The History of Calabi-Yau Metrics Revisited

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Calabi-Yau and Geometry

Conference honouring Shing Tung YAU at 70

Universita La Sapienza, Roma, 31 May 2019

Context of the Lecture

Differential Geometry has a long history, connecting it to a number of other fields, in particular Mechanics and Physics:

- In the 20^{th} century the discipline developed considerably thanks to the major contributions of Élie CARTAN, CHERN Shiing Shen, Eugenio CALABI, Erich KÄHLER, André LICHNEROWICZ, Isadore M. SINGER and Sir Michael ATIYAH, among many others;
- It underwent three major transformations:
  1. the use of bundles to formulate geometric concepts;
  2. the involvement with the theory of linear and non-linear partial differential equations to solve problems of geometric origin;
  3. an enhanced relation to Theoretical Physics, both in General Relativity and in String Theory.

In the last two of these transformations Shing Tung YAU played a major role, in particular through its epoch-making solution of the Calabi Conjecture, leading to the existence of Calabi-Yau metrics.

Outline of the Lecture

Here is an outline of the topics to be covered:

1. The Basic Setting of Kähler Geometry;
2. Towards the Calabi Conjecture;
3. When Holonomy Enters;
4. Calabi-Yau Metrics;
5. Some Concluding Remarks

The Basic Setting of Kähler Geometry

The Seminal Article by Erich KÄHLER

Erich KÄHLER's seminal 1933 article *Über eine bemerkenswerte Hermitesche Metrik* led to the development of Kähler Geometry which has grown into an essential domain of Mathematics:

1. The Basics of Kähler Geometry

Über eine bemerkenswerte Hermitesche Metrik.

Von ERICH KÄHLER in Hamburg.

1.

Bei der Untersuchung der Invarianten einer reell $2n$-dimensionalen HERMITEschen Metrik[1])

$$ds^2 = \sum g_{i\bar{k}}\, dx_i\, d\bar{x}_k \tag{1}$$

gegenüber den „pseudokonformen" Transformationen

$$\begin{aligned} x_i' &= \varphi_i(x_1, x_2, \cdots, x_n) \\ \bar{x}_i' &= \bar{\varphi}_i(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) \end{aligned} \qquad (i = 1, 2, \cdots, n) \tag{2}$$

ist es naheliegend, neben (1) die alternierende quadratische Differentialform (forme extérieure)

$$\omega = \sum g_{i\bar{k}}\, d(x_i, \bar{x}_k)$$

zu betrachten, in der $d(x_i, \bar{x}_k)$ das sog. äußere Produkt[2]) der Differentiale $dx_i, d\bar{x}_k$, also eine Differentialdeterminante $\frac{\partial(x_i, \bar{x}_k)}{\partial(s, t)}\, ds\, dt$ bedeutet. Diese invariant mit (1) verknüpfte Form ω gibt Gelegenheit, den eleganten Kalkül der symbolischen Differentialformen[3]) zur Herstellung von Invarianten zu verwenden. So ist z. B. die Ableitung

$$\omega' = \sum d(g_{i\bar{k}}, x_i, \bar{x}_k) = \sum \frac{\partial g_{i\bar{k}}}{\partial x_l}\, d(x_l, x_i, \bar{x}_k) + \frac{\partial g_{i\bar{k}}}{\partial \bar{x}_l}\, d(\bar{x}_l, x_i, \bar{x}_k)$$

bereits eine neue invariante Form, und durch Kombination von dieser mit ω ergeben sich weitere Invarianten.

The Seminal Article by Erich KÄHLER

Erich KÄHLER's seminal 1933 article *Über eine bemerkenswerte Hermitesche Metrik* led to the development of Kähler Geometry which has grown into an essential domain of Mathematics:

- It is difficult to pinpoint what made the domain so fruitful;
- As Eugenio CALABI once told me, he studied Kähler manifolds because *"elles sont vraiment simples"*;
- The subject, based on the interplay on a manifold between a Riemannian metric and a complex structure, now lies at the crossroads of many active branches of Mathematics (and of Theoretical Physics): Differential Geometry, the very rich interplay between Riemannian and Symplectic Geometries, Algebraic Geometry, Global Analysis, ...
- What is striking about this article is that, more or less every half page, Erich KÄHLER opens a new path that has later turned out to be crucial for the development of the subject.

The Basics of Kählerian Geometry

- On an m-dimensional complex manifold M one considers a triple (g, J, ω) where g is a Riemannian metric and J an almost complex structure for which g is Hermitian. For tangent vectors X and Y at a point $p \in M$, the 2-form ω defined by $\omega(X,Y) = g(JX,Y))$ is automatically skew.
- The triple is said to be Kählerian if $d\omega = 0$. Hence ω determines a cohomology class, which is peculiar because it contains a non-degenerate 2-form (making (M,ω) a symplectic manifold) corresponding, using J, to a positive definite metric, a so-called *Kähler class*.
- Already in the original article Erich KÄHLER considers the case where the class is trivial, i.e. when one can find a function u, a so-called Kähler potential, so that $\omega = i\,\partial\bar{\partial}u$.
- When the class is non-trivial, one can always consider deformations of a Kähler metric ω of the type $\tilde{\omega} = \omega + i\,\partial\bar{\partial}\phi$ with ϕ small enough so that $\tilde{\omega}$ remains positive definite.

The Ricci Curvature in Kählerian Geometry

Let us denote the Ricci curvature of the metric g by Ric_g .

- One can similarly introduce the 2-form ρ_ω by setting, for tangent vectors X and Y at a point $p \in M$, $\rho_\omega(X,Y) = \mathrm{Ric}_g(JX,Y)$ and ρ_ω is automatically skew too;
- In complex coordinates (z^α), the metric has a local expression $g = \sum_{\alpha,\beta=1}^m g_{\alpha\bar{\beta}}\, dz^\alpha\, dz^{\bar{\beta}}$ and Erich KÄHLER had already the following key formula:

$$\rho_\omega = -i\partial\bar{\partial} log(det\, g_{\alpha\bar{\beta}})\,;$$

  In modern terms, it says that ρ_ω is the curvature form of the canonical line bundle $\Lambda^m_{\mathbb{C}} T^*M \to M$, hence a closed form;
- Therefore one has ρ_ω is, up to a constant, a representative of $c_1(M)$, a fundamental fact for Kähler Geometry;
- Kähler Geometry is dominated by the question of prescribing the Ricci curvature under the constraint $[\rho_\omega] = 2\pi c_1(M)$.

2. Towards the Calabi Conjecture

Calabi's Decisive Claims

In 1953 Eugenio CALABI publishes two notes in the AMS Bulletin:

- The first one is entitled *The variation of Kähler metrics. I. The structure of the space*. Its Theorem 1 states that **the Ricci curvature of a Kähler metric can be prescribed**;
- He bases his argument on the geometry of the space of Kähler potentials: an infinite-dimensional open manifold with a complete natural metric claimed to be of positive curvature;
- The second one is entitled *The variation of Kähler metrics. II. A minimum problem* and contains two theorems:
  1. Theorem 1 states that ρ_ω is a harmonic 2-form with respect to ω if and only if the scalar curvature τ_ω of ω is constant;
  2. Theorem 2 states that, if a Kähler metric minimises the L^2 norm of ρ_ω, then $\partial\tau_\omega$ is dual to a holomorphic vector field. Therefore, if there is no non-trivial holomorphic vector fields on M, the existence of a minimum would guarantee the existence of a Kähler metric $\tilde{\omega}$ with $\tau_{\tilde{\omega}}$ constant.

Calabi's Decisive Claims (cont.)

In 1954 Eugenio CALABI presented the results at the International Congress of Mathematicians in Amsterdam. Here is the first one:

Calabi's Decisive Claims (cont.)

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THE SPACE OF KÄHLER METRICS

EUGENIO CALABI

Let M^n be a closed, n-dimensional complex manifold. We assume that M^n admits at least one Kähler metric $g_{\alpha\beta^*}$; its associated closed exterior form $\omega = \sqrt{-1}\, g_{\alpha\beta^*}\, dz^\alpha \wedge dz^{\beta^*}$ determines a real cohomology class, called the principal class of the metric. Consider the space Ω of all infinitely differentiable Kähler metrics in M^n with the same principal class; the topology of Ω is defined by the L^2 topology of the tensorial components of metrics in Ω in compact subregions of co-ordinate domains. If $R_{\alpha\beta^*}$ is the Ricci tensor of any metric in Ω, then the Ricci form $\sqrt{-1}\, R_{\alpha\beta^*}\, dz^\alpha \wedge dz^{\beta^*}$ is closed and its cohomology class is $2\pi C^{(1)}$ ($C^{(r)} = r$th Chern class).

Theorem 1. Given in M^n any real, closed, infinitely differentiable exterior form Σ of type $(1, 1)$ and cohomologous to $2\pi C^{(1)}$, there exists exactly one Kähler metric in Ω whose Ricci form equals Σ.

Calabi's Decisive Claims (cont.)

In the Proceedings of the International Congress of Mathematicians in Amsterdam he gives some evidence for the method used to get the theorems:

- He outlines the method, later coined as the *continuity method*, that should provide the solution. He says: "*The proof proceeds by joining the Ricci form of one metric in* Ω *with [the given form]* Σ *by a differentiably parametrized arc in the same linear space of forms (for example by linear interpolation) and constructing a corresponding parametrized path in* Ω*, then by proving that it is unique, and that its terminal point is independent of the path*";
- He then refers to the Riemannian geometry of Ω which, he claims, guarantees the existence of a unique geodesic;
- In this session chaired by Henri CARTAN also spoke among others Pierre DOLBEAULT, and Marcel BERGER reporting on his thesis on holonomy groups of Riemannian metrics.

The Calabi Conjecture

Eugenio CALABI acknowledged that his proof is incomplete in the article *On Kähler manifolds with vanishing canonical class* published in 1957 in the proceedings of a symposium to honour Solomon LEFSCHETZ. Still he outlined a possible approach:

- The equation to be solved is of Monge-Ampère type as the problem can be brought back to a scalar problem as mentioned already by Erich KÄHLER:
$$\frac{(\omega + i\partial\bar{\partial}\phi)^m}{\omega^m} = F \ ;$$
- The method proposed was to use the continuity method which requires that the sub-interval of [0,1] along which a solution is found is non-empty, open and closed (a priori estimates are crucial to guarantee that one gets the convergence in the appropriate function space).

The Solution of the Calabi Conjecture

The Calabi Conjecture remained open for almost 20 years generating doubts about its truth (see later for an attempt in this direction) until Shing Tung YAU solved it in a breakthrough paper.

- The strategy using the continuity method proved in the end to be the right one to provide a solution;
- For this method to work one needs to have a full set of a priori estimates. Eugenio CALABI provided third order estimates, and indications for the second order ones;
- The most difficult a priori estimate, the C^0 one, has been lacking for a long time; it was the real tour de force that Shing Tung YAU managed to do;
- Simplifications were obtained later first by Jerry KAZDAN (and me), and several others later because of the iconic value of this problem, and the need for more sophisticated estimates for the related problem of finding Kähler-Einstein metrics.

3. When Holonomy Enters

Holonomy Groups

The notion of *parallel transport* was introduced by Tullio LEVI-CIVITA in 1917 in the setting of submanifolds of Euclidean space. But soon after it was used in Riemannian Geometry.

- In the Riemannian setting the notion of a *holonomy group* was introduced by Élie CARTAN in 1926 in an article published in Acta Mathematica as *the group of all linear transformations on the tangent space at a point generated by parallel transport along all possible loops at this point*;
- In 1952 Armand BOREL and André LICHNEROWICZ proved that *the holonomy group is a Lie group*;
- Of course all quantities that are parallel under the covariant derivative are preserved by elements of the holonomy group;
- For the Riemannian covariant derivative the metric is preserved so that the holonomy group must be a subgroup of the orthogonal group; in symmetric spaces other structures are of course preserved.

Holonomy Groups

What is not so well known is that in 1962 André WEIL gave a Bourbaki seminar involving the notion of holonomy. Here it is:

Séminaire BOURBAKI
14e année, 1961/62, n° 239 Mai 1962

UN THÉORÈME FONDAMENTAL DE CHERN
EN GÉOMÉTRIE RIEMANNIENNE

par André WEIL

Soit V une variété riemannienne connexe de dimension n . Soit T(M) l'espace des vecteurs tangents à V en M , muni de la structure euclidienne déterminée par la structure riemannienne de V . Si α est un chemin, différentiable par morceaux, d'origine M et d'extrémité M' sur V , le transport parallèle suivant α détermine un isomorphisme $i(\alpha)$ de T(M) sur T(M') ; si β est un chemin analogue d'origine M' , on a $i(\alpha\beta) = i(\beta) \circ i(\alpha)$. L'ensemble des $i(\gamma)$, quand on prend pour γ tous les chemins (différentiables par morceaux) d'origine et d'extrémité M , forme un sous-groupe H(M) du groupe A(M) des automorphismes de T(M) , qui s'appelle, comme on sait, le groupe d'holonomie de V en M ; si α est un chemin allant de M à M' , on a $H(M') = \alpha H(M) \alpha^{-1}$.

Holonomy of Kähler Metrics

A list of possible holonomy groups of irreducible non symmetric Riemannian manifolds was identified by Marcel BERGER.

- Here is the list in the oriented case, SO_n, U_m, SU_m,Sp_q, $\mathrm{Sp}_q \times \mathrm{Sp}_1$, G_2, Spin_7 and Spin_9 (actually Alfred GRAY showed that this last case can be left out);
- Besides SO_n the other cases correspond to special Riemannian geometries: Kählerian, special Kählerian, hyperKählerian, quaternionic-Kähler, and two 'exotic' geometries;
- On a Kähler manifold, beyond the metric g, ω is also parallel (and therefore the almost complex structure J), hence the reduction of the holonomy group to U_m;
- Any special Kähler metric has one more parallel object, namely a complex volume element, which is actually equivalent to the vanishing of the Ricci curvature because of its interpretation as the curvature of the complex line bundle $\Lambda^m_{\mathbb{C}} T^*M \to M$ if the complex dimension of M is m.

The 2-dimensional Compact Complex Case

Actually in the 2-dimensional complex case there is a collision between the special Kählerian and the hyperKählerian geometries because $SU_2 = Sp_1$.

- This means that, at least on a simply connected manifold, having a Ricci-flat Kähler metric is equivalent to having three almost complex structures I, J and K behaving like the multiplication of quaternions which are parallel for the Kähler connection;
- All simply connected complex surfaces with vanishing first Chern class, the so-called K3 surfaces, are all diffeomorphic; they have a 20-dimensional deformation space of complex structures;
- Solving the Calabi conjecture implies that, for each Kähler class on a K3 surface (as a complex surface), there is a Ricci-flat metric in this class.

An Obstruction to SU_2-holonomy

This led Shing Tung YAU and myself to an attempt to challenge the existence of a hyperKälerian metric on a quotient of a K3 surface through the following theorem:

C. R. Acad. Sc. Paris, t. 277 (17 décembre 1973) Série A — 1175

GÉOMÉTRIE DIFFÉRENTIELLE. — *Sur les métriques riemanniennes à courbure de Ricci nulle sur le quotient d'une surface* K 3. Note (*) de MM. **Jean-Pierre Bourguignon** et **Shing Tung Yau**, présentée par M. André Lichnerowicz.

Sur un quotient d'une surface K 3 à courbure de Ricci nulle, nous établissons que le long d'une géodésique non nulle homotopiquement la courbure sectionnelle est identiquement nulle.

1. Motivation. — Aucun exemple de variété riemannienne compacte à courbure de Ricci nulle non plate n'est encore connu. D'après Cheeger-Gromoll [*cf.* (1), p. 126], on sait que ces exemples sont obtenus à partir d'une telle variété riemannienne simplement connexe à groupe d'isométries fini par produit avec $\mathbf{R}^n$ muni de sa métrique canonique, puis quotient par un sous-groupe du groupe d'isométries du produit agissant librement et de façon totalement discontinue.

En dimension 4, Hitchin [*cf.* (2)] prouve que les variétés à courbure de Ricci nulle non plates vérifient une inégalité entre leur caractéristique d'Euler et leur signature; celles pour lesquelles l'égalité a lieu sont revêtues par une surface K 3, les groupes fondamentaux éventuels étant $\mathbf{Z}_2$ et $\mathbf{Z}_2 \times \mathbf{Z}_2$.

4. Calabi-Yau Metrics

Calabi-Yau Metrics

In 1985 Philip CANDELAS, Gary T. HOROWITZ, Andrew STROMINGER and Edward WITTEN formulated a *Superstring Theory* in a $10(= 4 + 6)$ space-time. Since the extra spatial dimensions of the extended space-time M are not visible, they are assumed to form a compact manifold a extremely small size.

- Assuming that the metric on M must satisfy the equations of motion of General Relativity, and setting the vacuum to have zero energy, it must be Ricci-flat.
- On such a manifold any infinitesimal isometry leads to the existence of particles that have not been observed;
- This suggests to consider Kähler metrics on compact complex manifolds with vanishing first Chern class, whose existence is guaranteed by the solution by Shing-Tung YAU of the Calabi conjecture, hence the name given to them of *Calabi-Yau metrics*.

Calabi-Yau Metrics (final)

Although no explicit example of a Calabi-Yau metric is still available, this led to an incredible harvest of new objects and new problems. Google tells that the expression "Calabi-Yau" is present 494 000 times on Internet.

- The unraveling of Mirror Symmetry, showing how to pass smoothly between quantum theories on different Calabi-Yau manifolds, led to great results in Algebraic Geometry in the enumeration of rational curves in Calabi-Yau manifolds;
- It was also quite remarkable that the study of metrics with the exceptional holonomy groups G_2 could be connected to Kählerian geometry through a construction due to Christian BÄR, using the metric cone built on such manifolds;
- They have also appeared in Theoretical Physics in some supersymmetric models.

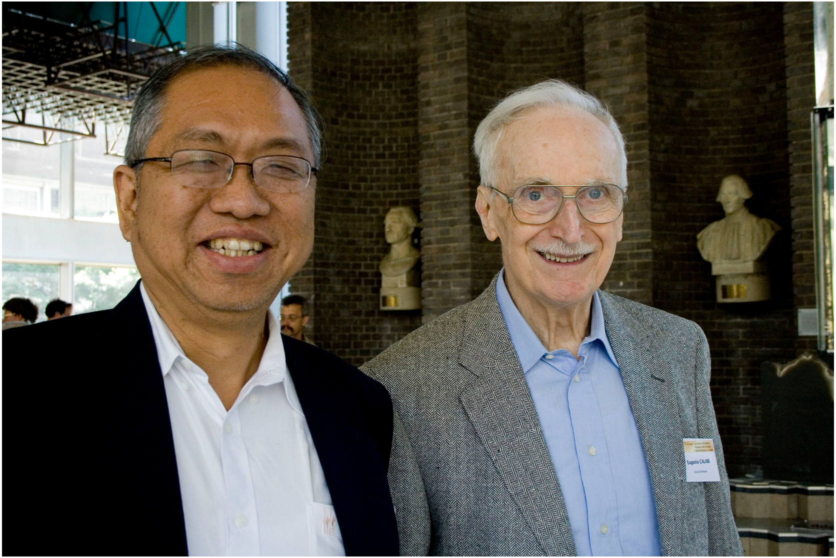
5. Concluding Remarks

Concluding Remarks

As often in Mathematics, the history of Calabi-Yau metrics does not follow a linear path:

- It started with Erich KÄHLER studying a remarkable metric that he developed having in mind some special non-compact manifolds as a possible tool for Number Theory;
- The use of the Fubini-Study metric on complex projective spaces was key to develop the link with Algebraic Geometry;
- The remarkable role of the Ricci curvature in Kählerian geometry led to the *optimistic* view that it should be very flexible, the case of Ricci-flat metrics when the first Chern class vanishes standing out as a special case to study;
- The translation of the geometric problem into a non-linear partial differential equation was a great challenge that played a significant role in the development of Global Analysis;
- And of course the unexpected link to Theoretical Physics was a great push with positive outcomes in Mathematics.

Calabi-Yau



Photograph courtesy of Jean-François DARS, 2007

I thank you for your attention.

Happy 70^{th} Birthday Shing Tung and many happy returns!

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