Algebraic index theorem for crossed products

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The term *index theorems* is usually used to describe the equality of, on one hand, analytic invariants of certain operators on smooth manifolds and, on the other hand, topological/geometric invariants associated to their "symbols". A convenient way of thinking about this kind of results is as follows.

One starts with a C^*-algebra of operators A associated to some geometric situation and a K-homology cycle (A, π, H, D), where $\pi \colon A \to B(H)$ is a $*$-representation of A on a Hilbert space H and D is a Fredholm operator on H commuting with the image of π modulo compact operators $\mathcal{K}$. The explicit choice of the operator D typically has some geometric/analytic flavour, and, depending on the parity of the K-homology class, H can have a $\mathbb{Z}/2\mathbb{Z}$ grading such that π is even and D is odd.

Given such a (say even) cycle, an index of a reduction of D by an idempotent in $A \otimes \mathcal{K}$ defines a pairing of K-homology and K-theory, i. e. the group homomorphism

$$KK_0(\mathbb{C}, A) \times KK_0(A, \mathbb{C}) \longrightarrow \mathbb{Z}. \tag{1}$$

One can think of this as a Chern character of D defining a map

$$K_0(A) \longrightarrow \mathbb{Z},$$

and the goal is to compute it explicitly in terms of some topological data extracted from the construction of D.

Example 1

$A = C(X)$, where X is a compact manifold and D is an elliptic pseudodifferential operator acting between spaces of smooth sections of a pair of vector bundles on X.

The number $< ch(D), [1] >$ is the Fredholm index of D, i. e. the integer

$$Ind(D) = \dim(Ker(D)) - \dim(Coker(D))$$

and the Atiyah–Singer index theorem identifies it with the evaluation of the $\hat{A}$-genus of T^*X on the Chern character of the principal symbol of D. This is the situation analysed in the original papers of Atiyah and Singer.

Example 2

$A = C^*(\mathcal{F})$, where $\mathcal{F}$ is a foliation of a smooth manifold and D is a transversally elliptic operator on X.

Everything is represented by concrete operators on a Hilbert space H.

Suppose that a $K_0(A)$ class is represented by a projection $p \in \mathcal{A}$, where $\mathcal{A}$ is a subalgebra of A closed under holomorphic functional calculus, so that the inclusion $\mathcal{A} \subset A$ induces an isomorphism on K-theory. For appropriately chosen $\mathcal{A}$, the fact that D is transversally elliptic implies that the operator pDp is Fredholm on the range of p. The corresponding integer

$$Ind\{pDp : rg(p) \to rg(p)\}$$

can be identified with a pairing of a certain cyclic cocycle $ch(D)$ on the algebra $\mathcal{A}$ with the Chern character of p in the cyclic periodic complex of $\mathcal{A}$.

For a special class of hypo-elliptic operators the computation of this integer is the context of the transversal index theorem of A. Connes and H. Moscovici.

A highly non-trivial technical part of their work is a construction of such an operator.

Example 3

Suppose again that X is a smooth manifold. The natural class of representatives of K-homology classes of $C(X)$ given by operators of the form

$$D = \sum_{\gamma \in \Gamma} P_\gamma \pi(\gamma),$$

where Γ is a discrete group acting on $L^2(X)$ by Fourier integral operators of order zero and P_γ is a collection of pseudodifferential operators on X, all of them of the same (non-negative) order.

Suppose that the group acts freely, i.e. $D \neq 0$ whenever any of P_γ's is non-zero. The principal symbol $\sigma_\Gamma(D)$ of such a D is an element of the C^*-algebra $C(S^*X) \rtimes_{max} \Gamma$, where S^*M is the cosphere bundle of M. Invertibility of $\sigma_\Gamma(D)$ implies that D is Fredholm and the index theorem in this case would express $Ind_\Gamma(D)$ in terms of some equivariant cohomology classes of M and an appropriate equivariant Chern character of $\sigma_\Gamma(D)$.

In the case when Γ acts by diffeomorphisms of M this example was studied by Savin, Schrohe, Sternin and by Perrot.

A typical computation of index proceeds via a reduction of the class of operators D under consideration to an algebra of (complete) symbols, which can be thought of as a "formal deformation" $\mathcal{A}^\hbar$. Let us spend a few lines on a sketch of the construction of $\mathcal{A}^\hbar$ in the case when the operators in question are differential operators on X.

Denote by $\mathcal{D}_X$ the algebra of differential operators on X.

Let $\mathcal{D}_X^\bullet$ be the filtration by degree of $\mathcal{D}_X$. One constructs the Rees algebra

$$R = \{(a_0, a_1, \ldots) \mid a_k \in \mathcal{D}_X^k\}$$

with the product

$$(a_0, a_1, \ldots)(b_0, b_1, \ldots) = (a_0 b_0, a_0 b_1 + a_1 b_0, \ldots, \sum_{i+j=k} a_i b_j, \ldots).$$

The shift

$$\hbar\colon (a_0, a_1, \ldots) \to (0, a_0, a_1, \ldots)$$

makes R into an $\mathbb{C}[[\hbar]]$-module.

The elements of $R/\hbar R$ have form of sequences

$$(\sigma_0, \sigma_1, \sigma_2, \dots) \text{ where } \sigma_k \in \mathcal{D}^k/\mathcal{D}^{k-1} = Pol_k(T^*X),$$

where $Pol_k(T^*X)$ is the space of smooth, fiberwise polynomial functions of degree k on the cotangent bundle T^*X. Hence

$$R/\hbar R \simeq \prod_k Pol_k(T^*X)$$

and a choice of a $\mathbb{C}[[\hbar]]$-linear isomorphism of R with $\prod_k Pol_k(T^*X)[[\hbar]]$ induces on $\prod_k Pol_k(T^*X)[[\hbar]]$ an associative, $\hbar$-bilinear product, easily seen to extend to $C^\infty(T^*X)[[\hbar]]$.

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This is a "formal deformation of T^*X". More generally, A formal deformation quantization of a symplectic manifold (M, ω) is an associative $\mathbb{C}[[\hbar]]$-linear product $\star$ on $C^\infty(M)[[\hbar]]$ of the form

$$f \star g = fg + \frac{i\hbar}{2}\{f, g\} + \sum_{k \geq 2} \hbar^k P_k(f, g);$$

where $\{f, g\} := \omega(I_\omega(df), I_\omega(dg))$ is the canonical Poisson bracket induced by the symplectic structure, I_ω is the isomorphism of T^*M and TM induced by ω, and the P_k denote bidifferential operators. We will also require that $f \star 1 = 1 \star f = f$ for all $f \in C^\infty(M)[[\hbar]]$. We will use $\mathcal{A}^\hbar(M)$ to denote the algebra $(C^\infty(M)[[\hbar]], \star)$. The ideal $\mathcal{A}^\hbar_c(M)$ in $\mathcal{A}^\hbar(M)$, consisting of power series of the form $\sum_k \hbar^k f_k$, where f_k are compactly supported, has a unique (up to a normalization) trace Tr with values in $\mathbb{C}[\hbar^{-1}, \hbar]]$.

Since the product in $\mathcal{A}_c^{\hbar}(M)$ is local, the computation of the pairing of K-theory and cyclic cohomology of $\mathcal{A}_c^{\hbar}(M)$ reduces to a differential-geometric problem and the result of the resulting computation is the “algebraic index theorem".

Back to the subject of this talk, the index of the operators of the type

$$D = \sum_{\gamma \in \Gamma} P_\gamma \pi(\gamma).$$

It is not difficult to see that the index computations reduce to the computation of the pairing of the trace (or some other cyclic cocycle) with the K-theory of the symbol algebra, which, in this case, is identified with a crossed product $\mathcal{A}^{\hbar}_c(M) \rtimes \Gamma$.

- As cyclic periodic homology is invariant under (pro-)nilpotent extensions, the result of the pairing depends only on the $\hbar = 0$ part of the K-theory of $\mathcal{A}^{\hbar}_c(M) \rtimes \Gamma$.
- The $\hbar = 0$ part of the symbol algebra $\mathcal{A}^{\hbar}_c(M) \rtimes \Gamma$ is just $C^{\infty}_c(M) \rtimes \Gamma$, hence the Chern character of D, originally an element of K-homology of the $C(M)$, enters into the final result only through a class in the equivariant cohomology $H^*_{\Gamma}(M)$.

- **Cyclic cocycle on $\mathcal{A}^{\hbar}_{c}(M) \rtimes \Gamma$.**
  For a group cocycle $\xi \in C^k(\Gamma, \mathbb{C})$, set

$$Tr_{\xi}(a_0\gamma_0 \otimes \ldots \otimes a_k\gamma_k) = \delta_{e,\gamma_0\gamma_1\ldots\gamma_k}\xi(\gamma_1, \ldots, \gamma_k)\, Tr(a_0\gamma_0(a_1)\ldots(\gamma_0\gamma_1\ldots\gamma_{k-1})(a_k)).$$

- The object of interest - the pairing

$$Tr_{\xi} : K_0(\mathcal{A}^{\hbar}_{c}(M) \rtimes \Gamma) \to \mathbb{C}[\hbar^{-1}, \hbar]]$$

The action of Γ on $\mathcal{A}^{\hbar}(M)$ induces (modulo $\hbar$) an action of Γ on M by symplectomorphisms. Let σ be the “principal symbol" map:

$$\mathcal{A}^{\hbar}(M) \to \mathcal{A}^{\hbar}(M)/\hbar\mathcal{A}^{\hbar}(M) \simeq C^{\infty}(M).$$

It induces a homomorphism

$$\sigma \colon \mathcal{A}^{\hbar}(M) \rtimes \Gamma \longrightarrow C^{\infty}(M) \rtimes \Gamma,$$

still denoted by σ. Let

$$\Phi \colon H^{\bullet}_{\Gamma}(M) \longrightarrow HC^{\bullet}_{per}\left(C^{\infty}_{c}(M) \rtimes \Gamma\right)$$

be the canonical map (first constructed by Connes).

The main result is the following.

Let $e,\ f \in M_N\left(\mathcal{A}^{\hbar}(M)\right)$ be a couple of idempotents such that the difference $e - f \in M_N\left(\mathcal{A}^{\hbar}_c(M) \rtimes \Gamma\right)$ is compactly supported. Let $[\xi] \in H^k(\Gamma, \mathbb{C})$ be a group cohomology class. Then $[e] - [f]$ is an element of $K_0(\mathcal{A}^{\hbar}_c(M) \rtimes \Gamma)$ and its pairing with the cyclic cocycle Tr_ξ is given by

$$< Tr_\xi, [e] - [f] >= \left\langle \Phi\left(\hat{A}_\Gamma e^{\theta_\Gamma}[\xi]\right), ch([\sigma(e)] - [\sigma(f)])\right\rangle . \quad (2)$$

Here $\hat{A}_\Gamma \in H^\bullet_\Gamma(M)$ is the equivariant $\hat{A}$-genus of M, $\theta_\Gamma \in H^\bullet_\Gamma(M)$ is the equivariant characteristic class of the deformation $\mathcal{A}^{\hbar}(M)$.

A deformation quantization of a symplectic manifold $\mathcal{A}^{\hbar}(M)$ can be seen as the space of flat sections of a flat connection ∇_F on the bundle of Weyl algebras over M constructed from the bundle of symplectic vector spaces $T^*M \to M$.

$$\begin{array}{ccc} \mathbb{W} & \longrightarrow & \mathcal{W} \\ & & \downarrow \\ & & M. \end{array}$$

The fiber of $\mathcal{W}$ is isomorphic to the Weyl algebra

$$\mathbb{W} = \{\hat{x}_1, \hat{x}_2 \ldots, \hat{x}_n, \hat{\xi}_1, \hat{\xi}_2, \ldots, \hat{\xi}_n \mid [\hat{\xi}_i, \hat{x}_j] = \hbar\delta_{i,j}\}$$

and ∇_F is a connection with values in the Lie algebra $\mathfrak{g}$ of derivations of $\mathbb{W}$, equivariant with respect to a maximal compact subgroup K of the structure group of T^*M.

Suppose that $\mathbb{L}$ is a $(\mathfrak{g}, K)$-module. The Gelfand–Fuks construction provides a sheaf $\mathcal{L}$ on M and a complex $(\Omega(M, \mathbb{L}), \nabla_F)$ of $\mathbb{L}$-valued differential forms with a differential ∇_F satisfying $\nabla_F^2 = 0$. Let us denote the corresponding cohomology spaces by $H^\bullet(M, \mathbb{L})$. The Gelfand–Fuks construction also provides a morphism of complexes

$$GF\colon C^\bullet_{Lie}(\mathfrak{g}, K; \mathbb{L})) \longrightarrow \Omega^\bullet(M, \mathcal{L})$$

In many of our examples $\mathbb{L}$ and, therefore, $\mathcal{L}$ is a complex.

Some examples.

- Quantization: $\mathbb{L} = \mathbb{W}$, $\mathcal{A}^{\hbar}(M) \simeq (\Omega(M, \mathbb{W}), \nabla_F)$
- Cohomology: $\mathbb{L} = \mathbb{C}$, $(\Omega(M), d) \simeq (\Omega(M, \mathbb{C}), \nabla_F)$
- $\mathbb{L} = CC^{per}_{\bullet}(\mathbb{O})$,
  $(CC^{per}_{\bullet}(C^{\infty}(M), b+uB) \simeq (\Omega(M, CC^{per}_{\bullet}(\mathbb{O})), b+uB+\nabla_F)$
- $\mathbb{L} = CC^{per}_{\bullet}(\mathbb{W})$,
  $(CC^{per}_{\bullet}(\mathcal{A}^{\hbar}(M), b+uB) \simeq (\Omega(M, CC^{per}_{\bullet}(\mathbb{W})), b+uB+\nabla_F)$.

First a "microlocal" version of the index theorem.

Let $\mathbb{L}^\bullet = \mathrm{Hom}^{-\bullet}(CC_\bullet^{per}(\mathbb{W}), \hat{\Omega}^{-\bullet}[\hbar^{-1}, \hbar]][u^{-1}, u]][2d])$. There exist two elements $\hat{\tau}_a$ and $\hat{\tau}_t$ in the hypercohomology group $\mathbb{H}^0_{Lie}(\mathfrak{g}, K; \mathbb{L}^\bullet)$ such that the following holds.

$$\hat{\tau}_a = \sum_{p \geq 0} \left[\hat{A} e^{\hat{\theta}}\right]_{2p} u^p \hat{\tau}_t,$$

where $\left[\hat{A} e^{\hat{\theta}}\right]_{2p}$ is the component of degree $2p$ of a certain hypercohomology class.

Note that $\mathbb{H}^\bullet_{Lie}(\mathfrak{g}, K; \mathbb{L}^\bullet)$ is a $H^\bullet_{Lie}(\mathfrak{g}, K; \mathbb{C}[[\hbar]])$-module. $\hat{\theta}$ isthe class of the Lie algebra extension

$$\frac{1}{\hbar}\mathbb{C}[[\hbar]] \to \mathbb{W} \to \mathfrak{g}.$$

Let $\mathcal{A}^\hbar(M)$ be the deformation of $M = T^*X$ associated to symbol calculus and $\hat{c} \in H^\bullet_{Lie}(\mathfrak{g}, K; \mathbb{C})$. Then

$$GF : \Omega^\bullet(M, \mathbb{L}^\bullet) \to Hom((CC^{per}_\bullet(\mathcal{A}^\hbar(M), b + uB)), (\Omega(M), d)),$$

and one checks the following.

1. $GF(\hat{\theta})$ is the characteristic class of the deformation
2. $GF(\hat{c}) =: c \in H^\bullet(M)$
3. $GF\left(\hat{A}e^{\hat{\theta}}\right) =: \hat{A}_M$
4. $GF(\hat{\tau}_t)(\sigma(p) - \sigma(q)) = ch(p_0) - ch(q_0))$
5. $\int_M GF(\hat{c}\hat{\tau}_a)(\sigma(p) - \sigma(q)) = Tr_c(p - q)$
6. Index Theorem

$$Tr_c(p - q) = \int_M c(ch(p_0) - ch(q_0))\hat{A}_M.$$

The proof of the theorem about the crossed product follows similar lines.
Let $E\Gamma$ be a simplicial model for the universal three action of Γ. The lift $\pi^*\mathcal{W}$ of the Weyl bundle of M under the projection $\pi : M \times E\Gamma \to M$ admits an action of Γ. Moreover the connection ∇_F has a Γ-equivariant flat extension ∇_Γ to $\pi^*\mathbb{W}$ and the Gelfand -Fuks map still exists in this context and all of the above constructions have parallels in this case.