Quantization of Poisson-Lie groups and a little bit beyond

Pavol Ševera

Joint work with Ján Pulmann

Deformation quantization problem for Hopf algebras

Ingredients

- a *commutative* Hopf algebra $(\mathcal{H}, m_0, \Delta_0, S_0, 1, \epsilon)$
- a compatible Poisson bracket $\{,\} : \mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$
  ($\Delta_0 : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ is a Poisson algebra morphism)

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The problem

Find "universal" (functorial) deformations

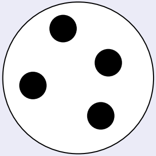
$$m_\hbar = \sum_{n=0}^{\infty} \hbar^n m_n \qquad \Delta_\hbar = \sum_{n=0}^{\infty} \hbar^n \Delta_n \qquad S_\hbar = \sum_{n=0}^{\infty} \hbar^n S_n$$

s.t. $(\mathcal{H}, m_\hbar, \Delta_\hbar, S_\hbar, 1, \epsilon)$ is a Hopf algebra and $m_1 - m_1^{op} = \{,\}$
[For $\mathcal{H} = (U\mathfrak{g})^*$: Etingof-Kazhdan 1995]

The method in a nutshell: holonomies on a surface

(not supposed to be understandable at this point)

Hopf holonomies on a disk . . .



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$H^1(\bigcirc, \bullet; G) \cong G^{n-1}$

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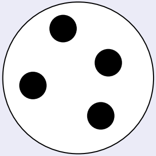

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$H^1(\bigcirc, \bullet; G) \cong G^{n-1}$ generalizes to

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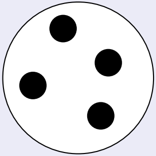

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Move the black disks $\rightsquigarrow$ B_n acts on $H_1(\bigcirc, \bullet; \mathcal{H})$

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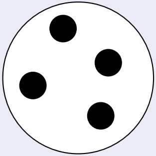

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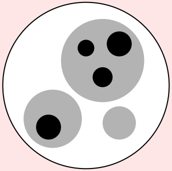

provided we know the maps (for nested disks)

$H_1(◍, ●; \mathcal{H}) \to H_1(○, ●; \mathcal{H}) \to H_1(○, ◍; \mathcal{H})$

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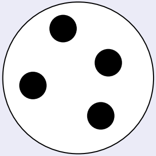

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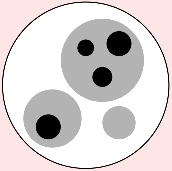

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Quantization: obtain the B_n action via the KZ connection (or from a Drinfeld associator)

The nerve of a group G

holonomies in the "commutative world"

X a finite set

$$F(X) = \{g : X \times X \to G \mid g_{ij} g_{jk} = g_{ik} \ \& \ g_{ii} = 1 \, (\forall i, j, k \in X)\}$$

$$F(X) \cong G^{|X|-1}, \text{ e.g. } \quad \bullet \xrightarrow{g_{12}} \bullet \xrightarrow{g_{23}} \bullet \xrightarrow{g_{34}} \bullet \quad (|X| = 4)$$

functoriality: $f : X \to Y \quad \rightsquigarrow \quad f^* : F(Y) \to F(X)$

$$F : \mathsf{FinSet}^{op} \to \mathsf{Set}$$

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From a nerve to its group

If F is the nerve of G then $G = F(\bullet\bullet)$

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If F is the nerve of G then $G = F(\bullet\bullet)$

F is a nerve iff $F(\bullet^n) \to F(\bullet\bullet)^{n-1}$ is a bijection

The product: $F(\bullet\bullet) \times F(\bullet\bullet) \cong F(\bullet\bullet\bullet) \to F(\bullet\bullet)$

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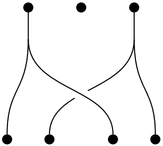
Colliding braids and Hopf algebras

BrSet - “braided maps”:



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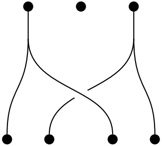

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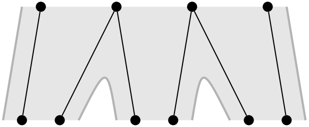
$F(\bullet^4)$



$F(\bullet\bullet)^3$

Theorem (The nerve of a Hopf algebra)

Hopf algebras (with invertible S) in a BMC $\mathcal{C}$ are equivalent to braided lax-monoidal functors $F : \mathrm{BrSet} \to \mathcal{C}$ such that $F(\bullet\bullet)^{n-1} \to F(\bullet^n)$ is an iso and $1_{\mathcal{C}} \to F() \to F(\bullet)$ are isos

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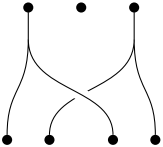

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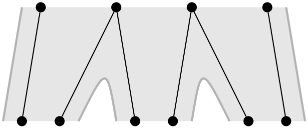

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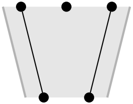
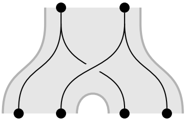
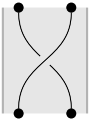
$\mathcal{H} = F(\bullet\bullet)$, $\Delta =$  , $m =$  , $S =$ 

Hopf holonomies at last

Constructing the nerve of a Hopf algebra

a Hopf algebra $\mathcal{H} \in \mathcal{C} \quad \rightsquigarrow \quad$ a functor $F : \mathsf{BrSet} \to \mathcal{C}$

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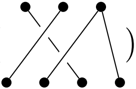
$F\Big($  $\Big) : \mathcal{H}^3 \to \mathcal{H}^2$ is

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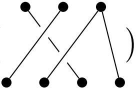
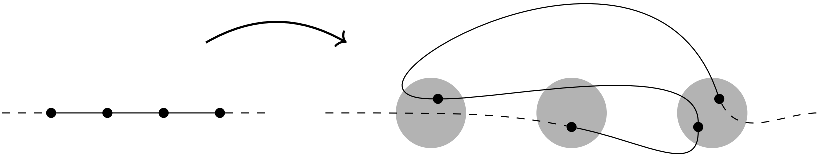
$F($  $) : \mathcal{H}^3 \to \mathcal{H}^2$ is



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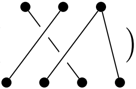

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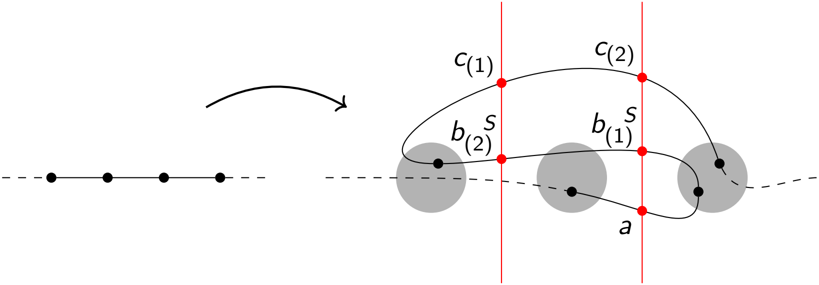

$a \otimes b \otimes c \quad \mapsto \quad b_{(2)}^{S} c_{(1)} \otimes a b_{(1)}^{S} c_{(2)}$

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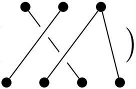

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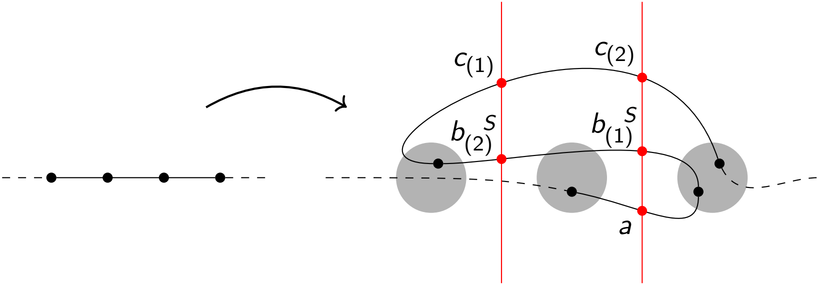

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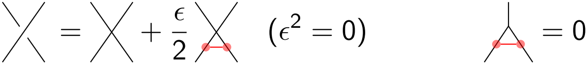
F is braided lax monoidal:

$F(\bullet^m)F(\bullet^n) = \mathcal{H}^{m-1}\mathcal{H}^{n-1} \to F(\bullet^{m+n-1}) = \mathcal{H}^{m+n-1}$: inserting 1

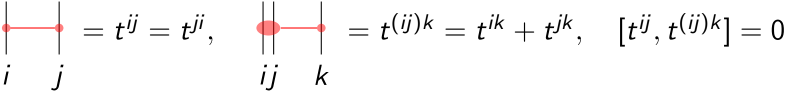
The semiclassical picture: FinSet + chord diagrams

Poisson Hopf algebras in terms of infinitesimal braids

ChordSet, the infinitesimally braided version of FinSet/BrSet:



$$= + \frac{\epsilon}{2} \quad (\epsilon^2 = 0) \qquad = 0$$



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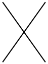

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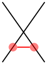

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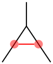

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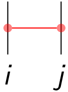

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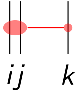

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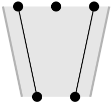

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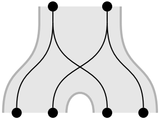

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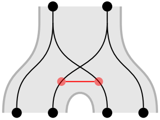


Quantization: KZ connection and associators

KZ connection becomes Gauss-Manin connection

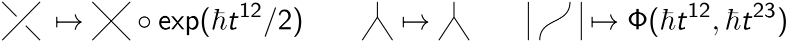
Knizhnik-Zamolodchikov connection

$$A_n^{KZ} = \hbar \sum_{1 \leq i < j \leq n} t^{ij} \frac{d(z_i - z_j)}{z_i - z_j} \qquad dA_n^{KZ} + [A_n^{KZ}, A_n^{KZ}]/2 = 0$$

Quantization of Poisson Hopf algebras

$$\begin{tikzcd}
\text{BrSet} \arrow[r, "P\exp\int A^{KZ}"] \arrow[rr, bend right, "\text{Hopf}"'] & \text{ChordSet} \arrow[r, "\text{Poisson Hopf}"] & \mathcal{C}
\end{tikzcd}$$

Better and easier: parenthesize the objects of BrSet, define the functor (Pa)BrSet → ChordSet via



$\mapsto \circ \exp(\hbar t^{12}/2) \qquad \mapsto \qquad \mapsto \Phi(\hbar t^{12}, \hbar t^{23})$

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(I guess it's still not understandable)

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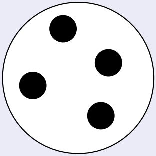

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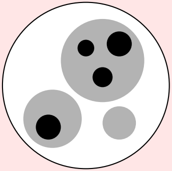

provided we know the maps (for nested disks)

$$H_1(◍, ●; \mathcal{H}) \xrightarrow[\text{(mon.)}]{} H_1(○, ●; \mathcal{H}) \xrightarrow[(F)]{} H_1(○, ◍; \mathcal{H})$$

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Little bit beyond 1: Easy Poisson groupoids

or glorified quantization of twists

Groupoids have nerves, too - which Poisson structures on Lie groupoids can we quantize via BrSet $\to$ ChordSet?

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Easy (or semi-commutative) Poisson groupoids

a Lie groupoids $\Gamma \rightrightarrows M$ with a Poisson structure on Γ
such that $\Gamma_{x,y} \subset \Gamma$ is a Poisson submanifold $\forall x, y \in M$
and s.t. the composition $\Gamma_{x,y} \times \Gamma_{y,z} \to \Gamma_{x,z}$ is a Poisson map

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a Lie groupoids $\Gamma \rightrightarrows M$ with a Poisson structure on Γ
such that $\Gamma_{x,y} \subset \Gamma$ is a Poisson submanifold $\forall x, y \in M$
and s.t. the composition $\Gamma_{x,y} \times \Gamma_{y,z} \to \Gamma_{x,z}$ is a Poisson map

A braided lax-monoidal functor $F : \mathrm{FinSet}, \mathrm{ChordSet}, \mathrm{BrSet} \to \mathcal{C}$
s.t. $F(\bullet\bullet) \otimes_{F(\bullet)} F(\bullet\bullet) \otimes_{F(\bullet)} \cdots \otimes_{F(\bullet)} F(\bullet\bullet) \to F(\bullet^n)$ is an iso

F is equivalent to a semi-commutative Hopf algebroid

Commutative algebra $B = F(\bullet)$, Poisson/NC algebra $A = F(\bullet\bullet)$,
$\epsilon : A \to B$ (units*), central maps $\eta_{L,R} : B \rightrightarrows A$ (source*,target*),
coassociative $\Delta : A \to A \otimes_B A$ (composition*), antipode $S : A \to A$

Little bit beyond 2: Braided Hopf algebras/oids

Braided Hopf algebras/oids

F : ChordSet $\to \mathcal{C}$ with $\mathcal{C}$ infinitesimally braided
$\rightsquigarrow$ quantization of quasi-Poisson groups/groupoids

Example (Manin quadruples)

$(\mathfrak{d}, \mathfrak{g}, \mathfrak{h}, \mathfrak{h}^*)$: $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{h} \oplus \mathfrak{h}^*$ as a vector space,
$\mathfrak{h}^\perp = \mathfrak{g} \oplus \mathfrak{h}$, ${\mathfrak{h}^*}^\perp = \mathfrak{g} \oplus \mathfrak{h}^*$

$C^\infty(H)$ is Poisson-Hopf in the iBMC $\mathcal{C} = U\mathfrak{g}$-Mod
(H is $\mathfrak{g}$-quasi-Poisson, $H \circledast H \to H$ is quasi-Poisson)

Quantization to a Hopf algebra in $\mathcal{C} = U\mathfrak{g}\text{-Mod}_\hbar^\Phi$

Farther beyond . . . maybe one day

(Every talk should mention higher structures)

Higher groupoids

A symmetric lax monoidal functor $F : \mathsf{FinSet} \to \mathcal{C}$
$=$ a functor $F : \mathsf{FinSet} \to \mathsf{CommAlg}(\mathcal{C})$
$=$ the algebra of functions on (the nerve of) a higher groupoid

Farther beyond . . . maybe one day

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“Poisson” structures

What is a braided lax monoidal functor $F : \mathrm{ChordSet} \to \mathcal{C}$*?*

$F(\text{a chord}) : F(X) \to F(X)$: *a second order differential operator*
($\Rightarrow$ a Poisson structure on $F(X)$, but more than that)

- What kind of “Poisson” structures are on the corresponding L_∞-algebras?
- *What kind of objects are* $F : \mathrm{BrSet} \to \mathcal{C}$*?*

Farther beyond . . . maybe one day

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THANKS!