\xi^2 + x^2 \geq 0$ fills the semiaxis $[0, \infty)$ but the operator $H \geq 1$. The latter inequality is sharp and the eigenfunction corresponding to the eigenvalue one is $e^{-x^2/2}$.

One has to be careful with factorizations of operators. Indeed, let us consider

$$B^*B = \left(-\frac{d}{dx} + x - \frac{1}{x}\right)\left(\frac{d}{dx} + x - \frac{1}{x}\right)$$

$$= -\frac{d^2}{dx^2} + x^2 - 2 + \frac{1}{x^2} - 1 - \frac{1}{x^2} = -\frac{d^2}{dx^2} + x^2 - 3 \geq 0.$$

This implies

$$H = -\frac{d^2}{dx^2} + x^2 \geq 3.$$

Question: Where is a mistake?

Remark

Note that after Fourier transform the operator $-\frac{d^2}{dx^2} + x^2$ becomes $\xi^2 - \frac{d^2}{d\xi^2}$.

Coherent state transform

Let us consider the map $\Phi : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ and defined by

$$\widetilde{\psi}(x,\xi) = (\Phi\,\psi)(x,\xi) = \int_{-\infty}^{\infty} e^{-2\pi i \xi y} g(x-y)\psi(y)\,dy,$$

where

$$g(x) = (1/\pi)^{1/4} e^{-x^2/2}.$$

Note that $\int_{-\infty}^{\infty} g^2(x)dx = 1$ and

$$\begin{aligned}\Phi^*\Phi\,\psi(x) &= \int_{\mathbb{R}^3} e^{2\pi i \xi x} g(x-z) e^{-2\pi i \xi y} g(z-y)\psi(y)\,d\xi dy dz \\ &= \int_{\mathbb{R}^2} \delta(x-y) g(x-z) g(z-y)\,\psi(y)\,dy dz \\ &= \psi(x) \int_{-\infty}^{\infty} g^2(x-z)\,dz = \psi(x).\end{aligned}$$

Theorem.

The map $\Phi : L^2(\mathbb{R}) \to L^2(\mathbb{R}^2)$ is an isometry, such that $\Phi^*\Phi = I$ and $P = \Phi\Phi^*$ is an orthogonal projection in $L^2(\mathbb{R}^2)$.

Action of the coherent state transform on the Harmonic oscillator

Let us compute $\Phi^*\xi^2\Phi$.

$$(\Phi^*\xi^2\Phi\psi,\psi) = \int_{\mathbb{R}^4} e^{2\pi i\xi x} g(x-z)\,\xi^2\, e^{-2\pi i\xi y} g(z-y)\psi(y)\overline{\psi(x)}\,d\xi dy dz dx$$

$$= \frac{1}{4\pi^2}\int_{\mathbb{R}^4} \frac{d}{dx}\left(e^{2\pi i\xi x}\right) g(x-z)\,\frac{d}{dy}\left(e^{-2\pi i\xi y}\right) g(z-y)\psi(y)\overline{\psi(x)}\,d\xi dy dz dx$$

$$= \frac{1}{4\pi^2}\int_{\mathbb{R}^4} e^{2\pi i\xi(x-y)}\frac{d}{dx}\left(g(x-z)\overline{\psi(x)}\right)\frac{d}{dy}\left(g(z-y)\psi(y)\right) d\xi dy dz dx$$

$$= \frac{1}{4\pi^2}\int_{\mathbb{R}^2}\left(\left(g'(x-z)\right)^2|\psi(x)|^2 + g^2(x-z)|\psi'(x)|^2\right) dz dx$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{\infty}(g'(z))^2 dz\,\|\psi\|_2^2 + \frac{1}{4\pi^2}\|\psi'\|_2^2.$$

Corollary.

$$\|\psi'\|_2^2 = \int_{-\infty}^{\infty}(2\pi)^2\xi^2|\tilde{\psi}(z,\xi)|^2\,dz d\xi - \frac{1}{2}\|\psi\|_2^2.$$

Similarly computing $\Phi^* z^2 \Phi$ we obtain

$$(\Phi^* z^2 \Phi\psi, \psi) = \int_{\mathbb{R}^4} e^{2\pi i\xi x} g(x-z)\, z^2\, e^{-2\pi i \xi y} g(z-y)\psi(y)\overline{\psi(x)}\, d\xi dy dz dx$$
$$= \int_{\mathbb{R}^2} z^2 g^2(x-z)|\psi(x)|^2\, dzdx = \int_{\mathbb{R}^2} (x-t)^2 g^2(t)|\psi(x)|^2\, dzdx$$
$$= \int_{-\infty}^{\infty} x^2|\psi(x)|^2\, dx + \int_{-\infty}^{\infty} t^2 g^2(t)\, dt \int_{-\infty}^{\infty} |\psi(x)|^2\, dx.$$

Corollary

$$\|x\,\psi\|_2^2 = \int_{-\infty}^{\infty} z^2|\widetilde{\psi}(z,\xi)|^2\, dzd\xi - \frac{1}{2}\|\psi\|_2^2.$$

Proposition. There is the following representation of the quadratic form of the Harmonic oscillator H

$$(H\psi, \psi) = \int_{-\infty}^{\infty} \left((2\pi)^2\xi^2 + z^2\right) |\widetilde{\psi}(z,\xi)|^2\, dzd\xi - \|\psi\|_2^2.$$

Further properties of the coherent state transform

Let us introduce the convolution

$$\varphi * \psi(x) = \int_{-\infty}^{\infty} \varphi(x-y)\psi(y)\,dy.$$

and let $\mathcal{F}$ be the Fourier transform

$$\widehat{\psi}(\xi) = (\mathcal{F}\psi)(\xi) = \int_{-\infty}^{\infty} e^{2\pi i \xi x}\psi(x)\,dx.$$

Lemma.

$$(\Phi\psi)(x,\xi) = \widetilde{\psi}(x,\xi) = e^{-2\pi i x\xi} \int_{-\infty}^{\infty} \widehat{\psi}(\eta)\, e^{2\pi i \eta x}\overline{\widehat{g}(\eta-\xi)}\,d\eta \qquad (*)$$

and

$$\int_{-\infty}^{\infty} |\widetilde{\psi}(x,\xi)|^2 dx = \int_{-\infty}^{\infty} |\widehat{\psi}(\xi-\eta)|^2 |\widehat{g}(\eta)|^2\,d\eta = |\widehat{\psi}|^2 * |\widehat{g}|^2(\xi). \quad (**)$$

$$\int_{\mathbb{R}} |\widetilde{\psi}(x,\xi)|^2\,d\xi = \int_{-\infty}^{\infty} |\psi|(x-y)|^2 |g(y)|^2\,dy = (|\psi|^2 * |g|^2)(x) \qquad (***)$$

Proof. Let us first show $(*)$

$$
\begin{aligned}
e^{-2\pi i x\xi} & \int_{-\infty}^{\infty} \widehat{\psi}(\eta)\, e^{2\pi i\eta x}\overline{\widehat{g}(\eta-\xi)}\, d\eta \\
&= e^{-2\pi i x\xi} \int_{\mathbb{R}^3} e^{-2\pi i\eta z}\psi(z)e^{2\pi i\eta x}e^{2\pi i(\eta-\xi)t}g(t)\, dt dz d\eta \\
&= e^{-2\pi i x\xi} \int_{\mathbb{R}^2} \delta(t+x-z)\psi(z)e^{-2\pi i\xi t}g(t)\, dt dz \qquad [t = z-x] \\
&= \int_{-\infty}^{\infty} e^{-2\pi i\xi z}\psi(z)g(x-z)\, dz = \widetilde{\psi}(x,\xi).
\end{aligned}
$$

In order to obtain $(**)$ we write by using $(*)$

$$
\begin{aligned}
\int_{-\infty}^{\infty} |\widetilde{\psi}(x,\xi)|^2 dx &= \int_{\mathbb{R}^3} \overline{\widehat{\psi}(\rho)}\, e^{-2\pi i\rho x}\widehat{g}(\rho-\xi)\ \widehat{\psi}(\eta)\, e^{2\pi i\eta x}\overline{\widehat{g}(\eta-\xi)}\, dx d\rho d\eta \\
&= \int_{-\infty}^{\infty} |\widehat{\psi}(\eta)|^2|\widehat{g}(\eta-\xi)|^2\, d\eta.
\end{aligned}
$$

Exercise. Prove $(***)$.

Weyl operators

We now consider a class of functional discrete operators that have some analogy with Harmonic oscillators, but whose spectrum is more complicated.

Let

$$U\psi(x) = \psi(x+i) \qquad \text{and} \qquad V\psi(x) = e^{2\pi x}\psi(x).$$

Then

$$UV\psi(x) = e^{x+i}\psi(x+i) = e^{i}VU\psi(x).$$

The respective domains of these operators are

$$D(U) = \left\{\psi \in L^2(\mathbb{R}) : \, e^{-2\pi\xi}\,\widehat{\psi}(\xi) \in L^2(\mathbb{R})\right\}$$

and

$$D(V) = \left\{\psi \in L^2(\mathbb{R}) : \, e^{2\pi x}\,\psi(x) \in L^2(\mathbb{R})\right\}.$$

Equivalently, $D(U)$ consists of those functions $\psi(x)$ which admit an analytic continuation to the strip

$$\{z = x + iy \in \mathbb{C} :\ 0 < y < 1\}$$

such that $\psi(x + iy) \in L^2(\mathbb{R})$ for all $0 \leq y < 1$ and there is a limit

$$\psi(x + i - i0) = \lim_{\varepsilon \to 0^+} \psi(x + i - i\varepsilon)$$

in the sense of convergence in $L^2(\mathbb{R})$, which we will denote by $\psi(x + i)$.

Question: Prove it.

The domains of U^{-1} and V^{-1}can be characterised similarly and obviously

$$U^{-1}\psi(x) = \psi(x - i) \qquad \text{and} \qquad V^{-1}\psi(x) = e^{-2\pi x}\psi(x).$$

Our main object of study is the operator H

$$H = U + U^{-1} + V + V^{-1}$$

whose symbol is

$$2\cosh\xi + 2\cosh x.$$

Remark.

It was discovered by M. Aganagic, R. Dijkgraaf, A. Klemm, M. Marino, and C. Vafa, that the functional-difference operators built from the Weyl operators U and V, appear in the study of local mirror symmetry as a quantisation of an algebraic curve, the mirror to a toric Calabi-Yau threefold. The spectral properties of these operators were considered in A. Grassi, Y. Hatsuda, and M. Marino.

Remark.

The operator

$$\mathcal{H}\psi(x) = (U + U^{-1} + V)\psi(x) = \psi(x+i) + \psi(x-i) + e^{2\pi x}\psi(x)$$

first appeared in the study of the quantum Liouville model on the lattice and plays an important role in the representation theory of the non-compact quantum group $SL_q(2;\mathbb{R})$. In the momentum representation it becomes the Dehn twist operator in quantum Teichmuller theory.

In particular, R. Kashaev obtained the eigenfunction expansion theorem for this operator in the momentum representation. It was stated as formal completeness and orthogonality relations in the sense of distributions. The spectral analysis of the functional-difference operator $\mathcal{H}$ was done in the recent paper of L. D. Faddeev and L. A. Takhtajan. The operator $\mathcal{H}$ was shown to be self-adjoint with a simple absolutely continuous spectrum $[2,\infty)$, and the authors proved eigenfunction expansion theorem for $\mathcal{H}$, by generalizing the classical Kontorovich-Lebedev transform.

Action of the coherent state transform on functional discrete operators

We aim to find representations of $(U\psi,\psi)$ and $(V\psi,\psi)$ in terms of coherent states.
It follows from $(**)$ that

$$\iint_{\mathbb{R}^2} 2\cosh(2\pi\xi)|\widetilde{\psi}(x,\xi)|^2\,d\xi\,dx = \iint_{\mathbb{R}^2} 2\cosh(2\pi\xi)|\widehat{\psi}(\xi-\eta)|^2|\widehat{g}(\eta)|^2\,d\xi\,d\eta,$$

and using

$$\cosh(x+y) = \cosh x\cosh y + \sinh x \sinh y$$

we obtain

$$\begin{aligned}\iint_{\mathbb{R}^2} & 2\cosh(2\pi\xi)|\widetilde{\psi}(x,\xi)|^2\,d\xi\,dy \\ &= \iint_{\mathbb{R}^2} 2\cosh\left(2\pi(\xi-\eta)\right)|\widehat{\psi}(\xi-\eta)|^2\cosh(2\pi\eta)|\widehat{g}(\eta)|^2\,d\xi\,d\eta \\ &\quad + \iint_{\mathbb{R}^2} 2\sinh\left(2\pi(\xi-\eta)\right)|\widehat{\psi}(\xi-\eta)|^2\sinh(2\pi\eta)|\widehat{g}(\eta)|^2\,d\xi\,d\eta\,.\end{aligned}$$

The first integral on the right-hand side can be computed to be

$$((U+U^{-1})\psi,\psi)((V+V^{-1})\widehat{g},\widehat{g}).$$

$$\iint_{\mathbb{R}^2} 2\cosh\big(2\pi(\xi-\eta)\big)\,|\widehat{\psi}(\xi-\eta)|^2 \cosh(2\pi\eta)|\widehat{g}(\eta)|^2\, d\xi\, d\eta \qquad (*)$$

$$+ \iint_{\mathbb{R}^2} 2\sinh\big(2\pi(\xi-\eta)\big)\,|\widehat{\psi}(\xi-\eta)|^2 \sinh(2\pi\eta)|\widehat{g}(\eta)|^2\, d\xi\, d\eta\,.$$

Indeed, note that

$$\iint_{\mathbb{R}^2} e^{2\pi i x\xi}\, 2\cosh(2\pi\xi)\, \widehat{\psi}(\xi)\, d\xi = \iint_{\mathbb{R}^2} e^{2\pi i x\xi}\left(e^{-2\pi\xi} + e^{2\pi\xi}\right) \widehat{\psi}(\xi)\, d\xi$$

$$= \iint_{\mathbb{R}^2} \left(e^{2\pi i(x+i)\xi} + e^{2\pi i(x-i)\xi}\right) \widehat{\psi}(\xi)\, d\xi$$

$$= \psi(x+i) - \psi(x-i) = (U+U^{-1})\psi(x).$$

Therefore the first integral $(*)$ equals

$$((U+U^{-1})\psi,\psi)\int_{\mathbb{R}} \cosh(2\pi\eta)|\widehat{g}(\eta)|^2\, d\eta = ((U+U^{-1})\psi,\psi)\,((V+V^{-1})\widehat{g},\widehat{g})/2.$$

$$\iint_{\mathbb{R}^2} 2\cosh(2\pi\xi)|\widetilde{\psi}(x,\xi)|^2\, d\xi\, dy$$

$$= \iint_{\mathbb{R}^2} 2\cosh\big(2\pi(\xi-\eta)\big)|\widehat{\psi}(\xi-\eta)|^2 \cosh(2\pi\eta)|\widehat{g}(\eta)|^2\, d\xi\, d\eta$$

$$+ \iint_{\mathbb{R}^2} 2\sinh\big(2\pi(\xi-\eta)\big)|\widehat{\psi}(\xi-\eta)|^2 \sinh(2\pi\eta)|\widehat{g}(\eta)|^2\, d\xi\, d\eta\,. \qquad (**)$$

Since $g(x) = g(-x)$, it holds that $\widehat{g}(\xi) = \widehat{g}(-\xi)$ and consequently the integral $(**)$

$$\iint_{\mathbb{R}^2} 2\sinh\big(2\pi(\xi-\eta)\big)|\widehat{\psi}(\xi-\eta)|^2 \sinh(2\pi\eta)|\widehat{g}(\eta)|^2\, d\xi\, d\eta$$

vanishes.

Thus for $\psi \in D(U)$ we obtain the representation

$$((U+U^{-1})\psi,\psi) = d_1 \iint_{\mathbb{R}^2} 2\cosh(2\pi\xi)|\widetilde{\psi}(x,\xi)|^2\, d\xi\, dx$$

where

$$d_1 = \frac{2}{((V+V^{-1})\widehat{g},\widehat{g})} = e^{-1/4} < 1.$$

Similarly, we can use $(***)$ to compute that

$$\iint_{\mathbb{R}^2} 2\cosh(2\pi x)\,|\widetilde{\psi}(x,\xi)|^2\,d\xi\,dx = \iint_{\mathbb{R}^2} 2\cosh(2\pi x)\,|\psi(x-z)|^2|g(z)|^2\,dx\,dz,$$

which with the help of the same trigonometric identity as above can be simplified to

$$\iint_{\mathbb{R}^2} 2\cosh(2\pi x)\,|\widetilde{\psi}(x,\xi)|^2\,d\xi\,dx = ((V+V^{-1})\psi,\psi)\,((V+V^{-1})g,g)/2.$$

Thus for $\psi \in D(V)$ we have the representation

$$((V+V^{-1})\psi,\psi) = d_2 \iint_{\mathbb{R}^2} 2\cosh(2\pi x)\,|\widetilde{\psi}(x,\xi)|^2\,d\xi\,dx,$$

where

$$d_2 = \frac{2}{((V+V^{-1})g,g)} = e^{-\pi^2} < 1.$$

Summary: Coherent state representation for H

Summarising, we obtain a remarkable identity

$$(H\psi,\psi) = ((U+U^{-1})\psi,\psi) + ((V+V^{-1})\psi,\psi)$$
$$= \iint_{\mathbb{R}^2} 2(d_1 \cosh(2\pi\xi) + d_2 \cosh(2\pi x))|\widetilde{\psi}(x,\xi)|^2 \, d\xi \, dx.$$

Deriving an Upper Bound

Let $\{\lambda_j\}_{j=1}^{\infty}$ be the eigenvalues of H and let $\{\psi_j\}_{j=1}^{\infty}$ be the corresponding orthonormal eigenfunctions which form a complete set. We first observe that the coherent state representation of H yields

$$\sum_{j\geq 1}(\lambda - \lambda_j)_+ = \sum_{j\geq 1}\left(\lambda - (H\psi_j, \psi_j)\right)_+$$

$$= \sum_{j\geq 1}\left(\lambda - \iint_{\mathbb{R}^2} 2\big(d_1 \cosh(2\pi\xi) + d_2 \cosh(2\pi x)\big)|\widetilde{\psi}_j(x,\xi)|^2 \, d\xi \, dy\right)_+ .$$

Note

$$\iint_{\mathbb{R}^2} |\widetilde{\psi}_j(x,\xi)|^2 \, dx d\xi = \|\psi_j\|_2^2 = 1.$$

Therefore

$$\begin{aligned}
&\sum_{j\geq 1}(\lambda-\lambda_j)_+ \\
&\quad= \sum_{j\geq 1}\left(\iint_{\mathbb{R}^2}(\lambda-2d_1\cosh(2\pi\xi)-2d_2\cosh(2\pi x))\,|\widetilde{\psi}_j(x,\xi)|^2\,d\xi\,dx\right)_+ \\
&\qquad\leq \iint_{\mathbb{R}^2}\big(\lambda-2d_1\cosh(2\pi\xi)-2d_2\cosh(2\pi x)\big)_+\sum_{j\geq 1}|\widetilde{\psi}_j(x,\xi)|^2\,d\xi\,dx\,.
\end{aligned}$$

Denote $e_{x,\xi}(y)=e^{2\pi i\xi y}g(x-y)$. Since the eigenfunctions ψ_j form an orthonormal basis in $L^2(\mathbb{R})$

$$\sum_{j=1}^{\infty}|\widetilde{\psi}_j(x,\xi)|^2=\sum_{j=1}^{\infty}|(e_{x,\xi},\psi_j)|^2=\|e_{x\xi}\|^2=1\quad\text{for all}\quad x,\xi\in\mathbb{R},$$

we arrive at the upper bound

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+\leq\iint_{\mathbb{R}^2}\big(\lambda-2d_1\cosh(2\pi\xi)-2d_2\cosh(2\pi x)\big)_+\,d\xi\,dx\,.$$

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \leq \iint_{\mathbb{R}^2} \big(\lambda - 2d_1\cosh(2\pi\xi) - 2d_2\cosh(2\pi x)\big)_+ \, d\xi\, dx\,.$$

To investigate the behaviour of the integral on the right-hand side as $\lambda \to \infty$, we first note that

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \leq 4\int_0^\infty \int_0^\infty \big(\lambda - 2d_1\cosh(2\pi\xi) - 2d_2\cosh(2\pi x)\big)_+ \, d\xi\, dx$$

$$\leq 4\int_0^\infty \int_0^\infty \big(\lambda - d_1 e^{2\pi\xi} - d_2\, e^{2\pi x}\big)_+ \, d\xi\, dx\,,$$

where we used that $2\cosh x > e^x$ for $x > 0$.

Changing the variables $u_1 = d_1 e^{2\pi\xi}, u_2 = d_2 e^{2\pi x}$ we arrive at

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \leq \frac{1}{\pi^2}\int_{d_1}^\infty \int_{d_2}^\infty \frac{(\lambda-u_1-u_2)_+}{u_1 u_2}\, du_2\, du_1$$

$$= \frac{1}{\pi^2}\int_{d_1}^{\lambda-d_2} \int_{d_2}^{\lambda-u_1} \frac{\lambda-u_1-u_2}{u_1 u_2}\, du_2\, du_1\,.$$

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \leq \frac{1}{\pi^2}\int_{d_1}^{\lambda-d_2}\int_{d_2}^{\lambda-u_1}\frac{\lambda-u_1-u_2}{u_1u_2}\,du_2\,du_1\,.$$

Here $\lambda \geq d_1+d_2$ since $\lambda \geq 2$ and $d_1, d_2 \leq 1/2$. Now we immediately obtain

$$\begin{aligned}\int_{d_1}^{\lambda-d_2}\int_{d_2}^{\lambda-u_1}\frac{\lambda-u_1-u_2}{u_1u_2}\,du_2\,du_1 &= \lambda\int_{d_1/\lambda}^{1-d_2/\lambda}\int_{d_2/\lambda}^{1-v_1}\frac{1-v_1-v_2}{v_1v_2}\,dv_2\,dv_1\\ &= \lambda\log^2\lambda + O(\lambda\log\lambda)\end{aligned}$$

as $\lambda\to\infty$, so that

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \leq \frac{\lambda\log^2\lambda}{\pi^2} + O(\lambda\log\lambda).$$

Deriving a Lower Bound

To obtain a lower bound, we use a different argument. Since $\|\psi_j\|_2 = \|\widetilde{\psi}_j\|_2 = 1$ we start from the identity

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ = \sum_{j\geq 1}(\lambda-\lambda_j)_+ \iint_{\mathbb{R}^2} |\widetilde{\psi}_j(x,\xi)|^2\, dx d\xi,$$

and observe that, if as before $e_{x,\xi}(y) = e^{2\pi i\, y\, \xi} g(x-y)$, we have

$$\widetilde{\psi}_j(x,\xi) = \int_{\mathbb{R}} \psi_j(y)\overline{e_{x,\xi}(y)}\, dy = (\psi_j, e_{x,\xi}).$$

This implies

$$\begin{aligned}\sum_{j\geq 1}(\lambda-\lambda_j)_+ &= \iint_{\mathbb{R}^2} \sum_{j\geq 1}(\lambda-\lambda_j)_+ (\psi_j, e_{x,\xi})\overline{(\psi_j, e_{x,\xi})}\, d\xi dx \\ &= \iint_{\mathbb{R}^2} \sum_{j\geq 1}(\lambda-\lambda_j)_+ \big((e_{x,\xi}, \psi_j)\psi_j, e_{x,\xi}\big)\, d\xi dx\,.\end{aligned}$$

Denoting by dE_μ the projection-valued spectral measure for H on $[2,\infty)$, we conclude that

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ = \iint_{\mathbb{R}^2}\sum_{j\geq 1}(\lambda-\lambda_j)_+\big((e_{x,\xi},\psi_j)\psi_j, e_{x,\xi}\big)\, d\xi dx$$
$$= \iint_{\mathbb{R}^2}\int_2^\infty (\lambda-\mu)_+(dE_\mu e_{x,\xi}, e_{x,\xi})\, d\xi dx\,.$$

Since by the spectral theorem

$$\int_2^\infty (dE_\mu e_{x,\xi}, e_{x,\xi}) = (e_{x,\xi}, e_{x,\xi}) = \|g\|_2^2 = 1,$$

we can apply Jensen's inequality with the convex function $x\mapsto(\lambda-x)_+$ and obtain the lower bound

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \geq \iint_{\mathbb{R}^2}\left(\lambda-\int_2^\infty \mu\,(dE_\mu\, e_{x,\xi}, e_{x,\xi})\right)_+ d\xi dx\,.$$

Computing

$$\int_2^\infty \mu\,(dE_\mu\, e_{x,\xi}, e_{x,\xi})\, d\xi dx.$$

It follows from the spectral theorem that

$$\int_2^\infty \mu\,(dE_\mu\, e_{x,\xi}, e_{x,\xi}) = (He_{x,\xi}, e_{x,\xi})$$
$$= ((U+U^{-1})e_{x,\xi}, e_{x,\xi}) + ((V+V^{-1})e_{x,\xi}, e_{x,\xi})\,.$$

The two terms on the right-hand side can be computed explicitly.
We first note that

$$g(x-y\pm i) = (1/\pi)^{1/4}\, e^{(x-y\pm i)^2} = e^{1/2} g(x-y) e^{\mp(x-y)i},$$

whence

$$((U+U^{-1})e_{x,\xi}, e_{x,\xi}) = \int_{-\infty}^{\infty} \left(e^{-2\pi\xi} g(x-y+i) + e^{2\pi\xi} g(x-y-i)\right) g(x-y)\, dy$$
$$= e^{1/2}\left(e^{-2\pi\xi}\int_{-\infty}^{\infty} g(z)^2 e^{-iz}\, dz + e^{2\pi\xi}\int_{-\infty}^{\infty} g(z)^2 e^{iz}\, dz\right)$$
$$= \frac{1}{d_1} 2\cosh(2\pi\xi)\,.$$

For the second term, $((V+V^{-1})e_{x,\xi}, e_{x,\xi})$, we get

$$((V+V^{-1})e_{x,\xi}, e_{x,\xi}) = \int_{-\infty}^{\infty} 2\cosh(2\pi y) g(x-y)^2\, dy$$
$$= \int_{-\infty}^{\infty} 2\cosh\big(2\pi(x-y)\big)\cosh(2\pi y) g(x-y)^2\, dy$$
$$+ \int_{-\infty}^{\infty} 2\sinh\big(2\pi(x-y)\big)\sinh(2\pi b y) g(x-y)^2\, dy = \frac{1}{d_2} 2\cosh(2\pi x)\,.$$

Therefore we finally arrive at

$$\sum_{j\geq 1}(\lambda-\lambda_j)_+ \geq \iint_{\mathbb{R}^2} \left(\lambda - \frac{2}{d_1}\cosh(2\pi\xi) - \frac{2}{d_2}\cosh(2\pi x)\right)_+ d\xi dx$$
$$= 4\int_0^\infty \int_0^\infty \left(\lambda - \frac{2}{d_1}\cosh(2\pi\xi) - \frac{2}{d_2}\cosh(2\pi x)\right)_+ d\xi dx\,.$$

Note that $2\cosh x \le 2e^x$ for $x \ge 0$ and thus

$$\sum_{j\ge1}(\lambda-\lambda_j)_+ \ge 4\int_0^\infty\int_0^\infty \left(\lambda - \frac{2}{d_1}e^{2\pi\xi} - \frac{2}{d_2}e^{2\pi x}\right)_+ d\xi dx\,.$$

The integral on the right-hand side is computed in the same way as previously. The only difference is that the numbers d_1, d_2 have been replaced by $2/d_1, 2/d_2$. These coefficients have no influence on the leading term for large λ as long as $\lambda \ge 2/d_1 + 2/d_2$, and we conclude

$$\sum_{j\ge1}(\lambda-\lambda_j)_+ \ge \frac{1}{\pi^2}\lambda\log^2\lambda + O(\lambda\log\lambda) \quad \text{as} \quad \lambda\to\infty.$$

Thus we have the following result

Theorem. For the Riesz mean of the eigenvalues of the operator H we have

$$\sum_{j\ge1}(\lambda-\lambda_j)_+ = \frac{1}{\pi^2}\lambda\log^2\lambda + O(\lambda\log\lambda) \quad \text{as} \quad \lambda\to\infty.$$

Theorem. For the number $N(\lambda) = \#\{j \in \mathbb{N} : \lambda_j < \lambda\}$ of eigenvalues of the operator H below λ we have

$$\lim_{\lambda \to \infty} \frac{N(\lambda)}{\log^2 \lambda} = \pi^{-2}.$$

Proof. To derive an upper bound on $N(\lambda)$, we let $\mu \geq \rho > 0$ and note that

$$\sum_{j \geq 1} (\mu - \lambda_j)_+ = \sum_{\lambda_j < \mu} (\mu - \lambda_j) \geq \sum_{\lambda_j < \mu - \rho} (\mu - \lambda_j) > \rho N(\mu - \rho).$$

We can now use the asymptotic behaviour of the Riesz mean to conclude that there exists a $C > 0$ such that

$$N(\mu - \rho) \leq \frac{\mu \log^2 \mu}{\rho \, \pi^2} + \frac{C}{\rho} \mu \log \mu \, .$$

With $\tau > 0$ we now choose $\mu = (1 + \tau)\lambda$ and $\rho = \tau\lambda$ such that $\mu - \rho = \lambda$ and

$$N(\lambda) \leq \frac{1}{\pi^2} \left(1 + \frac{1}{\tau}\right) \left(\log^2(\lambda + \lambda\tau) + C \log(\lambda + \lambda\tau)\right) .$$

$$N(\lambda) \leq \frac{1}{\pi^2}\left(1+\frac{1}{\tau}\right)\left(\log^2(\lambda+\lambda\tau)+C\log(\lambda+\lambda\tau)\right).$$

It remains to optimise this upper bound with respect to $\tau > 0$. The minimum is attained at τ_0 defined by the equation

$$2\tau_0 = \log(\lambda+\lambda\tau_0)\,.$$

Since $2\tau - \log(1+\tau)$ is bijective as a function from $[0,\infty)$ to $[0,\infty)$, a unique solution τ_0 exists for every λ. It clearly holds that $\tau_0 \to \infty$ as $\lambda \to \infty$ and thus $\tau_0 \leq \log\lambda$ for sufficiently large λ. We can conclude that

$$\limsup_{\lambda\to\infty} \frac{N(\lambda)}{\log^2\lambda} \leq \frac{1}{\pi^2}\,.$$

To find an analogous lower bound we note that by using the lower bound for the Riesz mean for $\lambda \geq 2$ we have

$$N(\lambda) \geq \sum_{j\geq 1}\left(1-\frac{\lambda_j}{\lambda}\right)_+ = \frac{1}{\lambda}\sum_{j\geq 1}(\lambda-\lambda_j)_+ \geq \frac{\log^2\lambda}{\pi^2} + C\log\lambda$$

with some constant $C > 0$. The proof is complete.

Remark.

Similar results could be obtained for the operators

$$H(\zeta) = U + U^{-1} + V + \zeta V^{-1}, \qquad \zeta > 0,$$

and

$$H_{m,n} = U + V + q^{-mn} U^{-m} V^{-n}, \quad m, n \in \mathbb{N},$$

where now U and V are self-adjoint Weyl operators satisfying

$$UV = q^2 VU, \qquad q = e^{i\pi b^2}, \quad b > 0.$$

Open Problems.

The spectrum of the problem

$$H\psi(x) = (U + U^{-1} + V + V^{-1})\,\psi(x)$$
$$= \psi(x+i) + \psi(x-i) + 2\cosh(x)\psi(x) = \lambda\psi(x)$$

is discrete.

- Find the first eigenvalue λ_1.

- As for Harmonic oscillator the operator $U + U^{-1} + V + V^{-1}$ maps by Fourier transform to $V + V^{-1} + U + U^{-1}$.
Is there a suitable factorisation of H as a product of creation and annihilation operators?

- Find the sharp constant C in the inequality

$$\sum_j (\lambda - \lambda_j)_+ \leq C\lambda \log^2 \lambda.$$

Remark.

The symbol of the operator $U + U^{-1} + V + V^{-1}$ equals

$$2\cosh\xi + 2\cosh x \geq 2.$$

However we can write

$$\begin{aligned} 2\cosh\xi + 2\cosh x &= 2\sum_{n=0}^{\infty}\frac{1}{(2n)!}(\xi^{2n} + x^{2n}) \\ &= 2 + (\xi^2 + x^2) + 2\sum_{n=2}^{\infty}\frac{1}{(2n)!}(\xi^{2n} + x^{2n}). \end{aligned}$$

Therefore using theft the the first eigenvalue of the harmonic oscillator equals 1 we have

$$\lambda_1(H) \geq 5.$$

Question.

Estimate from below the first eigenvalue of the operator

$$H_n = (-1)^n D^{2n} + x^{2n}.$$

Many thanks