# Calculus on symplectic and conformal Fedosov manifolds

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10 September, 2018
INdAM, Roma

# The structure of the lecture

# the original motivation: Eastwood – Goldschmidt

Eastwood, M.; Goldschmidt, H., Zero-energy fields on complex projective space. J. Differential Geom. 94 (2013), pp. 129-157.
$\mathbb{CP}_n$ comes with nice structures:

| | | | |
|---|---|---|---|
| Riemannian | $g_{ab}$ | Fubini-Study metric | $g_{ab} = J_a{}^c J_{bc}$ |
| complex | $J_a{}^b$ | complex structure | $J_a{}^b = g^{bc} J_{ac}$ |
| symplectic | $J_{ab}$ | Kähler form | $J_{ab} = J_a{}^c g_{bc}$ |

- symplectic form and Levi Civita connection are nicely linked.
- Special complexes of operators allow for strong theorems.
- The complexes are longer than the usual de Rahm complex.
- There are very few Kähler manifolds with the Ricci type holonomy (as symplectic manifold), see Proposition 4.3 in the paper on the c-projective geometry by Calderbank et al, arxiv1512.04516.

# The $\mathbb{CP}_n$ example

In dimension 4, $TM = \overset{0}{\times}\!-\!\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\bullet}$ $\quad \Lambda^1 = \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\bullet}$ $\quad \Lambda^2_\perp = \overset{-3}{\times}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{1}{\bullet}$.

$\mathbb{T} = \overset{1}{\times}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \oplus \overset{-1}{\times}\!-\!\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \oplus \overset{-1}{\times}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet}$

In particular the Rumin-Seshadri complex is (either at the contactization or pushed down)

$$0 \to \overset{1}{\times}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \xrightarrow{\nabla^2} \overset{-3}{\times}\!-\!\overset{2}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \xrightarrow{\nabla} \overset{-4}{\times}\!-\!\overset{1}{\bullet}\!\Leftarrow\!\overset{1}{\bullet} \xrightarrow{\nabla^2} \overset{-6}{\times}\!-\!\overset{1}{\bullet}\!\Leftarrow\!\overset{1}{\bullet} \xrightarrow{\nabla} \overset{-7}{\times}\!-\!\overset{2}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \xrightarrow{\nabla^2} \overset{-7}{\times}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \to 0.$$

Similarly, the initial portion

$$\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \xrightarrow{\nabla} \overset{-2}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \xrightarrow{\nabla^2} \overset{-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{2}{\bullet}\!-\cdots-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet}$$

on $\mathbb{CP}_n$ appears in the Eastwood-Goldschmidt paper, where it is shown that the second operator provides exactly the integrability conditions for the range of the Killing operator on $\mathbb{CP}_n$.
This means that the latter complex is exact at this point. We shall make this conclusion immediate consequence of the fact that $\mathbb{CP}_n$ is simply-connected.

Together with Michael Eastwood, we tried to understand the structure of the operators in general. After years, we updated the incomplete preprint:

*Conformally Fedosov manifolds*, (2016) arxiv1210.5597, 28 pp.

and added a new one:

*Calculus on symplectic manifolds*, (2017), to appear in Archivum Mathematicum, arxiv:1709.03059, 17pp.

Similarly to the parabolic tractor calculi, we couple the full analog of the Rumin complex with non-trivial representations and mimic the BGG machinary directly.

# Parallel development: Čap – Salač

Andreas Čap; Tomáš Salač, Pushing down the Rumin complex to conformally symplectic quotients, Diff. Geom. Appl., 35 (2014), 255-265.

- Contact manifold $M_\sharp$ together with a transversal infinitesimal automorphism $\xi$ provides a conformally symplectic structure on the quotient $M$.
- The Rumin complex on $M_\sharp$ can be pushed down to $M$.

A lot of further development in recent papers by Čap and Salač:
*Parabolic conformally symplectic structures I: Definition and distinguished connections.* Forum Math. 30 (2018), no. 3, 733-751.
*Parabolic conformally symplectic structures II: Parabolic contactification.* Ann. Mat. Pura Appl. (4) 197 (2018), no. 4, 1175-1199.
*Parabolic conformally symplectic structures III; Invariant differential operators and complexes.* arXiv:1701.01306, 36pp.

# Conformally symplectic manifolds

*A conformally symplectic* manifold is an even-dimensional manifold $M$ of dimension at least four equipped with a non-degenerate 2-form $J$ such that

$$dJ = 2\alpha \wedge J$$

for some closed 1-form $\alpha$. It is called the *Lee form* and it is automatically closed in dimensions $m \geq 6$.

If we rescale $\hat{J} = \Omega^2 J$ by a positive smooth function, then the existence of the Lee form remains valid with $\alpha$ replaced by $\hat{\alpha} = \alpha + \Upsilon$ for $\Upsilon \equiv d \log \Omega$.

## Definition (Reformulation)

A *conformally symplectic* manifold is a pair $(M, [J])$ where $[J]$ is an equivalence class of non-degenerate 2-forms with existing Lee forms, where $J$ and $\hat{J}$ are said to be equivalent if and only if $\hat{J} = \Omega^2 J$ for some positive smooth function $\Omega$.

# symplectically flat connections

## Definition

we say that a connection $\nabla_a$ on a given smooth vector bundle $E$ over a conformally symplectic manifold $(M, [J])$ is *symplectically flat* if and only if

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\sigma = 2J_{ab}\Theta\sigma$$

for some endomorphism $\Theta$ of $E$.

(As usual, one chooses an arbitrary torsion-free connection on $\Lambda^1$ to define the left hand side, which then does not depend on this choice.)
Evidently, if $J_{ab}$ is replaced by $\hat{J}_{ab} = \Omega^2 J_{ab}$, then symplectic flatness persists with $\Theta$ replaced by $\hat{\Theta} = \Omega^{-2}\Theta$.

# the pushed down Rumin comples

There are several ways to find the elliptic complex

$$\begin{array}{ccccccccccccccc}
0 & \rightarrow & \Lambda^0 & \xrightarrow{d-2\alpha} & \Lambda^1 & \rightarrow & \Lambda^2_\perp & \rightarrow & \Lambda^3_\perp & \rightarrow & \cdots & \rightarrow & \Lambda^n_\perp \\
 & & & & & & & & & & & & \downarrow \\
0 & \leftarrow & \Lambda^0 & \longleftarrow & \Lambda^1 & \leftarrow & \Lambda^2_\perp & \leftarrow & \Lambda^3_\perp & \leftarrow & \cdots & \leftarrow & \Lambda^n_\perp
\end{array}$$

on a conformally symplectic manifold, where all operators are first order except for the middle operator, which is second order.
(Here $\Lambda^k_\perp$ denotes the bundle of $k$-forms that are trace-free with respect to $J$.)
Notice, the length of such a complex is by one longer than that of the de Rham complex.
**For symplectically flat connections $\nabla_a$ on $E$, our first aim is to construct a version of the above complex coupled to $E$.**

The operator

$$D_a = \nabla_a - 2\alpha_a : E \to \Lambda^1 \otimes E$$

is a connection whose curvature is again

$$(D_a D_b - D_b D_a)\sigma = (\nabla_a \nabla_b - \nabla_b \nabla_a)\sigma = 2J_{ab}\Theta\sigma.$$

and it is quite clear how to continue:

$$E \xrightarrow{\nabla - 2\alpha\otimes\mathrm{Id}} \Lambda^1 \otimes E \longrightarrow \Lambda^2_\perp \otimes E,$$

where $\Gamma(\Lambda^1 \otimes E) \ni \varphi_a \mapsto \nabla_{[a}\varphi_{b]} - 2\alpha_{[a}\varphi_{b]}$ mod $J_{ab}$

**Lemma**

*The endomorphism $\Theta : E \to E$ has constant rank.*

**Proof.**

We may choose an auxiliary connection on $M$ and fix $J$ to be covariantly constant. Then the Bianchi identity for $\nabla_a$ implies $0 = \nabla_{[a}(J_{bc]}\Theta) = J_{[bc}\nabla_{a]}\Theta$. $\square$

Thus we may consider the bundles $\ker\Theta$ and $\operatorname{coker}\Theta = E/\operatorname{im}\Theta$. Remarkably, the connection $D_a$ provides a flat connection on both. We shall write $\underline{\ker\Theta}$ and $\underline{\operatorname{coker}\Theta}$ for the sheaf of germs of covariantly constant sections of the bundles, respectively.

## Lemma

*There is a natural elliptic complex:*

$$\begin{tikzcd}
E \arrow[r, "D"] \arrow[ddr] & \Lambda^1 \otimes E \arrow[r, "D"] \arrow[ddr] & \Lambda^2 \otimes E \arrow[r, "D"] \arrow[ddr] & \Lambda^3 \otimes E \arrow[r, "D"] \arrow[ddr] & \Lambda^4 \otimes E & \\
 & \oplus & \oplus & \oplus & \oplus & \cdots \\
 & E \arrow[r] \arrow[uur] & \Lambda^1 \otimes E \arrow[r] \arrow[uur] & \Lambda^2 \otimes E \arrow[r] \arrow[uur] & \Lambda^3 \otimes E &
\end{tikzcd}$$

*where the differentials are given by*

$$\sigma \mapsto \begin{bmatrix} D\sigma \\ \Theta\sigma \end{bmatrix} \quad \begin{bmatrix} \varphi \\ \eta \end{bmatrix} \mapsto \begin{bmatrix} D\varphi - J \otimes \eta \\ D\eta - \Theta\varphi \end{bmatrix} \quad \begin{bmatrix} \omega \\ \psi \end{bmatrix} \mapsto \begin{bmatrix} D\omega + J \wedge \psi \\ D\psi + \Theta\omega \end{bmatrix} \cdots$$

*It is locally exact, except for the zeroth and first cohomologies which may be identified with* $\underline{\ker \Theta}$ *and* $\underline{\operatorname{coker} \Theta}$*, respectively.*

## Theorem (The coupled Rumin–Seshadri complex)

*Suppose $(M,[J])$ is a conformally symplectic manifold and $\nabla_a$ is a symplectically flat connection on a vector bundle $E$ over $M$. Choose $J_{ab}\in[J]$ and the appropriate $\Theta : E\to E$. Then there is a natural elliptic complex*

$$\begin{array}{ccccccccccccc} 0 & \to & E & \to & \Lambda^1\otimes E & \to & \Lambda^2_\perp\otimes E & \to & \cdots & \to & \Lambda^n_\perp\otimes E \\ &&&&&&&&&& \downarrow \\ 0 & \leftarrow & E & \leftarrow & \Lambda^1\otimes E & \leftarrow & \Lambda^2_\perp\otimes E & \leftarrow & \cdots & \leftarrow & \Lambda^n_\perp\otimes E \end{array}$$

*where all operators are first order save for the middle operator, which is second order. This differential complex is locally exact save for its zeroth and first cohomologies, which may be identified with $\underline{\ker\Theta}$ and $\underline{\operatorname{coker}\Theta}$, respectively.*

## short proof

Rearranging the complex from the main Lemma as

$$\begin{tikzcd}
E \arrow[r] \arrow[drr] & \Lambda^1 \otimes E \arrow[r] & \Lambda^2 \otimes E \arrow[r] \arrow[dr] & \Lambda^3 \otimes E \arrow[r] \arrow[dr] & \Lambda^4 \otimes E \arrow[r] \arrow[dr] & \cdots \\
 & & E \arrow[u] \arrow[r] & \Lambda^1 \otimes E \arrow[u] \arrow[r] & \Lambda^2 \otimes E \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

one sees a filtered complex, the spectral sequence of which has as its $E_1$-level

$$\begin{array}{ll}
E \to \Lambda^1 \otimes E \to \Lambda^2_\perp \otimes E \to \cdots \to \Lambda^n_\perp \otimes E & 0 \\
 & 0 \quad \Lambda^n_\perp \otimes E \to \cdots \to \Lambda^2_\perp \otimes E \to \Lambda^1 \otimes E \to E.
\end{array}$$

Passing to the $E_2$-level constructs the requested complex and main Lemma gives its cohomology.

# Projective class of connections

A *projective structure* on a manifold $M$ is an equivalence class of torsion-free affine connections on $M$, where two connections $\nabla_a$ and $\hat{\nabla}_a$ are said to be projectively equivalent if and only if

$$\hat{\nabla}_a \varphi_b = \nabla_a \varphi_b - \nu_a \varphi_b - \nu_b \varphi_a$$

for some 1-form $\nu_a$.
If $J_{ab}$ is skew, then $\hat{\nabla}_{(a} J_{b)c} = \nabla_{(a} J_{b)c} - 3\nu_{(a} J_{b)c}$.

**Lemma**

*If $J_{ab}$ is skew, then the requirement that*

$$\nabla_{(a} J_{b)c} = \beta_{(a} J_{b)c}$$

*for some 1-form $\beta_a$ is projectively invariant.*

# The first version of Conformally Fedosov

For torsion-free $\nabla$ on a conformally symplectic manifold $(M, [J])$ we get $\nabla_{[a}J_{bc]} = 2\alpha_{[a}J_{bc]}$. Let us insist on $\nabla_{(a}J_{b)c} = \beta_{(a}J_{b)c}$.

A *conformally Fedosov* manifold is a triple $(M, [J], [\nabla])$ where

- $M$ is a smooth manifold of dimension $2n \geq 4$,
- $[J]$ is an equivalence class of non-degenerate 2-forms defined up to rescaling $J \mapsto \hat{J} = \Omega^2 J$ for some positive function $\Omega$,
- $[\nabla]$ is a projective structure, i.e. an equivalence class of torsion-free connections defined up to projective change for some 1-form $\nu_a$,
- the following equations hold

$$\nabla_{[a}J_{bc]} = 2\alpha_{[a}J_{bc]} \quad \nabla_{[a}\alpha_{b]} = 0 \quad \nabla_{(a}J_{b)c} = \beta_{(a}J_{b)c} \tag{1}$$

for some 1-forms $\alpha_a$ and $\beta_a$.

**Lemma**

*Let $(M, [J], [\nabla])$ be a conformally Fedosov manifold. Any representatives $J_{ab}$ and $\nabla_a$ of the structure uniquely determine the 1-forms $\alpha_a$ and $\beta_a$ and, conversely,*

$$\nabla_a J_{bc} = 2\alpha_{[a}J_{bc]} + \tfrac{2}{3}\beta_{(a}J_{b)c} - \tfrac{2}{3}\beta_{(a}J_{c)b} \tag{2}$$

*determines the full covariant derivative $\nabla_a J_{bc}$.*

**Lemma**

*For any conformally Fedosov manifold $(M, [J], [\nabla])$, if a representative 2-form $J_{ab}$ is chosen, then there is a unique torsion-free connection in the projective class such that*

$$\nabla_a J_{bc} = 2J_{a[b}\alpha_{c]}. \tag{3}$$

An alternative definition of a conformally Fedosov manifold is as follows. Firstly, define an equivalence relation on pairs $(J, \nabla)$ consisting of a non-degenerate symplectic form $J_{ab}$ and a torsion-free connection $\nabla_a$ by allowing simultaneous replacements

$$\begin{aligned} J_{ab} &\mapsto \hat{J}_{ab} = \Omega^2 J_{ab} \\ \nabla_a \varphi_b &\mapsto \hat{\nabla}_a \varphi_b = \nabla_a \varphi_b - \Upsilon_a \varphi_b - \Upsilon_b \varphi_a, \end{aligned} \tag{4}$$

where $\Upsilon_a = \nabla_a \log \Omega$.

**Definition**

Writing $[J, \nabla]$ for the equivalence class of such pairs, a conformally Fedosov manifold may then be defined as a pair $(M, [J, \nabla])$ such that $\nabla_a J_{bc} = 2J_{a[b}\alpha_{c]}$ holds.

We can check directly that (3) is invariant under (4) if one decrees that $\alpha_a \mapsto \hat{\alpha}_a = \alpha_a + \Upsilon_a$.

# Remarks

Any conformally symplectic manifold $(M, [J])$ can be extended to a conformally Fedosov structure $(M, [J, \nabla])$.
Equation $(3)$ is equivalent to

$$\nabla_a J^{bc} = 2\alpha^{[b}\delta_a{}^{c]}, \tag{5}$$

where $\alpha^b \equiv J^{bc}\alpha_c$.
As a corollary we see, that a projective structure $[\nabla]$ cannot necessarily be extended to a conformally Fedosov structure.
Indeed, the equation (5) hold for some vector field $\alpha^a$ is equivalent to requiring that

$$\text{the trace-free part of } (\nabla_a J^{bc}) = 0,$$

which is a system of finite type. Hence, there are obstructions to its solution (and writing it as (5) is the first step in its prolongation).

Choosing any representatives for $(M, [J, \nabla])$, the curvature $R_{ab}{}^c{}_d$ of $\nabla_a$ may be uniquely written as

$$R_{ab}{}^c{}_d = W_{ab}{}^c{}_d + \delta_a{}^c \mathrm{P}_{bd} - \delta_b{}^c \mathrm{P}_{ad},$$

where $\mathrm{P}_{ab}$ is a symmetric tensor and $W_{ab}{}^c{}_d$ satisfies

$$W_{ab}{}^c{}_d = W_{[ab]}{}^c{}_d \qquad W_{[ab}{}^c{}_{d]} = 0 \qquad W_{ab}{}^a{}_d = 0.$$

Under conformal rescaling $(4)$, the tensor $W_{ab}{}^c{}_d$ is unchanged whilst

$$\hat{\mathrm{P}}_{ab} = \mathrm{P}_{ab} - \nabla_a \Upsilon_b + \Upsilon_a \Upsilon_b.$$

Furthermore, the tensor $W_{abcd}$ may be uniquely decomposed as

$$W_{abcd} = V_{abcd} - \tfrac{3}{2n-1} J_{ac} \Phi_{bd} + \tfrac{3}{2n-1} J_{bc} \Phi_{ad} + J_{ad} \Phi_{bc} - J_{bd} \Phi_{ac} + 2 J_{ab} \Phi_{cd},$$

where

$$V_{abcd} = V_{[ab](cd)} \qquad V_{[abc]d} = 0 \qquad J^{ab} V_{abcd} = 0$$

and $\Phi_{ab}$ is symmetric.

# Back to $\mathbb{CP}^n$

The curvature of $\mathbb{CP}_n$ with its standard Fubini-Study metric is given by

$$R_{abcd} = g_{bd}J_{ac} - g_{ad}J_{bc} - g_{ac}J_{bd} + g_{bc}J_{ad} + 2J_{ab}g_{cd}$$

and one easily computes that

$$\mathsf{P}_{ab} = \tfrac{2(n+1)}{2n-1}g_{ab} \qquad \Phi_{ab} = g_{ab} \qquad V_{abcd} = 0.$$

# Fedosov gauge

It is often convenient locally to work in a gauge in which $\alpha_a = 0$ for then $\nabla_a J_{bc} = 0$ and the curvature $R_{abcd}$ decomposes in a more simple way into three components $\mathrm{Sp}(2n, \mathbb{R})$-irreducible parts,

$$V_{abcd} \in \overset{2}{\bullet}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\Leftarrow\overset{0}{\bullet} \quad \Phi_{ab} \in \overset{2}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\Leftarrow\overset{0}{\bullet} \quad \mathrm{P}_{ab} \in \overset{2}{\bullet}\!-\!\overset{0}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\Leftarrow\overset{0}{\bullet}$$

according to

$$R_{abcd} = V_{abcd} + 2J_{ab}\Phi_{cd} - 2\Phi_{c[a}J_{b]d} + \frac{6}{2n-1}J_{c[a}\Phi_{b]d} - 2J_{c[a}\mathrm{P}_{b]d}$$

with

$$(2n-1)\mathrm{P}_{ab} = 2(n+1)\Phi_{ab}.$$

We shall refer to a choice of pair $(J_{ab}, \nabla_a)$ from a conformally Fedosov structure $[J_{ab}, \nabla_a]$ for which $\nabla_a J_{bc} = 0$ as a *Fedosov gauge*. This is in accordance with the notion of Fedosov manifold.

# The Kähler case

Now, the Fedosov gauge $\nabla_a$ is the Levi-Civita connection of a metric $g_{ab}$ and $J_a{}^b \equiv J_{ac}g^{bc}$ is an almost complex structure on $M$ whose integrability is equivalent to the vanishing of $\nabla_a J_{bc}$.
The curvature decomposes as follows:

$$
\begin{aligned}
R_{ab}{}^c{}_d &= U_{ab}{}^c{}_d \\
&\quad + \delta_a{}^c \Xi_{bd} - \delta_b{}^c \Xi_{ad} - g_{ad}\Xi_b{}^c + g_{bd}\Xi_a{}^c \\
&\quad\quad + J_a{}^c \Sigma_{bd} - J_b{}^c \Sigma_{ad} - J_{ad}\Sigma_b{}^c + J_{bd}\Sigma_a{}^c + 2J_{ab}\Sigma^c{}_d + 2J^c{}_d\Sigma_{ab} \\
&\quad\quad\quad + \Lambda(\delta_a{}^c g_{bd} - \delta_b{}^c g_{ad} + J_a{}^c J_{bd} - J_b{}^c J_{ad} + 2J_{ab}J^c{}_d),
\end{aligned}
$$

where indices have been raised using $g^{ab}$ and

- $U_{ab}{}^c{}_d$ is totally trace-free with respect to $g^{ab}$, $J_a{}^b$, and $J^{ab}$,
- $\Xi_{ab}$ is trace-free symmetric
- $\Sigma_{ab} \equiv J_a{}^c \Xi_{bc}$ is skew.

Consequently,

$$R_{bd} \equiv R_{ab}{}^{a}{}_{d} = 2(n+2)\Xi_{bd} + 2(n+1)\Lambda g_{bd}$$

$$\Phi_{ab} = \tfrac{n+2}{n+1}\Xi_{ab} + \Lambda g_{ab}.$$

$$\begin{aligned} J_c{}^a R_{ab}{}^c{}_d &= J_c{}^a V_{ab}{}^c{}_d - J_{bd}\Phi_a{}^a - 2J_b{}^a\Phi_{da} \\ &= J_c{}^a V_{ab}{}^c{}_d - 2\tfrac{n+2}{n+1}\Sigma_{bd} - 2(n+1)\Lambda J_{bd}. \end{aligned}$$

$$J_c{}^a V_{ab}{}^c{}_d - 2\tfrac{n+2}{n+1}\Sigma_{bd} = -2(n+2)\Sigma_{bd}$$

and we have established:

**Lemma**

*Concerning the symplectic curvature decomposition on a Kähler manifold,*

$$J_c{}^a V_{ab}{}^c{}_d = -2\tfrac{n(n+2)}{n+1}\Sigma_{bd}.$$

# Conformal Tractors

The *standard tractor bundle* $\mathbb{T}$ on a conformal Riemannian manifold is defined in the presence of a chosen metric $g_{ab}$ to be the direct sum

$$\mathbb{T} = \Lambda^0[1] \oplus \Lambda^1[1] \oplus \Lambda^0[-1]$$

but if the metric is rescaled as $\hat{g}_{ab} = \Omega^2 g_{ab}$, then this decomposition is mandated to change according to

$$\begin{bmatrix} \hat{\sigma} \\ \hat{\mu}_b \\ \hat{\rho} \end{bmatrix} = \begin{bmatrix} \sigma \\ \mu_b + \Upsilon_b \sigma \\ \rho - \Upsilon^b \mu_b - \frac{1}{2} \Upsilon^b \Upsilon_b \sigma \end{bmatrix}, \quad \text{where } \Upsilon_a \equiv \nabla_a \log \Omega.$$

For a chosen metric $g_{ab}$ in the conformal class, the *tractor connection* can be computed or defined by

$$\nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a \sigma - \mu_a \\ \nabla_a \mu_b + g_{ab}\rho + \mathsf{P}_{ab}\sigma \\ \nabla_a \rho - \mathsf{P}_a{}^b \mu_b \end{bmatrix},$$

where $\nabla_a \mu_b$ is the Levi-Civita connection of $g_{ab}$.

We shall proceed analogously for the conformally Fedosov manifolds now.

# (Conformally) symplectic tractors

For chosen representatives, the vector bundle $\mathbb{T}$ is defined as

$$\mathbb{T} = \Lambda^0[1] \oplus \Lambda^1[1] \oplus \Lambda^0[-1]$$

but this splitting is decreed to change as

$$\begin{bmatrix} \hat{\sigma} \\ \hat{\mu}_b \\ \hat{\rho} \end{bmatrix} = \begin{bmatrix} \sigma \\ \mu_b + \Upsilon_b \sigma \\ \rho - \Upsilon^b \mu_b + \Upsilon^b \alpha_b \sigma \end{bmatrix} \tag{6}$$

under (4), where $\alpha_a$ is defined by (3). A direct check reveals that this decree is self-consistent.

There is a non-degenerate skew form defined on $\mathbb{T}$ by

$$\left\langle \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix}, \begin{bmatrix} \tilde{\sigma} \\ \tilde{\mu}_c \\ \tilde{\rho} \end{bmatrix} \right\rangle = \sigma\tilde{\rho} - J^{bc}\mu_b\tilde{\mu}_c - \rho\tilde{\sigma} = \sigma\tilde{\rho} + \mu^b\tilde{\mu}_b - \rho\tilde{\sigma}. \quad (7)$$

Let us first consider the connection $D_a$ on $\mathbb{T}$ defined by

$$D_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} = \begin{bmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu_b - J_{ab}\rho + \mathrm{P}_{ab}\sigma - J_{ab}\alpha^c\mu_c \\ \nabla_a\rho - \mathrm{P}_a{}^b\mu_b - \alpha^b(2\mathrm{P}_{ab} + \nabla_a\alpha_b)\sigma \end{bmatrix}.$$

This connection is well-defined, i.e. is independent of choice of representatives $(J_{ab}, \nabla_a)$, and preserves the skew form (7). (The check is straightforward but quite tedious.)

# Improving the tractor connection

The following two homomorphisms $\mathbb{T} \to \Lambda^1 \otimes \mathbb{T}$

$$\begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} \mapsto \begin{bmatrix} 0 \\ \Phi_{ab}\sigma \\ \Phi_{ab}\mu^b + 2(\nabla^b\Phi_{ab})\sigma \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 \\ 0 \\ (\nabla^b\Phi_{ab} + \alpha^a\Phi_{ab})\sigma \end{bmatrix}$$

are invariantly defined.

Thus we can change the connection $D_a$ by appropriate multiples of these. The *tractor connection* on $\mathbb{T}$ is defined by

$$\nabla_a \begin{bmatrix} \sigma \\ \mu_b \\ \rho \end{bmatrix} \equiv \begin{bmatrix} \nabla_a\sigma - \mu_a \\ \nabla_a\mu_b - J_{ab}\rho + \mathrm{P}_{ab}\sigma - \frac{3}{2n-1}\Phi_{ab}\sigma - J_{ab}\alpha^c\mu_c \\ \nabla_a\rho + \mathrm{P}_{ab}\mu^b - \frac{3}{2n-1}\Phi_{ab}\mu^b - \frac{1}{2n+1}(\nabla^b\Phi_{ab})\sigma \\ \qquad - (2\alpha^b\mathrm{P}_{ab} + \alpha^b\nabla_a\alpha_b - \frac{10n+7}{(2n+1)(2n-1)}\alpha^b\Phi_{ab})\sigma \end{bmatrix}$$

# Curvature

The tractor connection preserves the skew form $(7)$ and, in the Fedosov gauge, its curvature is given by

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\begin{bmatrix}\sigma\\ \mu_c\\ \rho\end{bmatrix} = \begin{bmatrix}0\\ V_{abcd}\mu^d + Y_{abc}\sigma\\ Y_{abc}\mu^c - \frac{1}{2n}(\nabla^c Y_{abc} - V_{abce}\Phi^{ce})\sigma\end{bmatrix}$$
$$-2J_{ab}\begin{bmatrix}\rho\\ S_c\sigma - \Phi_{cd}\mu^d\\ S_c\mu^c - \frac{1}{2n}(\Phi_{de}\Phi^{de} + \nabla^c S_c)\sigma\end{bmatrix}$$

Here the quantity $Y_{abc}$ stays for the gradient of $V_{abcd}$, while $(2n+1)S_a$ is the gradient of $\Phi_{ab}$.

**Theorem**

*The curvature of the tractor connection has the form*

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)\Sigma = 2J_{ab}\Theta\Sigma$$

*for some endomorphism $\Theta$ of $\mathbb{T}$ if and only if $V_{abcd} \equiv 0$.*

**Theorem**

*If $V_{abcd} = 0$, then*

$$(\nabla_a \Phi^{bc})_\circ = 0$$

*in Fedosov gauge, where $(\ )_\circ$ means to take the trace-free part.*

**Theorem**

*The symplectic tractor connection on a Kähler manifold is symplectically flat if and only if the metric has constant holomorphic sectional curvature.*

## Theorem (The coupled Rumin–Seshadri complex)

*Suppose $(M, [\nabla, J])$ is a conformally symplectic manifold with the curvature $V_{abcd}$ vanishing, $\nabla_a$ be the symplectically flat connection on any vector bundle $E$ over $M$ induced by the standard tractor bundle. Then there is a natural elliptic complex*

$$\begin{array}{ccccccccccc} 0 & \rightarrow & E & \rightarrow & \Lambda^1 \otimes E & \rightarrow & \Lambda^2_\perp \otimes E & \rightarrow & \cdots & \rightarrow & \Lambda^n_\perp \otimes E \\ & & & & & & & & & & \downarrow \\ 0 & \leftarrow & E & \leftarrow & \Lambda^1 \otimes E & \leftarrow & \Lambda^2_\perp \otimes E & \leftarrow & \cdots & \leftarrow & \Lambda^n_\perp \otimes E \end{array}$$

*where all operators are first order save for the middle operator, which is second order. This differential complex is locally exact save for its zeroth and first cohomologies, which may be identified with $\underline{\ker \Theta}$ and $\underline{\operatorname{coker} \Theta}$, respectively, where $\Theta$ is the endomorphism induced from the curvature of the tractor connection.*

## Theorem

*Suppose $(M, [J, \nabla])$ is a conformally Fedosov manifold of dimension $2n$ whose invariant curvature $V_{abcd}$ vanishes. Then for any $n+1$ non-negative integers $a, b, c, \cdots d, e$ there is a differential complex*

$$\overset{a}{\times}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}\cdots\overset{d}{\bullet}\!\Leftarrow\!\overset{e}{\bullet} \xrightarrow{\nabla^{a+1}} \overset{-a-2}{\times}\!-\!\overset{a+b+1}{\bullet}\!-\!\overset{c}{\bullet}\cdots\overset{d}{\bullet}\!\Leftarrow\!\overset{e}{\bullet}$$
$$\xrightarrow{\nabla^{b+1}} \overset{-a-b-3}{\times}\!-\!\overset{a}{\bullet}\!-\!\overset{b+c+1}{\bullet}\cdots\overset{d}{\bullet}\!\Leftarrow\!\overset{e}{\bullet} \xrightarrow{\nabla^{c+1}} \cdots,$$

*which is locally exact save at the 0th and 1st positions, where its local cohomology may be identified with the locally constant sheaves $\underline{\ker\Theta}$ and $\underline{\operatorname{coker}\Theta}$, respectively.*

Here, $\Theta \in \operatorname{Aut}\left(\overset{a}{\bullet}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}\cdots\overset{d}{\bullet}\!\Leftarrow\!\overset{e}{\bullet}(\mathbb{T})\right)$ is induced by $\Theta : \mathbb{T} \to \mathbb{T}$ and $\overset{a}{\bullet}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}\cdots\overset{d}{\bullet}\!\Leftarrow\!\overset{e}{\bullet}(\mathbb{T})$ is the bundle associated to $\mathbb{T}$ via the $\mathrm{Sp}(2n+2, \mathbb{R})$-module $\overset{a}{\bullet}\!-\!\overset{b}{\bullet}\!-\!\overset{c}{\bullet}\cdots\overset{d}{\bullet}\!\Leftarrow\!\overset{e}{\bullet}$, bearing in mind that the non-degenerate skew form (7) reduces the structure group of $\mathbb{T}$ to $\mathrm{Sp}(2n+2, \mathbb{R})$.

# A few examples

In dimension 4, $TM = \overset{0}{\times}\!-\!\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\bullet}$ $\quad \Lambda^1 = \overset{-2}{\times}\!-\!\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\bullet}$ $\quad \Lambda^2_\perp = \overset{-3}{\times}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{1}{\bullet}$.

$\mathbb{T} = \overset{1}{\times}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \oplus \overset{-1}{\times}\!-\!\overset{1}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \oplus \overset{-1}{\times}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet}$

In particular the Rumin-Seshadri complex is

$$0 \to \overset{1}{\times}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \xrightarrow{\nabla^2} \overset{-3}{\times}\!-\!\overset{2}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \xrightarrow{\nabla} \overset{-4}{\times}\!-\!\overset{1}{\bullet}\!\Leftarrow\!\overset{1}{\bullet} \xrightarrow{\nabla^2} \overset{-6}{\times}\!-\!\overset{1}{\bullet}\!\Leftarrow\!\overset{1}{\bullet} \xrightarrow{\nabla} \overset{-7}{\times}\!-\!\overset{2}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \xrightarrow{\nabla^2} \overset{-7}{\times}\!-\!\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \to 0.$$

Similarly, the initial portion

$$\overset{0}{\times}\!-\!\overset{1}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \xrightarrow{\nabla} \overset{-2}{\times}\!-\!\overset{2}{\bullet}\!-\!\overset{0}{\bullet}\cdots\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet} \xrightarrow{\nabla^2} \overset{-4}{\times}\!-\!\overset{0}{\bullet}\!-\!\overset{2}{\bullet}\cdots\overset{0}{\bullet}\!\Leftarrow\!\overset{0}{\bullet}$$

on $\mathbb{CP}_n$ appears in the Eastwood-Goldschmidt paper, where it is shown that the second operator provides exactly the integrability conditions for the range of the Killing operator on $\mathbb{CP}_n$. This conclusion is immediate from our Theorem here: since $\mathbb{CP}_n$ is simply-connected, there is no global cohomology arising from coker $\Theta$.