Connections:

Classical Fourier Series

$\Updownarrow$

Calderón-Zygmund Theory

Rome, September 2017

1950's

Golden Age of Fourier Series

Beurling - Helson

Sidon Sets

Birth of Calderón - Zygmund Thy

Fourier multipliers: $Tf = f * \Lambda$

$\widehat{Tf} = m \hat{f}$; $m = \hat{\Lambda}$

Beurling-Helson (1953)

$$\Phi : \mathbb{T} \to \mathbb{T}$$

$$\begin{array}{ccc} A(\mathbb{T}) & & A(\mathbb{T}) \\ f & \to & f \circ \Phi \end{array} \Longrightarrow \Phi(e^{it}) = e^{i(c+kt)}$$

affine

$$\Phi : \mathbb{T}^k \to \mathbb{T}^d$$

$$\begin{array}{ccc} A(\mathbb{T}^d) & & A(\mathbb{T}^k) \\ f & \to & f \circ \Phi \end{array} \Rightarrow \Phi \text{ affine}$$

Γ LCA $\qquad \widehat{G} = \Gamma$

$$A(\Gamma) := \widehat{L^1(G)}$$

$$\Psi : L^1(G_1) \to L^1(G_2) \longleftrightarrow \Phi : \Gamma_2 \to \Gamma_1$$

$$\left(\widehat{\Psi f}(\gamma) = \hat{f}(\Phi(\gamma)) \right)$$

$\boxed{\text{P. Cohen (1960)}}$

P. Cohen (1960)

$\Psi : L^1(G_1) \to L^1(G_2)$ homomorphism

$\Rightarrow$ Ψ given by a piecewise affine

$$\Phi : \Gamma_2 \to \Gamma_1$$

Kahane (late 1950's)

$$\Phi : \mathbb{T} \to \mathbb{T}$$

$A(\mathbb{T})$ $\quad$ $U(\mathbb{T})$ $\quad$ (Φ) $\quad$?

$$f \to f \circ \Phi$$

L. Alpár (1966)

(Φ) holds for any real-analytic map

(Φ) can fail for certain C^∞ maps

R. Kaufman (1975)

(Φ) holds for any finite-type map

6

$$\Phi : \mathbb{T}^k \to \mathbb{T}^d$$

$$A(\mathbb{T}^d) \qquad U(\mathbb{T}^k) \qquad (\Phi)$$

$$f \to f \circ \Phi$$

- $f(x) = \sum_n \hat{f}(n) e^{inx} \in A(\mathbb{T})$

- $\Phi : \mathbb{T} \to \mathbb{T}$

$$\boxed{\begin{aligned} &\Phi(e^{it}) = e^{i\psi(t)} \\ &\psi(t) = \varphi(t) + kt \\ &\text{periodic} \quad \text{linear} \end{aligned}}$$

- $f \circ \Phi = \sum_n \hat{f}(n) e^{in\psi} \in \mho(\mathbb{T})$

$$\Updownarrow$$

$$\| f \circ \Phi \|_{\mho} \le \sum |\hat{f}(n)| \, \| e^{in\psi} \|_{\mho} < \infty$$

$$\Updownarrow$$

$$\sup_n \| e^{in\psi} \|_{\mho} < \infty$$

$$\|g\|_V := \sup_N \|S_N g\|_\infty$$

$$S_N g(x) = \sum_{|n|\le N} \hat{g}(n) e^{inx} = \int_{\mathbb{T}} g(x+t)\, D_N(t)\,dt$$

$$S_N e^{in\psi}(x) \sim \int_{|t|\le 1} e^{in\psi(x+t)}\, e^{iNt}\, \frac{1}{t}\,dt$$

$$\boxed{\sup_n \|e^{in\psi}\|_V = \sup_{x,n,N} \left| \int_{|t|\le 1} e^{i[n\varphi(x+t)+Nt]}\, \frac{1}{t}\,dt \right| < \infty}$$

Hilbert transforms along curves

$$H_\varphi f(x,y) = \int_{|t|\le 1} f(x-t, y-\varphi(t))\,\frac{1}{t}\,dt$$

$$H_\varphi : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2) \quad \text{iff}$$

$$m(\xi,\eta) := \int_{|t|\le 1} e^{i[\eta\varphi(t)+\xi t]}\,\frac{1}{t}\,dt \in L^\infty(\mathbb{R}^2)$$

— Intermission —

Sidon Sets

$E \subseteq \mathbb{Z} \rightsquigarrow C_E = \{ f \in C(\mathbb{T}) : \hat{f}(n) = 0 \ \forall n \notin E \}$

Def'n $E \subseteq \mathbb{Z}$ Sidon if $C_E \subseteq A(\mathbb{T})$

Sidon (1920's) Lacunary sets $E = \{2^k\}$ are Sidon sets

$E \subseteq \mathbb{Z}^n$ or $E \subseteq \Gamma$ discrete LCA

E Sidon $\Rightarrow$

$$\left(\sum_{n \in E} |\hat{f}(n)|^2 \right)^{\frac{1}{2}} \leq C \|f\|_{L\sqrt{\log L}}$$

Il'yanov (1953) $A(\mathbb{T}) \to U(\mathbb{T})$

Def'n $E \subseteq \mathbb{Z}$ SUC (spectral sets of uniform convergence)

if $C_E \subseteq U(\mathbb{T})$

- E Sidon $\Rightarrow$ E SUC
- $E = \{3^k + 3^\ell\}$ SUC but not Sidon (Figà Telemonca)
- $E = \mathbb{Z}$ not SUC

How dense can SUC be?

Is $E = \{n^2\}$ a SUC?

Lemma $E \subseteq \mathbb{Z}$ SUC iff

$$L_N(E) := \sup_{\substack{f \in C_E \\ \|f\|_\infty = 1}} \|S_N f\|_\infty = O(1).$$

Pf E SUC $\Rightarrow$ $\forall f \in C_E$, $\sup_N \|S_N f\|_\infty < \infty$

$$\text{UBP} \Rightarrow \sup_N \sup_{\substack{f \in C_E \\ \|f\|_\infty = 1}} \|S_N f\|_\infty = \sup_N L_N(E) < \infty.$$

- $L_N(E)$ Lebesgue constants of E

- $E = \mathbb{Z} \Rightarrow L_N(\mathbb{Z}) \sim \log N$

$E_{sq} = \{ n^2 \}$

Consider

$$F_N(x) = \sum_{|n| \le N} \frac{e^{i(n+N)^2 x}}{n} \in C_{E_{sq}}$$

- $$S_{N^2} F_N(x) = \sum_{n=-N}^{-1} \frac{e^{i(n+N)^2 x}}{n}$$

$$\left(\Rightarrow S_{N^2} F_N(0) = \sum_{n=-N}^{-1} \frac{1}{n} \right)$$

- $$| S_{N^2} F_N(0) | \sim \log N$$

$$\Downarrow$$

$$\boxed{\log N \le \| S_{N^2} F_N \|_\infty \le L_{N^2}(E_{sq}) \, \| F_N \|_\infty}$$

If $\|F_N\|_\infty = \mathcal{O}(1)$, then

$$L_N(E_{sq}) \sim \log N$$

$$F_N(x) = \sum_{|n| \leq N} e^{i[xn^2 + 2xNn]} \frac{1}{n} \cdot e^{iN^2 x}$$

depends on the diophantine properties of x !

Thm (Arkhipov, Oskolkov 1988)

$$\sup_{\substack{p \in \mathcal{P}_d \\ N}} \left| \sum_{|n| \le N} \frac{e^{i P(n)}}{n} \right| < \infty$$

This is also a theorem of Stein & Wainger

$Q \in \mathbb{Z}[X]$

$$T_Q f(m) = \sum_{n \in \mathbb{Z}} \frac{f(m - Q(n))}{n}$$

$$T_Q : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) \quad \text{iff}$$

$$m(\theta) := \sum_{n \in \mathbb{Z}} \frac{e^{i Q(n) \theta}}{n} \in L^\infty(\mathbb{T})$$

$$\left(T_Q : \ell^1 \to \ell^1 \quad ? \right)$$