A problem of Kahane in higher dimensions

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Kahane

$$\Phi : \mathbb{T} \to \mathbb{T}$$

$$A(\mathbb{T}) \qquad\qquad U(\mathbb{T}) \qquad\qquad (\Phi)$$

$$f \to f \circ \Phi$$

$$\Phi(e^{it}) = e^{i(\varphi(t) + kt)}$$

φ periodic

$k \in \mathbb{Z}$ linear

Heart

$$I(x,n,N) = \int_{|t|\le 1} e^{i(n\varphi(x,t) + Nt)} \, \frac{1}{t} \, dt$$

$$\sup_{x,n,N} |I(x,n,N)| < \infty$$

Problem in C-Z theory

$$I(0,n,N) = \int_{|t|\leq 1} e^{i(n\varphi(t)+Nt)} \frac{1}{t}\, dt$$

$$\boxed{\text{Cancellation arises in 2 ways!}}$$

2 ways:

Oscillation $|\varphi^{(k)}(t)| \geq \lambda$, $a \leq t \leq b$, $k \geq 2$

$$\Rightarrow \left| \int_a^b e^{i\varphi(t)}\, dt \right| \leq C_k \lambda^{-\frac{1}{k}} \quad \text{(v.c.)}$$

C-Z Kernel $\frac{1}{t}$ $\widehat{\text{p.v.} \frac{1}{t}}(\xi) = \text{p.v.} \int e^{i\xi t} \frac{1}{t}\, dt \in L^\infty$

Hilbert transform $Hf(x) = \int f(x-t) \frac{1}{t}\, dt$ is bounded on L^2

- Let $k \geq 2$ be minimal w.r.t $\varphi^{(k)}(0) \neq 0$

(if $\varphi^{(\ell)}(0) = 0 \ \forall \ell \geq 2 \Rightarrow \varphi(t) \equiv a + bt$ ✓)

- $\exists\, \delta > 0$ s.t. $|\varphi^{(k)}(t)| \gtrsim 1, \quad |t| \leq \delta$

$$I = \int_{|t| \leq \delta} e^{i(n\varphi(t) + Nt)} \frac{1}{t} \, dt + O(\log 1/\delta)$$

$\psi(t) := n\varphi(t) + Nt \implies |\psi^{(k)}(t)| \gtrsim n, \quad |t| \le \delta$

(V.C) + I.B.P. $\implies$

$$\left| \int_{n^{-\frac{1}{k}} \le |t| \le \delta} e^{i(n\varphi(t)+Nt)} \frac{1}{t}\, dt \right| \le C_k$$

Left w/

$$\int_{|t| \le n^{-\frac{1}{k}}} e^{i(n\varphi(t)+Nt)} \frac{1}{t}\, dt$$

- $\varphi(t) = \varphi(0) + \varphi'(0)t + R_k(t)$, $\quad |R_k(t)| \lesssim |t|^k$

Compare

$$\left| \int_{|t| \le n^{-\frac{1}{k}}} e^{i(n\varphi(t) + Nt)} \frac{1}{t}\, dt - \int_{|t| \le n^{-\frac{1}{k}}} e^{i(n[\varphi(0) + \varphi'(0)t] + Nt)} \frac{1}{t}\, dt \right|$$

$$\lesssim n \int_{|t| \le n^{-\frac{1}{k}}} |t|^k \frac{1}{|t|}\, dt \lesssim_k 1$$

Finally left with

$$e^{in\varphi(0)} \int_{|t| \le n^{-\frac{1}{k}}} e^{i(n\varphi'(0)+N)t} \frac{1}{t} \, dt$$

but this is $O(1)$ by the

L^2 bddedness of the Hilbert transform!

Back to

$$I(x,n,N) = \int_{|t|\le 1} e^{i(n\varphi(x+t)+Nt)} \frac{1}{t}\, dt$$

$$\varphi(x+t) = \varphi(x) + \varphi'(x)t + \frac{1}{k!}\varphi^{(k)}(x)\, t^k + \dots$$

- $k \geq 2$ minimal w.r.t. $\varphi^{(k)}(x) \neq 0$

Newton diagram of φ at x

- $\exists\ \delta > 0$ s.t. $|\varphi^{(k)}(x+t)| \geq C$, $|t| \leq \delta$, etc...

BUT $k = k(x)$, $\delta = \delta(x)$, $C = C(x)$, ... !

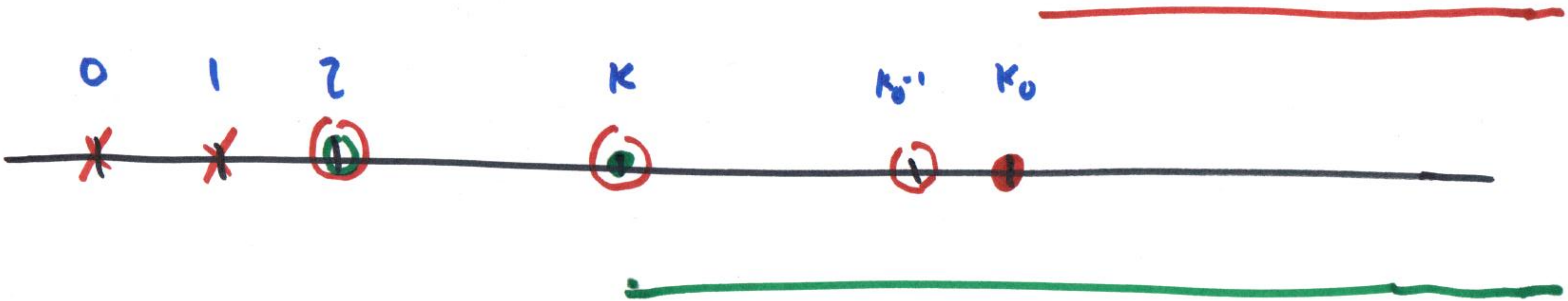
Cptness argument

$$\sup_{n, N, y \in I_{x_0}} \left| \int_{|t| \leq \delta_0} e^{i(n\varphi(x+t) + Nt)} \frac{1}{t}\, dt \right| < \infty$$

where

$$|\varphi^{(k_0)}(x+t)| \geq C_0, \quad |t| \leq \delta_0, \quad k_0 \geq 2$$

Newton diagram of φ at x_0



Newton diagram of φ at x

- Again (V.C) + I.B.P $\Rightarrow$

$$\left| \int_{n^{-\frac{1}{k_0}} \le |t| \le \delta_0} e^{i(n\varphi(x+t) + Nt)} \frac{1}{t}\, dt \right| \le C_0$$

- $$\varphi(x+t) = \sum_{l=0}^{k_0-1} \frac{1}{l!} \varphi^{(l)}(x)\, t^l + R_{x,k_0}(t)$$

$$= P_{x,k_0}(t) + O(|t|^{k_0})$$

Again compare

$$\int_{|t|\le n^{-\frac{1}{k_0}}} e^{i(n\varphi(x+t)+Nt)}\,\frac{1}{t}\,dt - \int_{|t|\le n^{-\frac{1}{k_0}}} e^{i(nP_{x,k_0}(t)+Nt)}\,\frac{dt}{t} = O(1)$$

Left w/

$$\int_{|t|\le n^{-\frac{1}{k_0}}} e^{i(Q(t)+Nt)}\,\frac{1}{t}\,dt$$

Q has bdd degree but we have No control on the coefficients!

Extend L^2 (L^p) theory of

$$Hf(x) = \int_{\mathbb{R}} f(x-t)\,\frac{1}{t}\,dt$$

to

$$H_Q f(x) = \int_{\mathbb{R}} f(x-t)\,\frac{e^{iQ(t)}}{t}\,dt \qquad ?$$

NB

$$\widehat{H_Q f}(\xi) = \int_{\mathbb{R}} e^{i[Q(t)+\xi t]}\,\frac{1}{t}\,dt \cdot \hat{f}(\xi)$$

Yes! Stein-Wainger, Stein, Ricci-Stein

$\underline{K}$ C-Z Kernel, Q poly.

$$T_Q f(x) := \int_{\mathbb{R}^n} f(y)\, e^{iQ(x,y)}\, K(x,y)\, dy$$

$$\sup_{Q \in \mathcal{P}_d} \| T_Q \|_{L^p \to L^p} < \infty, \quad 1 < p < \infty$$

$$\Phi : \mathbb{T} \to \mathbb{T}^d$$

$$\Phi(e^{it}) = \left(e^{i(\varphi_1(t) + k_1 t)}, \ldots, e^{i(\varphi_d(t) + k_d t)} \right)$$

$$\boxed{\begin{array}{ccc} A(\mathbb{T}^d) & \to & U(\mathbb{T}) \\ f & \to & f \circ \Phi \end{array} \quad (\Phi)}$$

Prop (Φ) holds for any real-analytic map

$$\sup_{n_1,\dots,n_d,N,x} \left| \int_{|t|\le 1} e^{i(n_1\varphi_1(x+t)+\dots+n_d\varphi_d(x+t)+Nt)} \frac{dt}{t} \right|$$

$$\sum_{j=1}^{d} n_j \varphi_j(t) = |\vec{n}|\, \varphi(t, \vec{\omega}) \qquad \vec{\omega} \in S^{d-1}$$

$$\varphi(t,\omega) := \vec{\omega}\cdot\vec{\varphi}(t) \qquad \vec{\varphi} = (\varphi_1, \dots, \varphi_d)$$

$$\Phi : \mathbb{T}^k \to \mathbb{T}^d, \quad k \geq 2$$

$$A(\mathbb{T}^d) \qquad U(\mathbb{T}^k) \qquad (\Phi)$$

$$f \quad \to \quad f \circ \Phi$$

$$U(\mathbb{T}^k) = \ ??$$

when $k \geq 2$

$K=2$

$$\sum_{|m|,|n|\le N} \hat{f}(m,n)\, e^{imx+ny} \quad \text{Square} \rightsquigarrow U_{sq}$$

$$\sum_{|m|\le M,\, |n|\le N} \hat{f}(m,n)\, e^{imx+ny} \quad \text{Rectangular} \rightsquigarrow U_{rect}$$

$$\sum_{m^2+n^2\le R} \hat{f}(m,n)\, e^{imx+ny} \quad \text{spherical} \rightsquigarrow U_{sph}$$

$$(\Phi)_{sq} \qquad (\Phi)_{rect} \qquad \cancel{(\Phi)}_{sph}$$

$K=2,\ d=1$ $\quad \Phi : \mathbb{T}^2 \to \mathbb{T}\ ; \quad \Phi = \Phi_{\varphi,(k,\ell)}$

$$\boxed{\Phi(e^{is}, e^{it}) = e^{i(\varphi(s,t) + ks + \ell t)}}$$

Thm

ⓐ For any r.a. periodic φ, $\exists$ ∞-many (k,ℓ) s.t.

$$\left(\Phi_{\varphi,(k,\ell)}\right)_{sq} \quad \text{holds}$$

ⓑ $\exists$ r.a. $\Phi = \Phi_{\varphi,(k,\ell)}$ s.t.

$$(\Phi)_{sq} \quad \text{fails}$$

C. Fefferman $\quad \exists\, f \in C(\mathbb{T}^2)$ s.t. $\overline{\lim_{M,N\to\infty}} |S_{M,N} f(x,y)| = \infty$
$\forall (x,y) \in \mathbb{T}^2$

Heart $\quad h_n(x,y) = e^{inxy} \quad : \quad \forall (x,y)$

$$\sup_{M,N} \left| S_{M,N} h_n(x,y) \right|$$

$$= \sup_{M,N} \left| \sum_{\pm\,\pm} \iint_{|s|,|t| \le 1} e^{in(x+s)(y+t)} \frac{e^{\pm iMs}\, e^{\pm iNt}}{s\ t}\, ds\, dt \right|$$

$$\gtrsim \log n$$

General $\varphi(s,t)$:

$$S_{M,N}\, e^{in\varphi}(x,y) = \iint_{\mathbb{T}^2} e^{in\varphi(x+s,\,y+t)}\, D_M(s)\, D_N(t)\, ds\, dt$$

$$\varphi(x+s,\,y+t) = \sum_{k,l} \frac{1}{k!}\frac{1}{l!}\,\partial^{k,l}\varphi(x,y)\, s^k t^l$$

$$\boxed{st \text{ arises if } \frac{\partial^2\varphi}{\partial x\partial y}(x,y) \neq 0}$$

$$\frac{\partial^2\varphi}{\partial x\partial y} \equiv 0 \implies \varphi(s,t) = f(s) + g(t) \quad ✓$$

$a, c \neq 0 \Rightarrow$

$$\sup_n \left| \iint_{|s|, |t| \leq 1} e^{in[as^2 + bst + ct^2]} \frac{1}{s} \frac{1}{t} \right| < \infty$$

$\frac{\partial^2 \varphi}{\partial x \partial y}(x,y) \neq 0 \Rightarrow$ Both $\frac{\partial^2 \varphi}{\partial s^2}(x,y)$, $\frac{\partial^2 \varphi}{\partial t^2}(x,y) \neq 0$?

Example

$\varphi(s,t) = f(ks + \ell t) \quad k, \ell \in \mathbb{Z}$

$$Z_{ss} \cup Z_{tt} \subseteq Z_{st}$$

Factorisation Hypothesis (FH)

$\varphi_{ss},\ \varphi_{tt} \mid \varphi_{st}$ as germs of r.a. functions

Locally $\dfrac{\partial^2\varphi}{\partial s\partial t} = K\,\dfrac{\partial^2\varphi}{\partial s^2}$, $\dfrac{\partial^2\varphi}{\partial s\partial t} = L\,\dfrac{\partial^2\varphi}{\partial t^2}$

- $\varphi(s,t) = f(s) + g(t)$ satisfies (FH)
- $\varphi(s,t) = f(ks + \ell t)$ satisfies (FH)
- If φ satisfies (FH), then for any r.a. $\mathbb{K}$, $\exists$ $\varepsilon > 0$ s.t.

$$\psi = \varphi + \varepsilon \mathbb{K} (\varphi_{ss})^3 (\varphi_{tt})^3$$

satisfies (FH)

$$\Phi : \mathbb{T}^2 \to \mathbb{T} \quad ; \quad \Phi = \Phi_{\varphi,(k,\ell)}$$

$$\Phi(e^{is}, e^{it}) = e^{i(\varphi(s,t) + ks + \ell t)}$$

<u>Thm</u> $(\Phi)_{rect}$ holds iff φ satisfies (FH)

$\Phi : \mathbb{T}^2 \to \mathbb{T}^d \qquad \Phi = \Phi_{\vec{\varphi}, \vec{K}}$

$\vec{\varphi}(s,t) = (\varphi_1(s,t), \dots, \varphi_d(s,t))$, $\vec{K} = (K_1, \dots, k_d) \in (\mathbb{Z}^2)^d$

$$\boxed{\begin{array}{l}\varphi(s,t,\vec{w}) := \vec{\varphi}(s,t)\cdot\vec{w} \text{ satisfies (FH) on } S^{d-1} \text{ if} \\ \qquad \varphi_{ss}, \varphi_{tt} \mid \varphi_{st} \text{ as germs of r.a. on } \mathbb{T}^2 \times S^{d-1}\end{array}}$$

<u>Thm</u> $(\Phi)_{rect}$ holds iff φ satisfies (FH) on S^{d-1}

$$\iint_{|s|,|t|\le 1} e^{i[n_1\varphi_1(x+s,y+t)+\dots+n_d\varphi_d(x+s,y+t)+Ms+Nt]}\,\frac{1}{s}\,\frac{1}{t}\,ds\,dt$$

Multi-parameter Singular Radon Transforms

E. Stein & B. Street