L_∞-formality question for the universal algebra of semisimple Lie algebras

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joint work with Martin Bordemann, Olivier Elchinger and Abdenacer Makhlouf,

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Formal deformations of associative algebras (Gerstenhaber)

A formal deformation $\mu + C$ of an associative algebra $(\mathcal{A}, \mu)$ is defined by a series $C = \sum_{r\geq 1} t^r C_r$ of bilinear maps $C_r : \mathcal{A} \times \mathcal{A}$ to $\mathcal{A}$ so that

$$(\mu+C)\big((\mu+C)(u,v),w\big)-(\mu+C)\big(u,(\mu+C)(v,w)\big) = 0 \qquad \forall u,v,w \in \mathcal{A}.$$

At order 1 : $\mu(C_1(u,v),w) + C_1(\mu(u,v),w) - C_1(u,\mu(v,w)) - \mu(u,C_1(v,w)) = 0$, hence C_1 is a 2-cocycle for the Hochschild cohomology of $\mathcal{A}$ with values in $\mathcal{A}$.

Two formal deformations $(\mu + C)$ and $(\mu + C')$ are equivalent if there exists of a series $T = \sum_{r\geq 1} t^r T_r$ of linear maps $T_r : \mathcal{A} \to \mathcal{A}$ such that

$$(\mu + C')(u,v) = e^T \left((\mu + C)(e^{-T}u, e^{-T}v)\right).$$

At order 1 : $C_1'(u,v) = C_1(u,v) + T_1(u,v) - \mu(T_1u,v) - \mu(u,T_1v)$, i.e. $C_1' - C_1$ is a Hochschild coboundary.

If $H^2_H(\mathcal{A},\mathcal{A}) = 0$, all formal deformations are trivial (i.e. equivalent to μ) and any deformation at order 1 can be prolongated into a deformation.

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Hochschild complex of an associative algebra

Let $(\mathcal{A}, \mu)$ be an associative algebra over $\mathbb{K}$ of characteristic 0.
The *Hochschild complex* of $\mathcal{A}$ with values in the bimodule $\mathcal{A}$ is $C_H(\mathcal{A}, \mathcal{A}) := \bigoplus_{n\in\mathbb{N}} C^n_H(\mathcal{A}, \mathcal{A})$, with grading by number of arguments.

The *Gerstenhaber multiplication* $\circ_G : C_H \times C_H \to C_H$ is the bilinear map of degree -1 defined for any $f \in C^k_H(\mathcal{A}, \mathcal{A})$ and any $g \in C^l_H(\mathcal{A}, \mathcal{A})$ by

$$(f \circ_G g)(a_1, \ldots, a_{k+l-1}) = \sum_{i=1}^{k} (-1)^{(i-1)(l-1)} f(a_1, \ldots, a_{i-1}, g(a_i, \ldots, a_{i+l-1}), a_{i+l}, \ldots, a_{k+l-1}).$$

One considers, on the shifted space $\mathfrak{G}(\mathcal{A}) := C_H(\mathcal{A}, \mathcal{A})[1]$ for which $k-1$ is the shifted degree of a k-cochain f, the graded commutator,

$$[f, g]_G = f \circ_G g - (-1)^{(k-1)(l-1)} g \circ_G f,$$

called the *Gerstenhaber bracket*.

Any bilinear map $\mu : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is of degree 1 in $\mathfrak{G}(\mathcal{A})$, and gives an associative multiplication iff $[\mu, \mu]_G = 0$. For any such μ the square of $b := [\mu, \ ]_G$ vanishes and defines, up to a global sign, the Hochschild coboundary operator on the complex $C_H(\mathcal{A}, \mathcal{A})[1]$.

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Differential graded Lie algebras

A *differential graded Lie algebra* $(\mathfrak{G}, b, [\ ,\ ])$, consists of a graded Lie algebra $(\mathfrak{G}, [\ ,\ ])$ and a $\mathbb{K}$-linear map $b : \mathfrak{G} \to \mathfrak{G}$ of degree 1 such that $b^2 = 0$, and b is a graded derivation of the graded Lie bracket $[\ ,\ ]$.
Ex: For $(\mathcal{A}, \mu)$ an associative algebra $\left(C_H(\mathcal{A},\mathcal{A})[1], b = [\mu,\ ]_G, [\ ,\ ]_G\right)$.

Its cohomology $\mathfrak{H}$ with respect to b, $\mathfrak{H}^n := \dfrac{Z^n\mathfrak{G} ;= \{C \in \mathfrak{G}^n \mid bC = 0\}}{B^n\mathfrak{G} := \{bC \mid C \in \mathfrak{G}^{n-1}\}}$ carries a canonical graded Lie bracket $[\ ,\ ]_H$ induced from $[\ ,\ ]$ so that $(\mathfrak{H}, 0, [\ ,\ ]_H)$ is again a graded Lie algebra.

A deformation $\mu + C$ of the associative algebra $(\mathcal{A}, \mu)$ yields an element $C \in C_H(\mathcal{A},\mathcal{A})[1]t[[t]]$ of degree 1 so that $[\mu + C, \mu + C]_G = 0$ i.e.

$bC + \frac{1}{2}[C, C]_G = 0$ hence $bC_1 = 0$ and $[C_1, C_1]_G = -2bC_2$ so $[[C_1], [C_1]]_H = 0$

Equivalence is given by the action of e^T with $T \in C_H(\mathcal{A},\mathcal{A})[1]t[[t]]$ of degree 0 via : $\mu + C' = (\exp[T,\ ]_G)(\mu + C)$. Then $C'_1 = C_1 - bT_1$. One defines the infinitesimal action $T \cdot C := -bT + [T, C]$.

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L_∞ algebras

Let $W = \oplus_{j\in\mathbb{Z}} W^j$ a $\mathbb{Z}$-graded vector space. Let $V = W[1]$ be the shifted graded vector space. The *graded symmetric bialgebra of V*, denoted $\mathcal{S}V$, is the quotient of the free algebra $\mathcal{T}V$ by the two-sided graded ideal generated by $x\otimes y - (-1)^{|x||y|} y\otimes x$ for any homog. elements x, y in V.

The graded cocommutative comultiplication Δ_{sh} is induced by the shuffle comultiplication $\Delta_{sh} : \mathcal{T}V \to \mathcal{T}V \otimes \mathcal{T}V$ which is the homomorphism of associative algebras so that $\Delta_{sh}(x) = 1\otimes x + x\otimes 1$ (with signs given by Koszul convention).

A *L_∞-structure on W* is defined to be a graded coderivation $\mathcal{D}$ of $\mathcal{S}(W[1])$ of degree 1 satisfying $\mathcal{D}^2 = 0$ and $\mathcal{D}(\mathbf{1}_{\mathcal{S}W[1]}) = 0$.

Such a $\mathcal{D}$ is determined by $D := pr_{W[1]} \circ \mathcal{D} : \mathcal{S}(W[1]) \to W[1]$ via $\mathcal{D} = \mu_{sh} \circ D \otimes \mathrm{Id} \circ \Delta_{sh}$ and we write $\mathcal{D} = \overline{D}$. The pair $(W, \mathcal{D})$ is called an *L_∞-algebra*.

Ex: $(\mathfrak{G}, b, [\ ,\ ])$ a dga $\Rightarrow \left(\mathfrak{G}, \mathcal{D} = \overline{b[1] + [\ ,\ ][1]} \text{ on } \mathcal{S}(\mathfrak{G}[1])\right)$.

A solution $bC + \frac{1}{2}[C, C]_G = 0$ corresponds to a $C' \in V^0 t[[t]]$ such that $\mathcal{D}(e^{C'}) = 0$.

For a linear map $\phi : V^{\otimes k} \to W^{\otimes \ell}$, $\phi[j] : V[j]^{\otimes k} \to W[j]^{\otimes \ell}$ via $\phi[j] := (s^{\otimes \ell})^{-j} \circ \phi \circ (s^{\otimes k})^j$ where $s : V \to V[-1]$ is the identity.

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L_∞–morphisms, quasi-isomorphisms and Formality

A *L_∞-morphism* from a L_∞-algebra $(W, \mathcal{D})$ to a L_∞-algebra $(W', \mathcal{D}')$ is a morphism of graded con. coalgebras $\Phi : \big(\mathcal{S}(W[1]), \mathcal{D}\big) \to \big(\mathcal{S}(W'[1]), \mathcal{D}'\big)$, intertwining differentials $\Phi \circ \mathcal{D} = \mathcal{D}' \circ \Phi$.

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A L_∞-map Φ is called an *L_∞-quasi-isomorphism* if its first component $\Phi_1 = \Phi|_{W[1]} = \varphi_1 : W[1] \to W'[1]$ –which is a chain map $(W[1], \mathcal{D}_1) \to (W'[1], \mathcal{D}'_1)$– induces an isomorphism in cohomology.

A formality for a differential graded Lie algebra $(\mathfrak{G}, b, [\ ,\ ])$ is a L_∞-quasi-isomorphism from the L_∞-algebra corresponding to $(\mathfrak{H}, 0, [\ ,\ ]_H)$ (the cohomology of $\mathfrak{G}$ with respect to b), to the L_∞-algebra corresponding to $(\mathfrak{G}, b, [\ ,\ ])$: $\Phi : \mathcal{S}(\mathfrak{H}[1]) \to \mathcal{S}(\mathfrak{G}[1])$, such that $\Phi \circ \overline{[\ ,\ ]_H[1]} = \overline{(b[1] + [\ ,\ ][1])} \circ \Phi$

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Low orders terms of a formality

We look for $\varphi : \mathcal{S}(\mathfrak{H}[1]) \to \mathfrak{G}[1]$ a degree 0 map vanishing on 1, such that $\Phi = e^{*\varphi} : \mathcal{S}(\mathfrak{H}[1]) \to \mathcal{S}(\mathfrak{G}[1])$ satisfies $\Phi \circ \overline{[\ ,\ ]_H[1]} = \overline{(b[1] + [\ ,\ ][1])} \circ \Phi$.
Denoting φ_n the restriction of φ to $Sym^n(\mathfrak{H}[1])$, we have in particular that $b[1] \circ \varphi_1 = 0$ and $\varphi_1 : (\mathfrak{H}[1], 0)] \to (\mathfrak{G}[1], b[1])$ must induce an isomorphism in cohomology.

The cohomology of $(\mathfrak{G}[1], b[1])$ identifies with $\mathfrak{H}[1]$.
We denote by $\pi : Z\mathfrak{G} = \{C \in \mathfrak{G} \mid bC = 0\} \to \mathfrak{H}$ the canonical projection.
We must have $b[1] \circ \varphi_1 = 0$ and $\pi[1] \circ \varphi_1 = \mathrm{Id}$.
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ΛV is the quotient of $\mathcal{T}V$ by the two-sided graded ideal gen. by $x \otimes y + (-1)^{|x||y|} y \otimes x$.

Low orders terms of a formality

We look for $\varphi : \mathcal{S}(\mathfrak{H}[1]) \to \mathfrak{G}[1]$ a degree 0 map vanishing on 1, such that $\Phi = e^{*\varphi} : \mathcal{S}(\mathfrak{H}[1]) \to \mathcal{S}(\mathfrak{G}[1])$ satisfies $\Phi \circ \overline{[\ ,\ ]_H[1]} = \overline{(b[1] + [\ ,\ ][1])} \circ \Phi$. Denoting φ_n the restriction of φ to $Sym^n(\mathfrak{H}[1])$, we have in particular that $b[1] \circ \varphi_1 = 0$ and $\varphi_1 : (\mathfrak{H}[1], 0)] \to (\mathfrak{G}[1], b[1])$ must induce an isomorphism in cohomology.

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A characteristic 3-class

Let $\varphi_1 : \mathfrak{H}[1] \to \mathfrak{G}[1]$ and $\varphi_2 : \mathcal{S}^2(\mathfrak{H}[1]) \to \mathfrak{G}[1]$ be degree 0 maps, such that their shifts $\phi_i = \varphi_i[-1] : \Lambda^i\mathfrak{H} \to \mathfrak{G}$ satisfy $0 = b \circ \phi_1, \quad \pi \circ \phi_1 = \mathrm{Id}_{\mathfrak{H}}$ and $0 = b \circ \phi_2 + [\ ,\ ]_G \circ (\phi_1 \otimes \phi_1) - \phi_1 \circ [\ ,\ ]_H$.

1. The linear map $w_3(\varphi) : \Lambda^3\mathfrak{H} \to \mathfrak{G}$ of degree -1 defined on homogeneous elements $y_1, y_2, y_3 \in \mathfrak{H}$ by $w_3(\varphi)(y_1, y_2, y_3) = (-1)^{|y_1|}[\phi_1(y_1), \phi_2(y_2, y_3)]_G - (-1)^{|y_2|}(-1)^{|y_2||y_1|}[\phi_1(y_2), \phi_2(y_1, y_3)]_G +$ $(-1)^{|y_3|}(-1)^{|y_3|(|y_1|+|y_2|)}[\phi_1(y_3), \phi_2(y_1, y_2)]_G - \phi_2([y_1, y_2]_H, y_3) +$ $(-1)^{|y_3||y_2|}\phi_2([y_1, y_3]_H, y_2)(-1)^{(|y_2|+|y_3|)|y_1|}\phi_2([y_2, y_3]_H, y_1)$ satisfies $b \circ w_3(\varphi) = 0$.
2. The trilinear map $z_3(\varphi) = \pi \circ w_3(\varphi) : \Lambda^3\mathfrak{H} \to \mathfrak{H}$ is a graded Chevalley-Eilenberg 3-cocycle of degree -1, i.e. $\delta_{\mathfrak{H}} z_3 = 0$, and its class $c_3 = c_3(\mathfrak{G}, b, [\ ,\ ]_G)$ does not depend on the chosen φ_1, φ_2.

   The Chevalley Eilenberg coboundary operator on a graded Lie algebra is given by the usual formulas with signs.
3. There is a L_∞-quis of order 3 between $\mathfrak{G}$ and its cohom. $\mathfrak{H}$ iff $c_3 = 0$.

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What is formality for the Hochschild complexe of $\mathcal{U}\mathfrak{g}$?

We are interested in checking whether there is formality for the graded Lie algebra $\big(C_H(\mathcal{A},\mathcal{A})[1], b=[\mu,\ ]_G, [\ ,\ ]_G\big)$ given by the Hochschild complex of the associative algebra $(\mathcal{A},\mu)$ when $\mathcal{A}=\mathcal{U}\mathfrak{g}$ is the universal enveloping algebra of a Lie algebra $\mathfrak{g}$, thus in checking whether one can build a quasi-isomophism

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Theorem[Cartan-Eilenberg] . Let $\mathfrak{g}$ be a finite dim Lie algebra and $\mathcal{M}$ be a $\mathcal{U}\mathfrak{g}$-bimodule. Then $H^n_H(\mathcal{U}\mathfrak{g},\mathcal{M}) \simeq H^n_{CE}(\mathfrak{g},\mathcal{M}_a)$ where $\mathcal{M}_a=\mathcal{M}$ with the action of $g\in\mathfrak{g}$ defined by $g\cdot m := gm - mg$. In particular

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The Chevalley-Eilenberg complex $(C_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g}), \delta_{\mathfrak{g}})$

$\mathcal{S}\mathfrak{g}$ is a $\mathfrak{g}$-module via the adjoint representation. $C_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})$ is canonically isomorphic to $\mathcal{S}\mathfrak{g} \otimes \Lambda\mathfrak{g}^*$ and is $\mathbb{Z}$-graded by the form degree of $\Lambda\mathfrak{g}^*$.

It is a graded commutative algebra by means of the tensor product of the commutative multiplication in $\mathcal{S}\mathfrak{g}$ and the usual exterior multiplication in $\Lambda\mathfrak{g}^*$ which we also denote by $\wedge$.

$f \in \mathcal{S}\mathfrak{g}$ is viewed as a polynomial function on the dual space $\mathfrak{g}^*$ so $C_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})$ is viewed as the space of all polynomial poly-vector-fields on $\mathfrak{g}^*$.

It is equipped with the usual *Schouten bracket* $[\ ,\ ]_s$:

$$[F, G]_s = \sum_{i=1}^n \iota_{e_i}(F) \wedge \partial^i G - (-1)^{(|F|-1)(|G|-1)} \sum_{i=1}^n \iota_{e_i}(G) \wedge \partial^i F.$$

where $e_1, \ldots, e_n$ is a basis of $\mathfrak{g}$, $\epsilon^1, \ldots, \epsilon^n$ the dual basis; $\iota_\xi : \Lambda\mathfrak{g}^* \to \Lambda\mathfrak{g}^*$, for each $\xi \in \mathfrak{g}$, is the usual interior product graded derivation and for each $y \in \mathfrak{g}^*$, $\iota_y : \mathcal{S}\mathfrak{g} \to \mathcal{S}\mathfrak{g}$ the corresponding derivation, writing $\iota_{\epsilon^i}(f) = \partial^i f$ for each $f \in \mathcal{S}\mathfrak{g}$ and extending these derivations to $\mathcal{S}\mathfrak{g} \otimes \Lambda\mathfrak{g}^*$.

Let $\pi = [\ ,\ ] = \frac{1}{2} \sum_{i,j,k} c^i_{jk} e_i \otimes (\epsilon^j \wedge \epsilon^k)$ be the *linear Poisson structure* of $\mathfrak{g}^*$, where $c^i_{jk} = \epsilon^i([e_j, e_k]) \in \mathbb{K}$ are the structure constants. Then $[\pi, \pi]_s = 0$ and $\delta_{\mathfrak{g}} = [\pi,\ ]_s$ is the (shifted) Chevalley Eilenberg coboundary operator.

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Kontsevich formality

Kontsevich gives the construction of a quasi-isomorphism

$$e^{*\varphi} : \mathcal{S}\big(C_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})[2]\big) \to \mathcal{S}\big(C_H(\mathcal{S}\mathfrak{g}, \mathcal{S}\mathfrak{g})[2]\big)$$

from the L_∞-algebra $\mathcal{S}\big(C_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})[2]\big)$ associated to the graded Lie algebra $\big(C_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})[1], 0, [\ ,\ ]_s\big)$ of all polynomial poly-vector-fields on the vector space $\mathfrak{g}^*$, equipped with zero differential and the usual Schouten bracket $[\ ,\ ]_S$, to the L_∞-algebra $\mathcal{S}\big(C_H(\mathcal{S}\mathfrak{g}, \mathcal{S}\mathfrak{g})[2]\big)$ associated to the graded Lie algebra $(C_H(\mathcal{S}\mathfrak{g}, \mathcal{S}\mathfrak{g})[1], b, [\ ,\ ]_G)$ of all poly-differential operators on $\mathfrak{g}^*$ with polynomial coefficients, equipped with the Hochschild differential b and the Gerstenhaber bracket $[\ ,\ ]_G$.

Quasi isomorphism between the Hochschild complex of $\mathcal{U}\mathfrak{g}$ and the Chevalley-Eilenberg complex of $\mathfrak{g}$ with values in $\mathcal{S}\mathfrak{g}$

Theorem (Kontsevich, also Bordemann and Makhlouf):

Let $(\mathfrak{g}, [\ ,\ ])$ be a finite-dimensional Lie-algebra.
There is a L_∞-quasi-isomorphism between the differential graded Lie algebra $\left(C_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})[1], \delta_\mathfrak{g}, [\ ,\ ]_s\right)$ and the differential graded Lie algebra $\left(C_H(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})[1], b, [\ ,\ ]_G\right)$.
In particular, this induces an isomorphism of the graded Lie algebras of their cohomologies (with respect to $\delta_\mathfrak{g}$ and b, respectively).
Hence, the L_∞-formality of $\left(C_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})[1], \delta_\mathfrak{g}, [\ ,\ ]_s\right)$ is equivalent to the L_∞-formality of $\left(C_H(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})[1], b, [\ ,\ ]_G\right)$.

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Abelian Lie algebras

In case the Lie algebra $\mathfrak{g}$ is abelian, $\mathcal{U}\mathfrak{g} = \mathcal{S}\mathfrak{g}$, the Chevalley-Eilenberg differential is zero, whence

$$H_H(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g}) \cong H_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g}) \cong C_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})$$

and formality of $\big(C_H(\mathcal{U}\mathfrak{g}, \mathcal{U}\mathfrak{g})[1], b, [\ ,\ ]_G\big) \cong \big(C_H(\mathcal{S}\mathfrak{g}, \mathcal{S}\mathfrak{g})[1], b, [\ ,\ ]_G\big)$ is the content of the Kontsevich formality theorem where one builds a quasi-isomorphism

$$e^{*\varphi} : \mathcal{S}\big(C_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})[2]\big) \to \mathcal{S}\big(C_H(\mathcal{S}\mathfrak{g}, \mathcal{S}\mathfrak{g})[2]\big).$$

Thus the Hochschild complex of the universal enveloping algebra of an abelian algebra is formal.

Cartan 3-regular quadratic Lie algebras

A triple $(\mathfrak{g}, [\ ,\ ], \kappa)$ is called a *quadratic Lie algebra* if the symmetric bilinear form $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{K}$ is *invariant and nondegenerate*. (A symmetric bilinear form is invariant if for all $\xi, \xi', \xi'' \in \mathfrak{g}$ we have $\kappa([\xi, \xi'], \xi'') = \kappa(\xi, [\xi', \xi''])$.)

The *Cartan 3-cocycle* $\Omega \in \Lambda^3 \mathfrak{g}^*$ is then defined by

$$\Omega(\xi, \xi', \xi'') = \kappa(\xi, [\xi', \xi''])$$

A quadratic Lie algebra $(\mathfrak{g}, [\ ,\ ], \kappa)$ is called a *Cartan-3-regular* if the cohomology class of the Cartan cocycle Ω, $[\Omega]$, is nonzero.

The Casimir is the element $q \in \mathcal{S}^2\mathfrak{g}$ which is the 'inverse' of κ ($q = \Sigma q^{ij} e_i \otimes e_j$, $\sum_j q^{ij} \kappa_{jr} = \delta^i_r$).

The space of polynomials in q, $\mathbb{K}[q]$, injects in the invariant polynomials $(\mathcal{S}\mathfrak{g})^{\mathfrak{g}} \cong H^0_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})$.

When $(\mathfrak{g}, [\ ,\ ], \kappa)$ is Cartan-3-regular,the map $\mathbb{K}[q] \to H^3_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g}) : \alpha \to [\alpha \wedge \Omega]$ is injective.

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A quadratic Lie algebra $(\mathfrak{g}, [\ ,\ ], \kappa)$ is called a *Cartan-3-regular* if the cohomology class of the Cartan cocycle Ω, $[\Omega]$, is nonzero.

The Casimir is the element $q \in \mathcal{S}^2\mathfrak{g}$ which is the 'inverse' of κ ($q = \Sigma q^{ij} e_i \otimes e_j$, $\sum_j q^{ij}\kappa_{jr} = \delta^i_r$).
The space of polynomials in q, $\mathbb{K}[q]$, injects in the invariant polynomials $\left(\mathcal{S}\mathfrak{g}\right)^{\mathfrak{g}} \cong H^0_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})$.
When $(\mathfrak{g}, [\ ,\ ], \kappa)$ is Cartan-3-regular,the map $\mathbb{K}[q] \to H^3_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g}) : \alpha \to [\alpha \wedge \Omega]$ is injective.

Non formality of $\mathcal{U}\mathfrak{g}$ for a Cartan 3-regular quadratic $\mathfrak{g}$

In $C_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})$ we have also the linear Poisson structure π and the *Euler field* $E = \sum_{i=1}^{n} e_i \otimes \epsilon^i$.

The Schouten brackets are given, for $\alpha, \beta, \gamma \in \mathbb{K}[q]$ and α' denoting the derivative of the polynomial α, by:

$\delta_{\mathfrak{g}}(\alpha) = [\pi, \alpha]_s = 0, \quad \delta_{\mathfrak{g}}(\alpha \wedge \Omega) = [\pi, \alpha \wedge \Omega]_s = 0, \quad \delta_{\mathfrak{g}}(\alpha \wedge E) = [\pi, \alpha \wedge E]_s = \alpha \wedge \pi,$

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Reductive Lie algebras

$$\mathfrak{g} = \mathfrak{z} \oplus [\mathfrak{g}, \mathfrak{g}]$$

where $\mathfrak{z}$ is its centre and the derived ideal $\mathfrak{l} = [\mathfrak{g}, \mathfrak{g}]$ is semisimple.

Pick any nondegenerate symmetric bilinear form on $\mathfrak{z}$
and the *Killing form* $\kappa_{\mathfrak{l}} : (\xi, \xi') \mapsto \mathrm{trace}(\mathrm{ad}_\xi \circ \mathrm{ad}_{\xi'})$ on $\mathfrak{l}$,
and let κ be the orthogonal sum of those two.

The Cartan 3-cocycle Ω w.r.t. $\mathfrak{g}$ is given by

$$\Omega(z_1 + l_1, z_2 + l_2, z_3 + l_3) = \Omega_{\mathfrak{l}}(l_1, l_2, l_3) = \kappa_{\mathfrak{l}}(l_1, [l_2, l_3])$$

where $\Omega_{\mathfrak{l}}$ is the Cartan 3-cocycle of $\mathfrak{l}$ which is well-known to be a nontrivial 3-cocycle.

Hence $(\mathfrak{g}, [\ ,\ ], \kappa)$ is Cartan-3-regular and so (the Hochschild complex of) its universal envelopping algebra is not L_∞-formal.

Semisimple Lie algebras

The above shows that (the Hochschild complex of) the universal envelopping algebra of a semisimple Lie algebra is not L_∞-formal.

Nonetheless, the deformation theory of $\mathcal{U}\mathfrak{g}$ is well known : $\mathcal{U}\mathfrak{g}$ is rigid because

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It is well-known that it is always possible to find a sequence of 'higher order brackets' $d_k : \mathcal{S}^k(H_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})[2]) \to H_{CE}(\mathfrak{g}, \mathcal{S}\mathfrak{g})[2]$ for $k \geqslant 3$ and a L_∞-quasi-isomorphism

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We do that for $\mathfrak{g} = \mathfrak{so}(3)$. We know that

$$\mathfrak{H} := H_{CE}(\mathfrak{so}(3), \mathcal{S}\mathfrak{so}(3))[1] = \mathbb{K}[q]\,\mathbf{1} \oplus \{0\} \oplus \{0\} \oplus \mathbb{K}[q][\Omega].$$

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The Lie algebra $\mathfrak{so}(3)$

Theorem

1. The Chevalley-Eilenberg complex $\mathfrak{G}$ of $\mathfrak{so}(3)$ is NOT formal.
2. $\mathfrak{G}_{\text{red}} := \mathbb{K}[q]\,\mathbf{1} \oplus \mathbb{K}[q]E \oplus \mathbb{K}[q]\pi \oplus \mathbb{K}[q]\Omega$ is a differential graded Lie subalgebra of $(\mathfrak{G} : H_{CE}(\mathfrak{so}(3), \mathcal{S}\mathfrak{so}(3))[1] =, \delta, [\ ,\ ]_s)$ and the injection $\mathfrak{G}_{\text{red}} \to \mathfrak{G}$ is a quasi-isomorphism of differential graded Lie algebras.
3. There is an L_∞ structure d on $\mathcal{S}(\mathfrak{H}[1])$ whose only nonvanishing Taylor coefficient is d_3 which is the shifted characteristic 3-class $d_3 = z_3[-1]$ and there is an L_∞-quasi-isomorphism $e^{*\varphi}$ from $(\mathcal{S}(\mathfrak{H}[1]), \overline{d_3})$ to $(\mathcal{S}(\mathfrak{G}_{\text{red}}[1]), \overline{\delta_{\mathfrak{g}}[1] + [\ ,\ ]_s[1]})$.
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The results follows from the L_∞ perturbation lemma. There is a (homotopy) contraction : the natural injection $i : \mathfrak{H} = \mathbb{K}[q]\,\mathbf{1} \oplus \{0\} \oplus \{0\} \oplus \mathbb{K}[q][\Omega] \to \mathfrak{G}_{\text{red}}$, the natural projection $p : \mathfrak{G}_{\text{red}} \to \mathfrak{H}$ with kernel $\mathbb{K}[q]E \oplus \mathbb{K}[q]\pi$, and the homotopy map h given by $h = h^1 : \mathbb{K}[q]\pi \to \mathbb{K}[q]E$, $h^1(\alpha \wedge \pi) = \alpha \wedge E$, for $\alpha \in \mathbb{K}[q]$, and is defined to vanish in degree $-1, 0, 2$. $(p \circ i = \mathrm{Id}_U,\ \mathrm{Id}_V - i \circ p = b_V \circ h + h \circ b_U,\ h^2 = 0,\ h \circ i = 0,\ p \circ h = 0)$.

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The results follows from the L_∞ perturbation lemma. There is a (homotopy) contraction : the natural injection $i : \mathfrak{H} = \mathbb{K}[q]\,\mathbf{1} \oplus \{0\} \oplus \{0\} \oplus \mathbb{K}[q][\Omega] \to \mathfrak{G}_{\text{red}}$, the natural projection $p : \mathfrak{G}_{\text{red}} \to \mathfrak{H}$ with kernel $\mathbb{K}[q]E \oplus \mathbb{K}[q]\pi$, and the homotopy map h given by $h = h^1 : \mathbb{K}[q]\pi \to \mathbb{K}[q]E$, $h^1(\alpha \wedge \pi) = \alpha \wedge E$, for $\alpha \in \mathbb{K}[q]$, and is defined to vanish in degree $-1, 0, 2$. $(p \circ i = \mathrm{Id}_U,\ \mathrm{Id}_V - i \circ p = b_V \circ h + h \circ b_U,\ h^2 = 0,\ h \circ i = 0,\ p \circ h = 0)$.

Theorem

1. The Chevalley-Eilenberg complex $\mathfrak{G}$ of $\mathfrak{so}(3)$ is NOT formal.
2. $\mathfrak{G}_{\text{red}} := \mathbb{K}[q]\,\mathbf{1} \oplus \mathbb{K}[q]E \oplus \mathbb{K}[q]\pi \oplus \mathbb{K}[q]\Omega$ is a differential graded Lie subalgebra of $(\mathfrak{G} : H_{CE}(\mathfrak{so}(3), \mathcal{S}\mathfrak{so}(3))[1] =, \delta, [\ ,\ ]_s)$ and the injection $\mathfrak{G}_{\text{red}} \to \mathfrak{G}$ is a quasi-isomorphism of differential graded Lie algebras.
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The Lie algebra $\mathfrak{so}(3)$

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Perturbation Lemma

Let (i, p, h) be a contraction between the complexes (U, b_U) and (V, b_V) (the differentials b_U and b_V have degree 1, $i : U \to V$ and $p : V \to U$ are chain maps, $h : V \to V$ has degree -1 and $p \circ i = \mathrm{Id}_U$, $\mathrm{Id}_V - i \circ p = b_V \circ h + h \circ b_U$, $h^2 = 0,\ h \circ i = 0,\ p \circ h = 0$) where U and V carry exhaustive and separated filtrations with V complete and such that the maps b_U, b_V, i, p and h are of filtration degree 0.

Moreover, let $\delta_V : V \to V$ be a perturbation of b_V, i.e. a morphism $\delta_V : V \to V$ of degree $+1$ such that $(b_V + \delta_V)^2 = 0$ and suppose that δ_V is of filtration degree -1.

Then the linear maps $(id_V + h \circ \delta_V)$ and $(id_V + \delta_V \circ h)$ from V to V are invertible, and we define

$$\tilde{\imath} = (id_V + h \circ \delta_V)^{-1} \circ i \qquad \tilde{h} = (id_V + h \circ \delta_V)^{-1} \circ h$$

$$\tilde{p} = p \circ (id_V + \delta_V \circ h)^{-1} \qquad \delta_U = p \circ (id_V + \delta_V \circ h)^{-1} \circ \delta_V \circ i.$$

Then δ_U is a perturbation of b_U of filtration degree -1, and ($\tilde{i}, \tilde{p}$ and $\tilde{h}$) define a new contraction between $(U, b_U + \delta_U)$ and $(V, b_V + \delta_V)$.

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L_∞-contraction

Let (i, p, h) a contraction between the complexes (U, b_U) and (V, b_V). The graded coderivations $\overline{b_U[1]}$ of $\mathcal{S}(U[1])$ and $\overline{b_V[1]}$ of $\mathcal{S}(V[1])$ are differentials. Setting $\varphi_1 := i[1]$ and $\psi_1 := p[1]$, the morphisms of graded coalgebras $e^{*\varphi_1} : \mathcal{S}(U[1]) \to \mathcal{S}(V[1])$ and $e^{*\psi_1} : \mathcal{S}(V[1]) \to \mathcal{S}(U[1])$ are chain maps satisfying $e^{*\psi_1} \circ e^{*\varphi_1} = id_{\mathcal{S}(U[1])}$.

Since $P = [h, b_V] : V \to V$ is an idempotent, let V_U be its kernel, and V_{acyc} its image; so $V = V_U \oplus V_{\mathrm{ac}}$, and $\mathcal{S}(V[1]) \cong \mathcal{S}(V_U[1]) \otimes \mathcal{S}(V_{\mathrm{ac}}[1])$ as graded bialgebras. Define $\beta : \mathcal{S}(V[1]) \to \mathcal{S}(V[1])$ of degree 0 by : for all $y_1, \ldots, y_k \in V_U[1]$ and $w_1, \ldots, w_l \in V_{\mathrm{acyc}}[1]$ where $k, l \in \mathbb{N}$:

$$\beta(y_1 \bullet \cdots \bullet y_k \bullet w_1 \bullet \cdots \bullet w_l) = \begin{cases} \frac{1}{l}(y_1 \bullet \cdots \bullet y_k \bullet w_1 \bullet \cdots \bullet w_l) & \text{if } l \neq 0, \\ 0 & \text{if } l = 0, \end{cases} \qquad (1)$$

and set $\eta = \overline{h[1]} \circ \beta = \beta \circ \overline{h[1]}$.

Then $(e^{*\varphi_1}, e^{*\psi_1}, \eta)$ is a contraction from $(\mathcal{S}(U[1]), \overline{b_U[1]})$ to $(\mathcal{S}(V[1]), \overline{b_V[1]})$.

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L_∞-perturbation Lemma (Bordemann Elchinger)

Let (i, p, h) be a contraction between the complexes (U, b_U) and (V, b_V). Let $(e^{*\varphi_1}, e^{*\psi_1}, \eta)$ be the corresponding contraction from $\left(\mathcal{S}(U[1]), \overline{b_U[1]}\right)$ to $\left(\mathcal{S}(V[1]), \overline{b_V[1]}\right)$.

Suppose $D = b_V[1] + D'_V$ with $D'_V = \sum_{k\geqslant 2} D'_k : \mathcal{S}(V[1]) \to V[1]$ of degree 1 defines an L_∞-structure, whence $\delta_{\mathcal{S}(V[1])} = \overline{D'_V}$ is a perturbation of $\overline{b_V[1]}$.

The maps $\widetilde{e^{*\varphi_1}}, \widetilde{e^{*\psi_1}}, \delta_{\mathcal{S}(U[1])}$, and $\widetilde{\eta}$ of the Perturbation Lemma so that $(\widetilde{e^{*\varphi_1}}, \widetilde{e^{*\psi_1}}, \widetilde{\eta})$ is homotopy contraction between $\left(\mathcal{S}(U[1]), \overline{b_U[1]} + \delta_{\mathcal{S}(U[1])}\right)$ and $\left(\mathcal{S}(V[1]), \overline{b_V[1]} + \overline{D'_V}\right)$ automatically preserve the structure of graded connected coalgebras, i.e. $\widetilde{e^{*\varphi_1}}$ and $\widetilde{e^{*\psi_1}}$ are morphism of graded differential connected coalgebras, and $\delta_{\mathcal{S}(U[1])}$ will be a graded coderivation of degree 1. This entails in particular that $\widetilde{e^{*\varphi_1}} =: e^{*\varphi}$ is a L_∞-quasi-isomorphism with quasi-inverse $\widetilde{e^{*\psi_1}} =: e^{*\psi}$.

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Thank you for your attention!