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Large Data Scattering for Supercritical Generalized KdV

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$\boxed{\textbf{Plan of the Talk}}$

- Introduction of the problem, basic properties and **nonlinear scattering**.

- Classical result on **small data scattering** (following C. Kenig-G.Ponce-L.Vega '93).

- **Large data scattering** by combining C. Kenig-F. Merle **concentration-compactness/rigidity** plus a basic **calculus inequality** by T. Tao.

Introduction to the Problem

Let us consider the following Cauchy problems:

$$\mathbf{(gKdV)_k} \qquad \begin{cases} \partial_t u + \partial_x^3 u = \partial_x(u^{k+1}), & k \geq 1 \ k \text{ even integer, } (t,x) \in \mathbb{R} \times \mathbb{R} \\ u(0,x) = \varphi(x) \in H^1(\mathbb{R}) \end{cases}$$

where $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

- For $k = 1$ this is the usual KdV (Kortweg and de Vries equation). It is completely integrable, solitons, multisolitons, exactly solvable by inverse scattering..... It is related to the mathematical description of one dimensional waves on shallow water surfaces (for instance waves in a channel);

- For $k = 2$ it is the modified KdV (mKdV);

- There is a connection between KdV and mKdV via the Miura map;

- We are interested in the (non integrable) case $k \geq 4$.

$\boxed{\textbf{Conserved Quantities}}$

We assume the nonlinearity **defocusing**, namely the following **positive energy**, is preserved along the flow:

$$E(u(t,x)) = E(u(0,x))$$

where

$$E(u) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x u|^2 dx + \frac{1}{k+2}\int_{\mathbb{R}} u^{k+2} dx$$

as well as the mass

$$\int_{\mathbb{R}} |u(t,x)|^2 dx = \int_{\mathbb{R}} |u(0,x)|^2 dx.$$

- Notice that it makes no sense to speak about **defocusing nonlinearity** for k **odd**.

We have the following relations:

$$\partial_t \rho + \partial_{xxx} \rho = \partial_x j$$

$$\partial_t e + \partial_{xxx} e = \partial_x k$$

where

$$\rho(t,x) = \rho(u(t,x)) = u^2$$

$$e(t,x) = e(u(t,x)) = \frac{1}{2}(\partial_x u)^2 + \frac{1}{k+2} u^{k+2}$$

$$j(t,x) = j(u(t,x)) = 3(\partial_x u)^2 + \frac{2(k+1)}{k+2} u^{k+2}$$

$$k(t,x) = k(u(t,x)) = \frac{3}{2}(\partial_{xx} u)^2 + 2(\partial_x u)^2 u^k + \frac{1}{2} u^{2k+2}$$

$\boxed{\textbf{Local and Global Existence Results}}$

- Based on the classical work by Kenig-Ponce-Vega one can show the following **Local Existence Result**

  $\forall k \geq 1 \exists ! u(t,x) \in X_T$ sol. to $(gKdV)_k$, where $T = T(\|\varphi\|_{H^1_x}) > 0$

  and $X_T \subset \mathcal{C}([-T,T]; H^1_x)$;

- As a consequence of the **Local Existence Result**, in conjunction with the **defocusing** character of the nonlinearity (for k even) one can show the existence of an **Unique Global Solution**.

$\boxed{\textbf{Nonlinear Scattering}}$

Question: how the solutions to $(gKdV)_k$ look like as $t \to \pm\infty$?

Nonlinear scattering: for large times the nonlinear solutions look like linear solutions. Namely:

$$\forall \varphi \in H^1_x \exists \varphi_\pm \in H^1_x \text{ s.t. } \|u(t,x) - U(t)\varphi_\pm\|_{H^1_x} \to 0 \text{ as } t \to \pm\infty$$

where $U(t) = e^{-t\partial_x^3}$ is the group associated with the (linear) Airy equation, namely $v_\pm(t,x) = U(t)\varphi_\pm$ solve:

$$\textbf{(Airy)} \quad \begin{cases} \partial_t u + \partial_x^3 u = 0, & (t,x) \in \mathbb{R} \times \mathbb{R} \\ u(0,x) = \varphi_\pm \end{cases}$$

Small Data Scattering

Based on the work by Kenig-Ponce-Vega one can prove

Theorem 1

Let $k \geq 4$, *then* $\exists \epsilon = \epsilon(k) > 0$ *such that* $u(t,x)$ *solution to* $(gKdV)_k$ *and* $\|u(0,x)\|_{H^1_x} < \epsilon$ *then* $u(t,x)$ *scatters as* $t \to \pm\infty$

- What about **small data scattering for** $k < 4$**?**
  Unknown for small data in the energy space H^1_x. Moreover for $k = 2$ it has been proved the **modified scattering** in weighted Sobolev spaces (see Hayashi-Naumkin).

- What about **large data scattering for** $k \geq 4$?
  It has been established by Dodson for $k = 4$ and by Farah-Linares-Pastor-V. for $k > 4$ **even** (for k odd there are problems with global well posedness since energy is not positive definite).

$\boxed{\textbf{Idea to prove Small Data Scattering}}$

The following estimates are available for the linear propagator

$$\|U(t)\varphi\|_{L_x^5 L_t^{10}} \leq C\|\varphi\|_{L_x^2} \text{ (Strichartz)}$$

$$\|\partial_x U(t)\varphi\|_{L_x^\infty L_t^2} \leq C\|\varphi\|_{L_x^2} \text{ (Smoothing)}$$

$$\|U(t)\varphi\|_{L_x^{\frac{5k}{4}} L_t^{\frac{5k}{2}}} \leq C\|D^{s_k}\varphi\|_{L_x^2} \text{ where } s_k = \frac{k-4}{2k} \text{ (Strichartz with loss)}$$

along with the corresponding versions for the Duhamel operator:

$$\int_0^t U(t-s)F(s,x)ds$$

Next we notice that the integral formulation of $(gKdV)_k$ is the following one:

$$u(t,x) = U(t)\varphi + \int_0^t U(t-s)\partial_x(u^{k+1}(s,x))ds = T_\varphi(u(t,x))$$

then one can perform a fixed point argument in the space

$$\|u(t,x)\|_X = \sup_t \|u(t,x)\|_{H^1_x} + \|D^{s_k}u(t,x)\|_{L^5_x L^{10}_t} + \|\partial_x u(t,x)\|_{L^5_x L^{10}_t}$$

$$+\|u(t,x)\|_{L^5_x L^{10}_t} + \|u\|_{L_x^{\frac{5k}{4}} L_t^{\frac{5k}{2}}}$$

One can prove

$$\|T_\varphi v\|_X \leq C\|\varphi\|_{H^1_x} + \|v\|_X^{k+1}$$

and hence for $\|\varphi\|_{H^1_x} << 1$ we have $T_\varphi : B_X(0,R) \to B_X(0,R)$ and moreover it is a contration where $R = R(\|\varphi\|_{H^1_x}) > 0$.

By the Duhamel formulation we get

$$U(-t)u(t,x) = \varphi + \int_0^t U(-s)\partial_x(u^{k+1}(s,x))ds$$

and hence we conclude scattering (by Cauchy criterion) once we show

$$\| \int_{t_1}^{t_2} U(-s)\partial_x(u^{k+1}(s,x))ds\|_{H^1_x} \to 0 \text{ as } t_1, t_2 \to \infty$$

By using the dual version of smoothing and Strichartz estimates we get

$$\| \int_{t_1}^{t_2} U(-s)\partial_x(u^{k+1}(s,x))ds\|_{H^1_x}$$

$$\leq C\|u^{k+1}(s,x)\|_{L^1_x L^2_{(t_1,t_2)}} + C\|\partial_x(u^{k+1}(s,x))\|_{L^1_x L^2_{(t_1,t_2)}}$$

$$\leq C\|u\|_{L^5_t L^{10}_x}\|u\|^k_{L_x^{\frac{5k}{4}} L^{\frac{5k}{2}}_{(t_1,t_2)}} + C\|\partial_x u\|_{L^5_x L^{10}_{(t_1,t_2)}}\|u\|^k_{L_x^{\frac{5k}{4}} L^{\frac{5k}{2}}_{(t_1,t_2)}}$$

Strategy to prove large data scattering for $k > 4$ even

Two fundamental ingredients:

$$\begin{cases} \textbf{C.Kenig-F.Merle conc./comp.-rigidity} \\ (\textbf{to get a minimal object}) \\ \\ \\ \textbf{Extinction of the minimal object} \\ \textbf{via a functional inequality due to T.Tao} \end{cases}$$

The Kenig-Merle technique

Consider

$$\alpha_0 = \sup\{\alpha > 0 | u(t,x) \text{ scatter provide that } E(u(0,x)) \le \alpha\}$$

- By small data scattering $\alpha_0 > 0$.
- **Aim**: to show that $\alpha_0 = \infty$.

Step 1 Assume by **the absurd** $\alpha_0 \in (0,\infty)$ then there exists $\varphi \in H^1(\mathbb{R})$ such that:

$$E(\varphi) = \alpha_0$$

and the corresponding non-linear solution $u(t,x)$ **does not scatters**. Moreover the **"minimal"** solution $u(t,x)$ is such that

$$\{u(t,x-x(t)) | t \in \mathbb{R}\} \text{ is } \textbf{compact in } H^1(\mathbb{R})$$

for a suitable selection of translations $x(t) \in \mathbb{R}$ (this property is due to the **minimality** of α_0!)

Key tool is the **Profile Decomposition**

Theorem 2

Let $\{\phi_n\}_{n\in\mathbb{N}}$ *be a bounded sequence in* H^1. *There exists (up to subsequence)* $\{\psi^j\}_{j\in\mathbb{N}} \subset H^1$, $\{W_n^j\}_{j,n\in\mathbb{N}} \subset H^1$, $\{t_n^j\}_{j,n\in\mathbb{N}} \subset \mathbb{R}$ *and* $\{x_n^j\}_{j,n\in\mathbb{N}} \subset \mathbb{R}$, *such that for every* $l \geq 1$,

$$\phi_n = \sum_{j=1}^{l} U(t_n^j)\psi^j(\cdot - x_n^j) + W_n^l$$

$$\|\phi_n\|^2_{\dot{H}^\lambda} - \sum_{j=1}^{l} \|\psi^j\|^2_{\dot{H}^\lambda} - \|W_n^l\|^2_{\dot{H}^\lambda} \to 0, \text{ as } n \to \infty, \text{ for all } 0 \leq \lambda \leq 1,$$

Furthermore, the time and space sequence have a pairwise divergence property: for $1 \leq i \neq j \leq l$, *we have*

$$\lim_{n\to\infty} |t_n^i - t_n^j| + |x_n^i - x_n^j| = \infty.$$

Finally, the reminder sequence has the following asymptotic smallness property

$$\limsup_{n\to\infty} \|U(t)W_n^l\|_{L_x^{5k/4} L_t^{5k/2}} = 0, \text{ as } l \to \infty.$$

Step 2 To prove the extinction of the minimal element: namely **the unique solution to** $(gKdV)_k$ **which is compact (up to space translations) is the trivial one**.

Basic tool is the following inequality (due to T. Tao)

$$\left(\int \rho(x)dx\right)\cdot\left(\int k(x)dx\right) > \left(\int e(x)dx\right)\cdot\left(\int j(x)dx\right)$$

for every $u \in H^2(\mathbb{R})$ not identically zero.

Introduction of a suitable functional

We introduce the quantity

$$\int\int Q(x-y)\rho(t,x)e(t,y)dxdy$$

and then we get

$$\frac{d}{dt}\int\int Q(x-y)\rho(t,x)e(t,y)dxdy$$

$$= \int\int Q(x-y)\partial_t\rho(t,x)e(t,y)dxdy + \int\int Q(x-y)\rho(t,x)\partial_t e(t,y)dxdy$$

$$= -\int\int Q(x-y)\partial_{xxx}\rho(t,x)e(t,y)dxdy - \int\int Q(x-y)\rho(t,x)\partial_{yyy}e(t,y)dxdy$$

$$+\int\int Q(x-y)\partial_x j(t,x)e(t,y)dxdy + \int\int Q(x-y)\rho(t,x)\partial_y k(t,y)dxdy$$

and, by integration by parts,

$$\frac{d}{dt}\int\int Q(x-y)\rho(t,x)e(t,y)dxdy$$

$$= -\int\int Q^{'}(x-y)j(t,x)e(t,y)dxdy + \int\int Q^{'}(x-y)\rho(t,x)k(t,y)dxdy.$$

and it implies

$$[\int\int Q(x-y)\rho(t,x)e(t,y)dxdy]_{-T}^{T}$$

$$= -\int_{-T}^{T}\int\int Q^{'}(x-y)j(t,x)e(t,y)dxdydt + \int_{-T}^{T}\int\int Q^{'}(x-y)\rho(t,x)k(t,y)dxdydt$$

Notice that by conservation laws

$$\sup_{T}\left|[\int\int Q(x-y)\rho(t,x)e(t,y)dxdy]_{-T}^{T}\right| < \infty$$

provided that $Q \in L^{\infty}$

We introduce a function $\Phi \in C^\infty(\mathbb{R})$ such that:

- $\Phi(x) = x \quad \forall\, |x| < 1$;

- $\Phi'(x) \geq 0$;

- $|\Phi'(x)| = 0$ for $|x| > 2$.

Next, we consider for $R > 0$ the rescaled functions $Q_R(x) = 2R\Phi(\frac{x}{2R})$

By combining compactness with the T. Tao inequality we get

$$-\int\int_{Q_R(x(t),x(t))} j(t,x)e(t,y)dxdy + \int\int_{Q_R(x(t),x(t))} \rho(t,x)k(t,y)dxdy \geq \epsilon_0 > 0$$

where

$$Q_R(x(t),x(t)) = \{(x,y) \in (-R+x(t), R+x(t)) \times (-R+x(t), R+x(t))\}$$

and hence

$$-\int\int Q_R'(x-y)j(t,x)e(t,y)dxdy + \int\int Q_R'(x-y)\rho(t,x)k(t,y)dxdy \geq \eta_0 > 0$$

In particular

$$-\int_{-T}^{T}\int\int Q_R'(x-y)j(t,x)e(t,y)dxdydt$$

$$+\int_{-T}^{T}\int\int Q_R'(x-y)\rho(t,x)k(t,y)dxdydt \to \infty \text{ as } T \to \infty$$

hence we get an absurd.

Thank you for your attention!