Discrete Analogues in Harmonic Analysis

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Are the squares $E_{sq} = \{n^2\}$ a SUC

(set of uniform convergence) ?

- $E \subseteq \mathbb{Z}$ is a SUC iff $L_N(E) = O(1)$

$\Downarrow$

$$S(P) := \sum_{n \in \mathbb{Z}}' \frac{e^{iP(n)}}{n} \quad ; \quad P \in \mathbb{R}[X]$$

$$\sup_{P \in \mathcal{P}_d} | S(P) | < \infty$$

Arkhipov, Oskolkov / Stein, Wainger

$$T f(\underline{x}) = \int_{\mathbb{R}} f(x_1 - t, x_2 - t^2, \ldots, x_d - t^d) \frac{1}{t}\, dt$$

$$\rightsquigarrow$$

$$T_D f(\underline{m}) = \sum_{n \in \mathbb{Z}}{}' f(m_1 - n, m_2 - n^2, \ldots, m_d - n^d) \frac{1}{n}$$

$$\Updownarrow$$

$$m(\underline{\theta}) = \sum_{n \in \mathbb{Z}} \frac{e^{i(\theta_1 n + \theta_2 n^2 + \cdots + \theta_d n^d)}}{n} \in L^\infty(\mathbb{T}^d)$$

$$\left(P(n) = \theta_1 n + \cdots + \theta_d n^d \in \mathbb{R}[X] \right)$$

$$HF(x) = \int_{\mathbb{R}} F(x-t)\frac{1}{t}\,dt$$

$$\Updownarrow$$

$$H_D f(m) = \sum_{n\in\mathbb{Z}}{}' f(m-n)\frac{1}{n}$$

$$\left(F(x) := f(n), \quad n \le x < n+1 \right)$$

$$Tf(x) = \int_{\mathbb{R}'} f(x-s^3)\frac{1}{s}\,ds = \frac{1}{3}\int_{\mathbb{R}'} f(x-t)\frac{1}{t}\,dt$$

$$T_D f(m) = \sum_{n\in\mathbb{Z}}{}' f(m-n^3)\frac{1}{n} \;\not\Leftrightarrow\; H_D f(m)$$

$$H \longleftrightarrow m(\xi) = \int_{\mathbb{R}} e^{i\xi t} \frac{1}{t}\, dt = i\, \mathrm{sgn}(\xi)$$

$$H_D \longleftrightarrow m(\theta) = \sum_{n \in \mathbb{Z}} \frac{e^{i\theta n}}{n} = i\left(\theta - \lfloor \theta \rfloor - \tfrac{1}{2}\right)$$

$$T \longleftrightarrow m(\xi) = \int_{\mathbb{R}} e^{i\xi s^3} \frac{1}{s}\, ds = \frac{1}{3} \int_{\mathbb{R}} e^{i\xi t} \frac{1}{t}\, dt$$

$$T_D \longleftrightarrow m(\theta) = \sum_{n \in \mathbb{Z}} \frac{e^{in^3\theta}}{n} \quad ??$$

- $$\sum_{a \le n < b} e^{in\theta} = \frac{e^{ib\theta} - e^{ia\theta}}{e^{i\theta} - 1}$$ geometric series

- $$\sum_{a \le n < b} e^{in^2\theta} \sim \frac{b-a}{q} \sum_{r=1}^{q} e^{ir^2 p/q} = \frac{b-a}{\sqrt{q}} \left(\frac{p}{q}\right)$$

$\theta = p/q$ Jacobi symbol

Birkhoff / Bourgain

Thm (Birkhoff, 1931) (X, T, μ) dynamical system

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^n x) = \int_X f \, d\mu \quad \mu \text{ a.e. } x \in X$$

holds for every $f \in L^1(X, \mu)$.

Thm (Bourgain, 1988) (X, T, μ) dynamical system

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{n^2} x) = \int_X f \, d\mu \quad \mu \text{ a.e. } x \in X$$

holds for $f \in L^p(X, \mu)$, $p > 1$

Maximal functions

$$\mathcal{M} f(x) = \sup_{N>0} \left| \frac{1}{N} \sum_{n=1}^{N} f(T^n x) \right|$$

$$\mathcal{M}_{sq} f(x) = \sup_{N>0} \left| \frac{1}{N} \sum_{n=1}^{N} f(T^{n^2} x) \right|$$

transference

$$Mf(m) = \sup_{N>0} \left| \frac{1}{N} \sum_{n=1}^{N} f(m+n) \right|$$

$$M_{sq} f(m) = \sup_{N>0} \left| \frac{1}{N} \sum_{n=1}^{N} f(m+n^2) \right|$$

$$M \longleftrightarrow \sup_{h>0} \left| \frac{1}{h} \int_0^h f(x+t)\, dt \right|$$

Hardy-Littlewood max'l function

$$\boxed{\sup_{h>0} \left| \frac{1}{h} \int_0^h f(x+s^2)\, ds \right| \le C \sup_{h>0} \frac{1}{h} \int_0^h |f(x+t)|\, dt}$$

$$M_{sq} \sim \sup_{h>0} \left| \frac{1}{h} \int_0^h f(x+t, y+t^2)\, dt \right|$$

inspiration for Bourgain

max'l funct. along a parabola

Hilbert trans.

$$\int_{\mathbb{R}} f(x+t, y+t^2)\, \frac{dt}{t}$$

Transference

$$f_0^*(x) = \sup_{N \le N_0} \left| \frac{1}{N} \sum_{n=1}^{N} f(T^{n^2} x) \right|$$

Goal: $\|M_{sq}\varphi\|_{\ell^p} \le C_p \|\varphi\|_{\ell^p}$

$$\Downarrow$$

$$\int_X f_0^*(x)^p \, d\mu(x) \le C_p^p \int_X |f(x)|^p \, d\mu(x)$$

Fix $x \in X$, f and $J \gg N_0^2$

Define

$$\varphi(j) = \begin{cases} f(T^j x), & 1 \le j \le J \\ 0, & \text{otherwise} \end{cases}$$

NB for $1 \le j \le J - N_0^2$,

$$\frac{1}{N}\sum_{n=1}^{N}\varphi(j+n^2) = \frac{1}{N}\sum_{n=1}^{N} f(T^{n^2}(T^j x)), \quad N \le N_0$$

$$\Downarrow$$

$$M_{sq}\varphi(j) \geq \sup_{N \le N_0}\left|\frac{1}{N}\sum_{n=1}^{N} f(T^{n^2}(T^j x))\right|$$

$$= f_0^*(T^j x), \qquad 1 \le j \le J - N_0^2$$

$$\sum_{j=1}^{J-N_0^2} M_{sq}\varphi(j)^p \le C_p^p \sum_{j=1}^{\infty}\varphi(j)^p$$

$$\Downarrow$$

$$\sum_{j=1}^{J-N_0^2} f_0^*(T^j x)^p \le C_p^p \sum_{j=1}^{J} f(T^j x)^p$$

BUT $\displaystyle\int_X g(Tx)\,d\mu(x) = \int_X g(x)\,d\mu(x) \Rightarrow$

$$(J - N_0^2) \int_{X} f_0^*(x)^p \, d\mu(x) \le C_p^p \, J \int_{X} f(x)^p \, d\mu(x)$$

$$\Big\Downarrow \; J \to \infty$$

$$\boxed{\int_{X} f_0^*(x)^p \, d\mu(x) \le C_p^p \int_{X} f(x)^p \, d\mu(x)}$$

Singular and Maximal Radon Transforms

- $\mathbb{P} = (P_1, \dots, P_d) \quad P_j \in \mathbb{R}[X_1, \dots, X_k]$ ($\mathbb{Z}$)
- $\mathbb{K}$ Calderón-Zygmund Kernel on $\mathbb{R}^k$

$$Tf(\underline{x}) = \int_{\mathbb{R}^k} f(\underline{x} - \mathbb{P}(\underline{t}))\, \mathbb{K}(\underline{t})\, d\underline{t}$$

$$Mf(\underline{x}) = \sup_{h>0} \left| \frac{1}{h^k} \int_{|\underline{t}| \le h} f(\underline{x} - \mathbb{P}(\underline{t}))\, d\underline{t} \right|$$

$$\Updownarrow$$

$$T_D f(\underline{m}) = \sum_{\underline{n} \in \mathbb{Z}^k} f(\underline{m} - \mathbb{P}(\underline{n}))\, \mathbb{K}(\underline{n})$$

$$M_D f(\underline{m}) = \sup_{N>0} \left| \frac{1}{N^k} \sum_{|\underline{n}| \le N} f(\underline{m} - \mathbb{P}(\underline{n})) \right|$$

Thm T, M, T_D, M_D bdd on $L^p(\ell^q)$, $1 < p < \infty$.

Stein, Wainger, Ionescu, Mirek, ...
Bourgain

Dunford, Zygmund (1950's)

- (X, μ) $\quad S, T : X \to X$ meas. preserving (ergodic)

$$\lim_{M,N\to\infty} \frac{1}{MN} \sum_{m=1}^{M}\sum_{n=0}^{N} f(S^m T^n x) = \int_X f \, d\mu$$

μ a.e. $x \in X$ holds $\forall f \in L^p(X,\mu)$
$p > 1$

$$M_{strong} f(j,k) = \sup_{M,N} \left| \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} f(j+m, k+n) \right|$$

$$\Big\downarrow \text{transference}$$

$$\mathcal{M}^* f(x) = \sup_{M,N} \left| \frac{1}{MN} \sum_{m=1}^{M} \sum_{n=1}^{N} f(S^m T^n x) \right|$$

Question Given $P, Q \in \mathbb{Z}[X,Y]$, do we have pointwise ergodic results for

$$\frac{1}{M} \frac{1}{N} \sum_{m=1}^{M} \sum_{n=1}^{N} f(S^{Q(m,n)} \circ T^{P(m,n)} x) \, ?$$