Higher brackets on cyclic and negative cyclic (co)homology

Niels Kowalzig

(joint work with D. Fiorenza)

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Higher structures arising from deformations

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for $a, b, c \in A$, as well as a bracket:

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- One can generalise $\{.,.\}$ to general elements in $C^n(A)$ and obtains a graded Lie bracket $C^p(A) \otimes C^q(A) \to C^{p+q-1}(A)$, the **Gerstenhaber bracket**.

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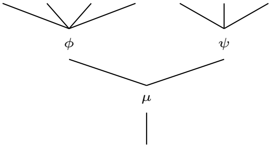

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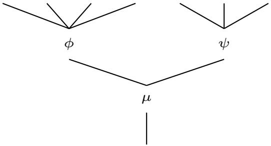

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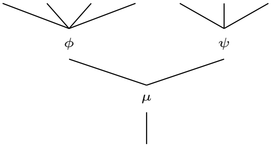

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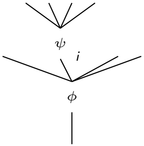

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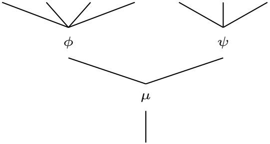

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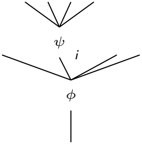

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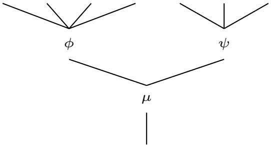

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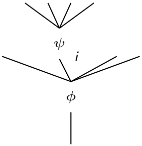

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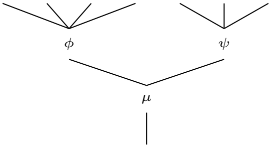

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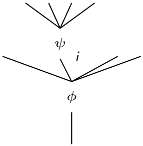
$\sum_{i=1}^{p} \pm$ 

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- Hence, on cohomology $\smile_0$ is graded commutative.

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- In algebra, the answer is in general: no. As you possibly noticed, the graded commutator $[.,.]_{\smile_1}$ is the **Gerstenhaber bracket** we talked about before $\{.,.\}$ for which there is in general no reason to vanish.

- However, in some situations, there is a homotopy formula

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- The appurtenant cohomology theory is **Gerstenhaber-Schack cohomology**. Recall (or define) $H(D(H), k) =: H_{GS}(H, H)$, and hence the question to answer is: what is the deg -2 bracket on Gerstenhaber-Schack cohomology?

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- Define the $k[[u, u^{-1}]]$-linear differential

$$d_u = \delta + uB$$

on $M^\bullet[[u, u^{-1}]]$, which somehow explains the term "perturbation".

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Remember that negative cyclic homology is the $k[u]$-dual to cyclic cohomology and the right receptacle for the Chern character $\mathrm{ch} : K_n \to HC^-_n$.

$$
\begin{array}{r|ccccccccc}
 & u^{-2} & & u^{-1} & & u^{0} & & u^{1} & & u^{2} \\
\hline
 & M^3 & \xrightarrow[uB]{} & M^2 & \xrightarrow[uB]{} & M^1 & \xrightarrow[uB]{} & M^0 & \xrightarrow[uB]{} & M^{-1} \\
 & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
CC^\bullet_{per}(M): & M^2 & \xrightarrow[uB]{} & M^1 & \xrightarrow[uB]{} & M^0 & \xrightarrow[uB]{} & M^{-1} & \xrightarrow[uB]{} & M^{-2} \\
 & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
 & M^1 & \xrightarrow[uB]{} & M^0 & \xrightarrow[uB]{} & M^{-1} & \xrightarrow[uB]{} & M^{-2} & \xrightarrow[uB]{} & M^{-3} \\
 & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
 & M^0 & \xrightarrow[uB]{} & M^{-1} & \xrightarrow[uB]{} & M^{-2} & \xrightarrow[uB]{} & M^{-3} & \xrightarrow[uB]{} & M^{-4}
\end{array}
$$

$$
\begin{array}{ccccc}
u^{-2} & u^{-1} & u^{0} & u^{1} & u^{2} \\
\hline
 & & M^{1} & \xrightarrow[uB]{} \; M^{0} & \xrightarrow[uB]{} \; M^{-1} \\
 & & \uparrow{\scriptstyle \delta} & \uparrow{\scriptstyle \delta} & \uparrow{\scriptstyle \delta} \\
CC^{\bullet}(M): & & M^{0} & \xrightarrow[uB]{} \; M^{-1} & \xrightarrow[uB]{} \; M^{-2} \\
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$$

$$
\begin{array}{ccccc}
u^{-2} & u^{-1} & u^{0} & u^{1} & u^{2} \\
\hline
\end{array}
$$

$$
CC_{-}^{\bullet}(M): \quad
\begin{array}{ccccc}
M^3 & \xrightarrow[uB]{} & M^2 & \xrightarrow[uB]{} & M^1 \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} \\
M^2 & \xrightarrow[uB]{} & M^1 & \xrightarrow[uB]{} & M^0 \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} \\
M^1 & \xrightarrow[uB]{} & M^0 & \xrightarrow[uB]{} & M^{-1} \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle \delta} \\
M^0 & \xrightarrow[uB]{} & M^{-1} & \xrightarrow[uB]{} & M^{-2}
\end{array}
$$

$$
CC^{per}_\bullet(N): \quad
\begin{array}{ccccccccc}
u^{-2} & & u^{-1} & & u^{0} & & u^{1} & & u^{2} \\
\hline
N_{-3} & \xrightarrow[uB]{} & N_{-2} & \xrightarrow[uB]{} & N_{-1} & \xrightarrow[uB]{} & N_{0} & \xrightarrow[uB]{} & N_{1} \\
\uparrow b & & \uparrow b & & \uparrow b & & \uparrow b & & \uparrow b \\
N_{-2} & \xrightarrow[uB]{} & N_{-1} & \xrightarrow[uB]{} & N_{0} & \xrightarrow[uB]{} & N_{1} & \xrightarrow[uB]{} & N_{2} \\
\uparrow b & & \uparrow b & & \uparrow b & & \uparrow b & & \uparrow b \\
N_{-1} & \xrightarrow[uB]{} & N_{0} & \xrightarrow[uB]{} & N_{1} & \xrightarrow[uB]{} & N_{2} & \xrightarrow[uB]{} & N_{3} \\
\uparrow b & & \uparrow b & & \uparrow b & & \uparrow b & & \uparrow b \\
N_{0} & \xrightarrow[uB]{} & N_{1} & \xrightarrow[uB]{} & N_{2} & \xrightarrow[uB]{} & N_{3} & \xrightarrow[uB]{} & N_{4}
\end{array}
$$

$$
\begin{array}{cccccccccc}
 & u^{-2} & & u^{-1} & & u^{0} & & u^{1} & & u^{2} \\
\hline
 & N_{-3} & \xrightarrow{uB} & N_{-2} & \xrightarrow{uB} & N_{-1} & & & & \\
 & \uparrow b & & \uparrow b & & \uparrow b & & & & \\
CC_\bullet(N): & N_{-2} & \xrightarrow{uB} & N_{-1} & \xrightarrow{uB} & N_{0} & & & & \\
 & \uparrow b & & \uparrow b & & \uparrow b & & & & \\
 & N_{-1} & \xrightarrow{uB} & N_{0} & \xrightarrow{uB} & N_{1} & & & & \\
 & \uparrow b & & \uparrow b & & \uparrow b & & & & \\
 & N_{0} & \xrightarrow{uB} & N_{1} & \xrightarrow{uB} & N_{2} & & & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & u^{-2} & u^{-1} & u^{0} & & u^{1} & & u^{2} \\
\hline
 & & & N_{-1} & \xrightarrow[uB]{} & N_0 & \xrightarrow[uB]{} & N_1 \\
 & & & \uparrow b & & \uparrow b & & \uparrow b \\
CC^-_\bullet(N): & & & N_0 & \xrightarrow[uB]{} & N_1 & \xrightarrow[uB]{} & N_2 \\
 & & & \uparrow b & & \uparrow b & & \uparrow b \\
 & & & N_1 & \xrightarrow[uB]{} & N_2 & \xrightarrow[uB]{} & N_3 \\
 & & & \uparrow b & & \uparrow b & & \uparrow b \\
 & & & N_2 & \xrightarrow[uB]{} & N_3 & \xrightarrow[uB]{} & N_4
\end{array}
$$

degrees:

$$
\begin{array}{ccccccccc}
u^{-2} & & u^{-1} & & u^{0} & & u^{1} & & u^{2} \\
\hline
-1 & \xrightarrow[uB]{} & 0 & \xrightarrow[uB]{} & 1 & \xrightarrow[uB]{} & 2 & \xrightarrow[uB]{} & 3 \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
-2 & \xrightarrow[uB]{} & -1 & \xrightarrow[uB]{} & 0 & \xrightarrow[uB]{} & 1 & \xrightarrow[uB]{} & 2 \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
-3 & \xrightarrow[uB]{} & -2 & \xrightarrow[uB]{} & -1 & \xrightarrow[uB]{} & 0 & \xrightarrow[uB]{} & 1 \\
\uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} & & \uparrow{\scriptstyle\delta} \\
-4 & \xrightarrow[uB]{} & -3 & \xrightarrow[uB]{} & -2 & \xrightarrow[uB]{} & -1 & \xrightarrow[uB]{} & 0
\end{array}
$$

SBI sequences (Connes' long exact sequences)

- For a mixed (**cochain**) complex $(M^\bullet, \delta, B)$, there is a short exact sequence of complexes

$$0 \to CC^\bullet(M)[-2] \xrightarrow{u} CC^\bullet(M) \xrightarrow{\mathrm{ev}_0} M^\bullet \to 0,$$

  where the first map is multiplication by u and the second map is evaluation at $u = 0$. This induces a cohomological long exact sequence

$$\cdots \to HC^{n-2}(M) \xrightarrow{S} HC^n(M) \xrightarrow{\pi} H^n(M) \xrightarrow{\beta} HC^{n-1}(M) \to \cdots,$$

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- Similarly, one has the following cohomological long exact sequences
$$\cdots \to H^n(M) \xrightarrow{j} HC^n_-(M) \xrightarrow{S} HC^{n+2}_-(M) \xrightarrow{\beta} H^{n+1}(M) \to \cdots,$$
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with connecting homomorphism given by $\beta[f] = [Bf_0]$, where f_0 is the coefficient of u^0 in the Laurent series f.
- For a mixed (**chain**) complex $(N_\bullet, b, B)$, by putting $M^i = N_{-i}$ into the above, one obtains in homology
$$\cdots \to HC^-_{n+2}(N) \to HC^-_n(N) \xrightarrow{\pi} H_n(N) \xrightarrow{\beta} HC^-_{n+1}(N) \to \cdots.$$

Cartan calculi I

Definition

Let $(M^\bullet, \delta, B)$ be a mixed cochain complex, and $(\mathfrak{g}^\bullet, d, \{\cdot,\cdot\})$ a DGLA. A **homotopy pre-Cartan calculus** of $\mathfrak{g}^\bullet$ on $CC^\bullet_{\mathrm{per}}(M)$ is the datum of a *contraction operator* (or *cap product*)

$$\iota\colon \mathfrak{g}^\bullet \otimes M^\bullet \to M^\bullet[1],$$

of a *Lie derivative*: $$\mathcal{L}\colon \mathfrak{g}^\bullet \otimes M^\bullet \to M^\bullet,$$

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such that
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Extending all operators by $k[[u, u^{-1}]]$-linearity to $CC_{\mathrm{per}}(M)$ and with $\mathcal{I} := \iota + u\mathcal{S}$, baptised **cyclic cap product**, one has the single equation

$$u\mathcal{L}_f = [d_u, \mathcal{I}_f] + \mathcal{I}_{df}.$$

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$$[\mathcal{I}_f, \mathcal{L}_g] - \mathcal{I}_{\{f,g\}} = [d_u, \mathcal{T}(f,g)] - \mathcal{T}(df, g) - (-1)^f \mathcal{T}(f, dg),$$

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and altogether this means that $\mathcal{L}$ defines a $\mathfrak{g}^\bullet$-dg-module structure on $CC^\bullet_{\rm per}(M)[n]$, inducing one on $CC^\bullet(M)[n]$ and $CC^\bullet_-(M)[n]$.

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- Finally, a (noncommutative differential) **calculus** is one where the homotopies vanish (usually obtained by descending to (co)homology).

Example (Classical geometric example)

- For a smooth manifold P, consider $(\mathcal{X}(P), 0, [.,.]_{\mathrm{SN}})$ acting on the mixed (chain) complex $(\Omega(P), 0, d_{\mathrm{dR}})$. Choose $\iota = i$, $\mathcal{L} = L$, whereas $\mathcal{S}$ and $\mathcal{T}$ can be chosen almost arbitrarily (since $\delta = 0$): take $\mathcal{S} = \mathcal{T} = 0$: this gives "fields acting on forms" with the customary formulae

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- The case "fields acting on fields" is obtained by $(\mathcal{X}(P), 0, [.,.]_{\mathrm{SN}})$ acting on $(\mathcal{X}(P), 0, d_{\mathrm{CE}})$ with $\iota_X Y := X \wedge Y$, the Lie derivative for multivector fields, and the differential d_{CE} from Lie algebra homology.

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Example (Classical algebraic example)

The pair of Hochschild cochains & chains forms a homotopy calculus s.t.

$$\big(H^\bullet(A, A), H_\bullet(A, A)\big)$$

of Hochschild cohomology and homology forms a calculus (Rinehart 1963, Connes, Getzler, Goodwillie, Nest-Tsygan (80/90s)).

Example (Sort-of universal example)

For a (left) Hopf algebroid U and (somehow technically complicated) coefficient modules M, N,

$$\big(\mathrm{C}^\bullet(U, N), \mathrm{C}_\bullet(U, M)\big)$$

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  forms a homotopy calculus (K. 2013).
- Let $\mathcal{O}$ be a cyclic operad with multiplication and $\mathcal{M}$ a cyclic module over $\mathcal{O}$: e.g., the operad itself. Then there is a homotopy calculus on

  $$\big(C^\bullet(\mathcal{O}), C^\bullet(\mathcal{O})\big)$$

  which leads to BV-algebras.

Induced Lie brackets on cyclic cohomology

- The semi-direct product DGLA $\mathfrak{g}^\bullet \ltimes CC^\bullet(M)[-2]$ is the cochain complex $\mathfrak{g}^\bullet \oplus CC^\bullet(M)[-2]$ endowed with the Lie bracket
$$\big[(f,x),(g,y)\big] := \big(\{f,g\}, \mathcal{L}_f y \pm \mathcal{L}_g x\big).$$

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Lemma

$\Psi_\xi : (\mathfrak{g}^\bullet \ltimes CC^\bullet(M)[-2], \partial_\xi) \to CC^\bullet(M),\ (f,x) \mapsto \pm \mathcal{I}_f \xi + ux$, is a morphism of complexes fitting into a diagram of SES of cochain complexes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & CC^\bullet(M)[-2] & \longrightarrow & (\mathfrak{g}^\bullet \ltimes CC^\bullet(M)[-2], \partial_\xi) & \longrightarrow & \mathfrak{g}^\bullet & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \Psi_\xi} & & \downarrow{\scriptstyle \iota_{(\cdot)}\xi_0} & & \\ 0 & \longrightarrow & CC^\bullet(M)[-2] & \xrightarrow[u]{} & CC^\bullet(M) & \xrightarrow[\mathrm{ev}_0]{} & M^\bullet & \longrightarrow & 0. \end{array}$$

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Theorem (First main result)

For a mixed complex M, the Lie bracket on $HC^\bullet(M)$ induced by a homotopy Cartan-Gerstenhaber calculus with a duality cocycle reads

$$[z, w] = (-1)^{z-1}\beta((\pi z) \smile (\pi w)),$$

where $\pi : HC^\bullet(M) \to H^\bullet(M)$ and $\beta : H^\bullet(M) \to HC^{\bullet-1}(M)$ are the canonical maps appearing in the SBI sequence and the cup product has been transported via the isomorphism $H^\bullet(\mathfrak{g}) \simeq H^\bullet(M)$.

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- If N is a mixed chain complex, by $M^i := N_{-i}$ and $HC^-_{-\bullet}(N) := HC^\bullet(M)$ as before, this yields a bracket on **negative cyclic homology**.

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- This generalises the one found by Van den Bergh *et al.* for Calabi-Yau algebras.

BV algebras arising from calculi I

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Let X be a smooth manifold of dimension d with a volume form ν, equipped with an orientation, that is, a volume form $\nu \in \Omega^d(X)$.

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Have this in mind when looking at the following.

BV algebras arising from calculi II

- The Gelfan'd-Daletskiĭ-Tsygan homotopy reduces on $H^\bullet(M)$ to

$$0 = \iota_f \mathcal{L}_g \pm \mathcal{L}_g \iota_f - \iota_{\{f,g\}} = \iota_f \mathcal{L}_g \pm \mathcal{L}_{g \smile f} \pm \iota_g \mathcal{L}_f - \iota_{\{f,g\}},$$

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- Let $p : H^\bullet(\mathfrak{g}) \to H^\bullet(M)$ denote the isomorphism induced by Poincaré duality w.r.t. ξ_0, by which we obtain a bracket $\{.,.\}$ and a product $\smile$ on $H^\bullet(M)$. Observe that one has for any d-cocycle f and δ-cocycle x:

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Theorem

In case of Poincaré duality, the degree -1 differential B on $H^\bullet(M)[-1]$ satisfies

$$\{x,y\} = (-1)^x B(x \smile y) - (-1)^x (Bx \smile y) - (x \smile By),$$

for any homogeneous x, y in $H^\bullet(M)[-1]$. Therefore, when $H^\bullet(\mathfrak{g})[-1]$ is a Gerstenhaber algebra, $(H^\bullet(M)[-1], \{\cdot,\cdot\}, \smile, B)$ is a BV algebra.

The string topology bracket of Chas-Sullivan

- Free loop space $LM := \mathrm{Map}(S^1, M^d)$ (of continuous closed paths without common base point) on a d-dimensional (closed oriented smooth) manifold: a topological space with the compact-open topology; its singular homology is called *loop homology*.

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- These maps fit into a LES: basically the SBI-sequence ($\beta = \mathrm{lift}$, $I = \mathrm{project}$, and $S = \frown c$, where c is the Euler class of the circle bundle).

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The string topology bracket arising from calculi

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Theorem (Third main result)

A homotopy C.-G. calculus with duality cocycle induces a BV algebra structure $(H^\bullet(M)[-1], \{\cdot,\cdot\}, \smile, B)$ *for a mixed complex* M.
The negative cyclic cohomology $HC^\bullet_-(M)$ *carries the deg* -2 **string topology bracket** *(or Chas-Sullivan-Menichi bracket)*

$$[x, y] := (-1)^x j((\beta x) \smile (\beta y)),$$

with the property

$$\beta[\cdot,\cdot] = \{\beta(\cdot), \beta(\cdot)\},$$

where $j : H^\bullet(M) \to HC^\bullet_-(M)$ *and* $\beta : HC^\bullet_-(M) \to H^{\bullet-1}(M)$ *are the maps appearing in the SBI sequence relating Hochschild to negative cyclic cohomology.*

3- (or e_3-)algebras

- More precisely, one obtains a homotopy formula

$$\{\phi, \psi\} = B(\phi \smile_0 \psi) \pm B\phi \smile_0 \psi \pm \phi \smile_0 B\psi \pm \delta(\psi \smile_2 \phi) \pm \delta\psi \smile_2 \phi \pm \psi \smile_2 \delta\phi.$$

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Theorem (Fourth main result)

If $\{\cdot, \cdot\} = 0$ on $H^\bullet(M)[-1]$, then

$$\{\!\{x, y\}\!\} := (-1)^x (Bx) \smile (By)$$

defines a degree -2 Lie bracket on $H^\bullet(M)[-1]$ with $j\{\!\{x, y\}\!\} = [jx, jy]$ and $B\{\!\{x, y\}\!\} = 0$, turning $\big(H^\bullet(M)[-1], \smile, \{\!\{\cdot, \cdot\}\!\}\big)$ into an e_3-algebra, that is,

$$\begin{aligned}
\{\!\{x, y\}\!\} &= -(-1)^{xy} \{\!\{y, x\}\!\}, \\
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- So far, it is not clear how $\smile_2$ and $\{\!\{\cdot,\cdot\}\!\}$ are related and what the appurtenant pre-Lie structure would be.

Examples: (cyclic) operads and (opposite) modules

- An **operad** is a collection of trees with a vertical composition, subject to a certain associativity (think of $\mathrm{Hom}_k(V^{\otimes \bullet}, V)$ for $V \in k$-**Mod**):

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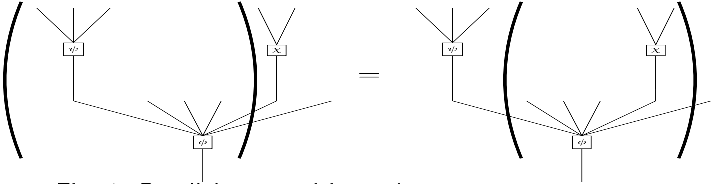

Fig. 1: Parallel composition axiom.

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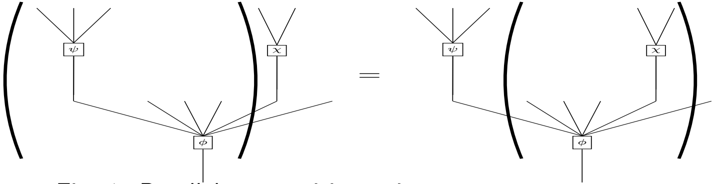

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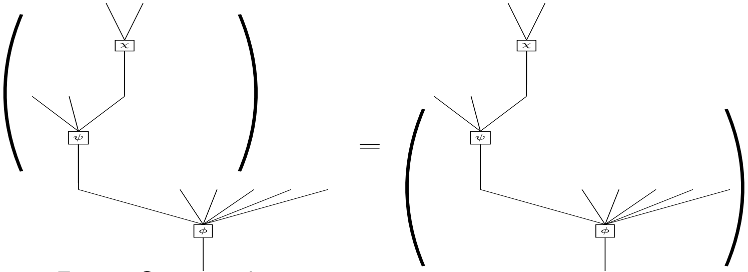

Fig. 2: Sequential composition axiom.

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- A **module** $\mathcal{M}$ over an operad $\mathcal{O}$ is a collection of trees with an action of the operad on it, again subject to a certain associativity: in the pictures just seen, replace one of the three ϕ, ψ, or χ by an element $m \in \mathcal{M}$.

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- An **opposite module** over an operad is an upside-down tree with an action of the operad on it, again subject to a certain associativity.

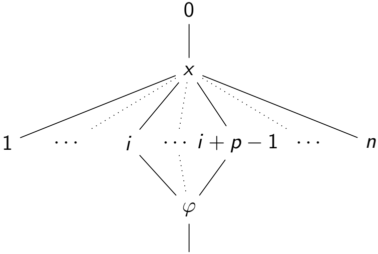

Fig. 3: Opposite modules

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- A **cyclic operad** is an operad with a cyclic action on it, which bends the trunk to become the last branch, and the first branch to become the trunk, subject to compatibility conditions.

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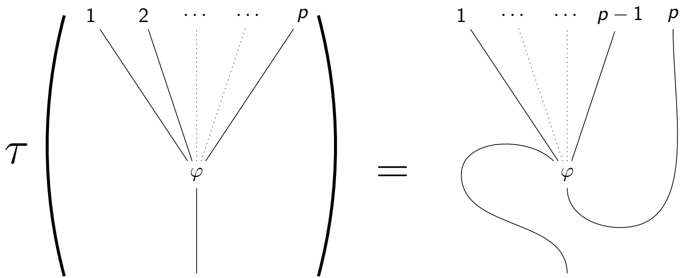

Fig. 4: Cyclic operads

Examples: (cyclic) operads and (opposite) modules

- A **cyclic opposite module** over a (not necessarily cyclic) operad is a module with a cyclic action on it, with an analogous bending as above, subject to conditions.

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- A **cyclic opposite module** over a (not necessarily cyclic) operad is a module with a cyclic action on it, with an analogous bending as above, subject to conditions.

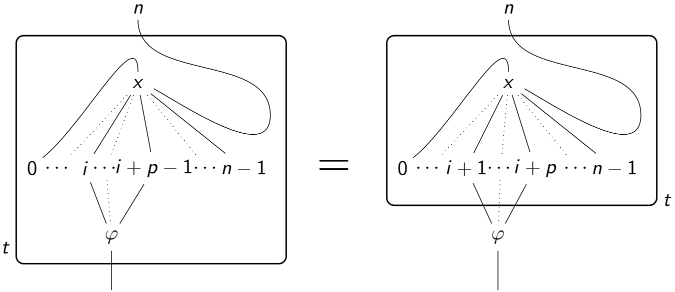

Fig. 5: The relation $t(\varphi \bullet_i x) = \varphi \bullet_{i+1} t(x)$ *for cyclic opposite modules*

Calculi for cyclic opposite modules over operads

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- A new achievement is the homotopy $\mathcal{T}$:

Theorem

For a cyclic opposite module $(\mathcal{N}, t)$ over an operad $\mathcal{O}$ with multiplication, define the Gel'fand-Daletskiĭ-Tsygan *homotopy as*

$$\mathcal{T} : \mathcal{O}(p) \otimes \mathcal{O}(q) \otimes \mathcal{N}(n) \to \mathcal{N}(n - p - q + 2),$$

$$(\varphi, \psi, x) \mapsto \sum_{j=1}^{p-1} \sum_{i=j}^{p-1} \pm(\varphi \circ_{p-i+j} \psi) \bullet_0 t^{j-1}(x).$$

With $\mathcal{T}(\varphi, \psi)(x) := \mathcal{T}(\varphi, \psi, x)$ and as before $d_u = b + uB$, one has

$$[\mathcal{I}_\psi, \mathcal{L}_\varphi] - \mathcal{I}_{\{\psi,\varphi\}} = [d_u, \mathcal{T}(\varphi, \psi)] - \mathcal{T}(\delta\varphi, \psi) - (-1)^{p-1}\mathcal{T}(\varphi, \delta\psi)$$

on $\overline{\mathcal{N}}$ for $\varphi, \psi \in \overline{\mathcal{O}}$.

Brackets on cyclic opposite modules

Definition

We say that there is *(Poincaré) duality* between an operad $\mathcal{O}$ and a cyclic opposite module $\mathcal{N}$ if there is a cocycle $\zeta \in \mathcal{N}(d)$ (the *fundamental class* $[\zeta]$) such that $\mathcal{O} \to \mathcal{N}$, $\varphi \mapsto i_\varphi \zeta = \varphi \frown \zeta$ induces an isomorphism $H^n(\mathcal{O}) \cong H_{d-n}(\mathcal{N})$.

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Geometrically, think of, as mentioned before, the volume form on a smooth manifold.

Corollary

If Poincaré duality holds, $HC^-_\bullet(\mathcal{N})$ carries a deg $(1-d)$ bracket

$$[z, w] = (-1)^{z+d} \beta((\pi z) \smile (\pi w)),$$

where $\pi : HC^-_n(\mathcal{N}) \to H_n(\mathcal{N})$ and $\beta \colon H_n(\mathcal{N}) \to HC^-_{n+1}(\mathcal{N})$.

Examples

Example (inside the example: Calabi-Yau algebras)

This happens for d-Calabi-Yau algebras: a homologically smooth algebra A in which Poincaré duality holds: $\cdot \frown \omega : H^i(A, A) \simeq H_{d-i}(A, A)$ with fundamental class $[\omega] \in H_d(A, A)$. Then $HC^-_\bullet(A, A)$ carries a bracket of degree $-d$ (Van den Bergh *et al.*).

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- The well-known calculus "fields acting on forms" can be obtained by the aforementioned calculus structure: $\iota_X \omega$ *contracts* forms (reduces in length) and hence can be described by opposite $\mathcal{O}$-modules.
- **Problem**: the [somewhat less] well-known calculus of "fields acting on fields" cannot be described this way: $\iota_X Y = X \wedge Y$ increases the length and hence should be described by $\mathcal{O}$-modules instead.
- Only that $\mathcal{O}$-modules are obviously not opposite $\mathcal{O}$-modules, not even in negative degree.

Brackets on cyclic modules

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- However, the sequence $\{\mathcal{M}^*(q)\}_{q \geq 0}$ with $\mathcal{M}^*(q) := \mathrm{Hom}_k(\mathcal{M}(q), k)$, is an opposite $\mathcal{O}$-module if $\mathcal{M}$ is an $\mathcal{O}$-module.

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- Hence, if $\mathcal{M}$ is cyclic, $\mathcal{M}^*$ is so as well and the explicit calculus operations on $\mathcal{M}$ can be obtained by considering adjoints. Define

$$\langle x, Bm \rangle := \langle Bx, m \rangle, \qquad \langle \iota_\varphi x, m \rangle := \langle x, \iota_\varphi m \rangle,$$
$$\langle \mathcal{L}_\varphi x, m \rangle := \langle x, \mathcal{L}_\varphi m \rangle, \qquad \langle \mathcal{S}_\varphi x, m \rangle := \langle x, \mathcal{S}_\varphi m \rangle,$$
$$\langle \mathcal{T}(\varphi, \psi)(x), m \rangle \quad := \quad \langle x, \mathcal{T}(\varphi, \psi)(m) \rangle.$$

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Theorem

If $\mathcal{M}$ is a cyclic module over a cyclic operad with multiplication, then there is the structure of a homotopy Cartan-Gerstenhaber calculus on $\mathcal{M}^*$ resp. $CC^\bullet_{\mathrm{per}}(\mathcal{M}^*)$ and therefore also one on $\mathcal{M}$ resp. $CC^\bullet_{\mathrm{per}}(\mathcal{M})$

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- In particular, a cyclic operad with multiplication $(\mathcal{O}, t, \mu, e)$ is a cyclic module over itself and hence carries a calculus structure. Therefore,

$$[\mathcal{I}_\psi, \mathcal{L}_\varphi] - \mathcal{I}_{\{\psi,\varphi\}} = [d_u, \mathcal{T}(\varphi,\psi)] - \mathcal{T}(\delta\varphi,\psi) - (-1)^{p-1}\mathcal{T}(\varphi,\delta\psi)$$

holds on $\mathcal{O}$ itself.

- By applying it to the special element "e" and observing things like $\mathcal{I}_{(\cdot)} e = \mathrm{id}_{\mathcal{O}}$, $\iota_\varphi \psi = \varphi \smile \psi$ and some more, one obtains

$$\{\psi, \varphi\} = -\psi \smile B(\varphi) \pm \mathcal{L}_\varphi \psi \pm \delta(S_\psi \varphi) \pm S_\psi \delta\varphi \pm S_{\delta\psi} \varphi.$$

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- This fallout of our general approach was first proven by Menichi.

— Bonus material —

Gerstenhaber algebras

- A **Gerstenhaber algebra** is now (in a not quite exact sense) a graded **Poisson algebra**, that is, an algebra with a graded Lie bracket $\{.,.\}$ and a (graded commutative) product $\smile$ such that

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- **Algebraic example**: as just seen, Hochschild cohomology $H^\bullet(A, A)$ is a Gerstenhaber algebra.
- **Geometric example**: for a smooth manifold M, the space $\mathcal{X}^p(M)$ of polyvector fields is a Gerstenhaber algebra. The product $\smile$ is the wedge product, and the bracket is the **Schouten-Nijenhuis** bracket, which is the commutator on vector fields.

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- You might want to comment that for *you* this is not really a problem as Hochschild cohomology $H^\bullet(A, A)$ is **not** functorial in A (an algebra map $A \to B$ does not induce a map $H^\bullet(A, A) \to H^\bullet(B, B)$), whereas $H^\bullet(A, A^*)$ is so, so: so what?

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- Let me, however, repeat that the groups $H^\bullet(A, A)$ are interesting objects to study as they are related to deformation theory.

Cyclic objects

- A **cyclic** k-**module** is a simplicial object $(X_\bullet, d_\bullet, s_\bullet)$ together with morphisms $t_n : X_n \to X_n$ subject to

$$d_i t_n = \begin{cases} t_{n-1} d_{i-1} & \text{if } 1 \le i \le n \\ d_n & \text{if } i = 0, \end{cases} \qquad s_i t_n = \begin{cases} t_{n+1} s_{i-1} & \text{if } 1 \le i \le n \\ t_{n+1}^2 s_n & \text{if } i = 0. \end{cases}$$

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- Define Hochschild operator, norm operator, extra degeneracy:

$$b := \sum_{j=0}^{n} (-1)^j d_j, \quad N := \sum_{j=0}^{n} (-1)^n t_{n+1}^j, \quad s_{-1} := t_{n+1} s_n,,$$

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- These operators fulfill $B^2 = 0$, $Bb + bB = 0$, and $b^2 = 0$, hence each cyclic object gives rise to a **mixed complex**.

- For a smooth (commutative) k-algebra A (with $\mathrm{char}(k) = 0$), the **cyclic HKR-map**

$$\big(C_\bullet(A, A), b, B\big) \to (\Omega^\bullet_{A|k}, 0, d_{dr})$$

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  Since the LHS is also defined for noncommutative A, one might see cyclic homology as a noncommutative generalisation of de Rham cohomology and also introduce noncommutative differential forms.
- For example, for the algebra $C^\infty(M)$ of smooth functions on a compact manifold M, one has

$$HH_\bullet(C^\infty(M)) \simeq \Omega^\bullet(M), \qquad HP^\bullet(C^\infty(M)) \simeq H^{\mathrm{even}}_{dR}(M) \oplus H^{\mathrm{odd}}_{dR}(M).$$

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$$\big(C_\bullet(A, A), b, B\big) \to (\Omega^\bullet_{A|k}, 0, d_{dr})$$

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- For a vector bundle E and the space of E-valued differential operators D,

$$HC^\bullet(D) \simeq H^{CE}_\bullet(\Gamma^\infty(E), k),$$

where the right hand side refers to Lie algebroid homology.

More details on the string topology bracket

- The *Borel construction* associates to a G-space X (Hausdorff with a continuous left action) an associated fibre bundle $X_G := EG \times_G X = (EG \times X)/G$ to the (universal) principle fibre bundle $G \to EG \to BG$ and equivariant homology is defined to be the homology of X_G (if points and closed sets can be separated by continuous functions and the G-action is free, this is isomorphic to the homology of X/G).

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- These maps fit into a long exact sequence which is basically the *SBI*-sequence ($\beta = m$, $I = e$, and $S = \frown c$, where $c \in H^2(X_{S^1})$ is the Euler class of the circle bundle).