Poisson structures on differentiable stacks

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September 13, 2018

Overview

1. Quasi-Poisson groupoids and their Morita equivalences
   - Definitions
   - Polyvector fields on a differentiable stack
2. From cotangent to tangent: VB groupoids and representations up to homotopy
3. Invertible Poisson structures on a differentiable stacks, and ranks

Motivation

Claim: *differentiable stacks are a good manner to represent singular spaces*.

As spaces, they should come with a notion of:

1. vector fields,
2. polyvector fields
3. **Poisson structures**
4. non-degenerate Poisson structures (rank of a Poisson structure)
5. cotangent and tangent
6. ... and Poisson structures should induce a map from cotangent to tangent.

Last, examples should exist.

Differentiable stacks

A *differentiable stack* is a Morita equivalence class of Lie groupoids.
How to define a notion of *blablabla* on a differentiable stack?
First, define the *blablabla* of a Lie groupoid $\Gamma \rightrightarrows M$. Then

1. Check that ME Lie groupoids have the <u>same</u> *blablabla*.
   - Ex: dimension of the stack, coming from $\dim(\Gamma \rightrightarrows M) := 2\dim(M) - \dim(\Gamma)$
2. Check that ME Lie groupoids have the <u>isomorphic</u> *blablabla*'s.
   - Ex: most cohomologies of Lie groupoids.
3. Associate a *blablabla* to a Lie groupoid $\Gamma \rightrightarrows M$. Check that different choices give <u>homotopy equivalent</u> *blablabla*'s. Check that a ME induces a homotopy equivalence of *blablabla*'s.

Towards Poisson structures on differentiable stacks

Idea one

Take *blablabla* to be multiplicative Poisson structures, as in Mackenzie-Xu.

Does it work?

No, this is not well behaved under Morita equivalence.

Idea two.

Consider quasi-Poisson Lie groupoids.

Does it work?

No, this is not well behaved under Morita equivalence.

Idea three

Consider quasi-Poisson Lie groupoids up to twists.

Quasi-Poisson structures and twists

Definition (ILX)

Let $\Gamma \rightrightarrows M$ be a Lie groupoid.

1. A quasi-Poisson structure on $\Gamma \rightrightarrows M$ is a pair (Π, Λ), with Π multiplicative bivector field and $\Lambda \in \Gamma(\wedge^3 A)$ s.t.

$$\frac{1}{2}[\Pi, \Pi] = \overleftarrow{\Lambda} + \overrightarrow{\Lambda}\ , \quad [\overleftarrow{\Lambda}, \Pi] = 0\ . \tag{1}$$

2. Two quasi-Poisson structures (Π_1, Λ_1) and (Π_2, Λ_2) on Γ are said to be *twist equivalent* if there exists a section $B \in \Gamma(\wedge^2 a)$, called the *twist*, such that

$$\Pi_2 = \Pi_1 + \overleftarrow{B} + \overrightarrow{B}, \quad \Lambda_2 = \Lambda_1 - \delta_{\Pi_1}(B) - \frac{1}{2}[B, B]. \tag{2}$$

A possible definition

A first result

Theorem

Morita equivalent Lie groupoids have isomorphic sets of quasi-Poisson structures up to twists.

Enough to define Poisson structures on a differentiable stack.

Unfortunatly...

Meaning is still missing.

Let us have L_∞-algebras into the picture.

Theorem

There is a $\mathbb{Z}$-graded crossed module structure on:

$$\Gamma(\wedge A)[2] \oplus \mathcal{X}_{mult}(\Gamma)[1].$$

Definition

See it as a $\mathbb{Z}$-graded Lie-2 algebra called *polyvector fields on* $\Gamma \rightrightarrows M$.

Remark: its Maurer-Cartan elements are quasi-Poisson structures on $\Gamma \rightrightarrows M$. Its gauges are the twists.

Theorem (Non-trivial result)

A Morita equivalence of Lie groupoids induces an unique up to homotopy $\mathbb{Z}$-graded Lie-2 algebra quasi-isomorphisms between their polyvector fields.

The previous theorem is a Corollary.

VB-groupoids

Definition

A *VB groupoid* is a groupoid in the category of vector bundles.

Examples. Let $\Gamma \rightrightarrows M$ be a Lie groupoid:

Name	Tangent groupoid	Cotangent groupoid
	$T\Gamma \rightrightarrows TM$	$T^*\Gamma \rightrightarrows A^*$
Units	TM	A^*
Core	A	T^*M

Definition

VB groupoids Morita equivalence is a Morita equivalence + linearity conditions.

Morita equivalent Lie groupoids have VB Morita equivalent tangent and cotangent groupoids. This defines the tangent and cotangent stacks of a stack.

VB groupoids 2-category

VB groupoids come with natural notions of:

1. morphisms
2. homotopies of morphisms
3. Morita equivalences of morphisms

Proposition

1. *For* (π, Λ) *quasi-Poisson on* $\Gamma \rightrightarrows M$, $\pi^{\#} : T^*\Gamma \to T\Gamma$ *is a morphism of BV groupoids.*
2. *For* (π_i, Λ_i), $i = 1, 2$ *quasi-Poisson on* $\Gamma \rightrightarrows M$, $\pi_i^{\#} : T^*\Gamma \to T\Gamma$ *are homotopic morphisms.*
3. *For* (π_i, Λ_i) *quasi-Poisson on* $\Gamma_i \rightrightarrows M_i$, *that correspond one to the other through a Morita equivalence,* $\pi_i^{\#} : T^*\Gamma_i \to T\Gamma_i$ *are Morita equivalent morphisms.*

Hence Poisson structures on stacks induce a map from cotanegtn stack to tangent stacks.

VB groupoids and representations up to homotopy

Gracia-Saz and Mehta found a dictionnary between

(i) VB-groupoids over $\Gamma \rightrightarrows M$

(ii) representations up to homotopies of $\Gamma \rightrightarrows M$.

Tangent groupoid/cotangent groupoids correspond to tangent complex and cotangent complex.

Proposition

For (π_i, Λ_i) quasi-Poisson on $\Gamma_i \rightrightarrows M_i$, that correspond one to the other through a Morita equivalence, the induced maps from cotangent to tangent complexes are homotopy equivalent.

Hence Poisson structures on stacks induce a map from cotangent complex to tangent complex.