Group Invariant Shape Regularisers, Feature Manifolds and Principal Bundles

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Outline

Introduction

Approaches for Enforcing Invariance in Shape priors

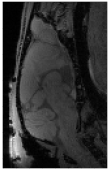
Invariance via Equivariance

Denormalisation/Normalisation

And some examples.

Introduction

- Original problem: Segmentation of some 3D shapes with some known training examples. Here rat brains.



- Variation in shapes and poses
- A few examples have been manually segmented.
- They have somewhat similar appearances.
- Goal: include training knowledge supporting
  - invariance to pose
  - "reasonable variations" to model shapes

Segmentation Objective

- Find a segmentation/object A in image I, A should be similar to some examples $B_1, \dots, B_n$. Ingredients for a segmentation objective with shape priors:

$$\mathcal{F}(A; I, B_1, \dots, B_n) = \mathcal{E}(A; I) + \lambda \mathcal{L}(A; B_1, \dots, B_n)$$

- Image similarity term $\mathcal{E}(A; I)$: modality dependent.
- Shape prior term $\mathcal{L}(A; B_1, \dots, B_n)$. Should be
  - Invariant to poses: no change if A is scaled, translated, rotated, or more
  - tolerant to reasonable variations from examples.

Why this work

- Cremers, Osher and Soatto, *Kernel Density Estimation and Intrinsic Alignment for Shape Priors in LevelSet Segmentation*. IJCV 2006. A simple mechanism that fixes position and scale in 2D. Why did they not introduce rotational invariance? Actually because things become more complicated....

- Hansen, Lauze, *Segmentation of 2D and 3D Objects with Intrinsically Similarity Invariant Shape Regularisers*. SSVM 2019. Adds rotations to obtain invariance by similarities. Correct algorithm but dubious maths...

- Then, what did I miss?

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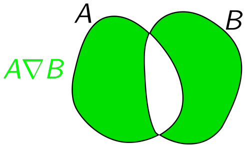
Distances

- Compare candidate shape A to reference B.
- Most common in literature: L^p-types distances

$$d_p(A,B) = \left(\int_{\mathbb{R}^n} \|\chi_A - \chi_B\|\, dx\right)^{\frac{1}{p}} = |A \nabla B|^{\frac{1}{p}}$$



$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

- Already clear that there is no invariance: similarity transform

$$\tilde{A} = sR.A + \vec{t} = \{sRx + \vec{t}, x \in A\} : \quad d(\tilde{A}, B) \neq d(A, B)$$

Pose Invariance

Variations in pose may depend on experimental context:

- Position, Scale, Rotation, More general affine transformations

In general, a (closed subgroup G of the Special Affine Group $\mathbb{A}_n^+ = GL_n^+ \rtimes \mathbb{R}^n$, $n = 2, 3$.

- Group of translations: $G \simeq \mathbb{R}^n$
- Positive Scalings: $G = \mathbb{R}_+^* \,\mathrm{id}$
- Positive Scalings and Translations: $G \simeq \mathbb{R}_+^* \rtimes \mathbb{R}^n$
- Special Euclidean group: $G = SE(n) := SO(n) \rtimes \mathbb{R}^n$
- Special Euclidean similarities: $G = S(n) := \mathbb{R}_+^* \times SO(n) \rtimes \mathbb{R}^n$
- Everything: $G = \mathbb{A}_n^+$.

What is a shape?

- Kendall 1984: all the geometrical information that remains when location, scale and rotational effects are filtered out from an object.
- Kendall shapes are classes of objects modulo the group of Euclidean Similarities.
- To us: classes of objects modulo one of the subgroups listed above.

- Objects: Compact connected subsets of $\mathbb{R}^n$ with non empty interior. May need extra hypotheses (boundary regularity)

What is a shape?

- *Action* of an affine transformation on an object: $g = (L, \vec{t})$ transforms $A \subset \mathbb{R}^n$ by

$$g.A = \{Lx + \vec{t}, \quad x \in A\}$$

- A shape is an **orbit**: $\{g.A, g \in G\} : G.A$
- Work on space of shapes = space of orbits X/G, quotient space.

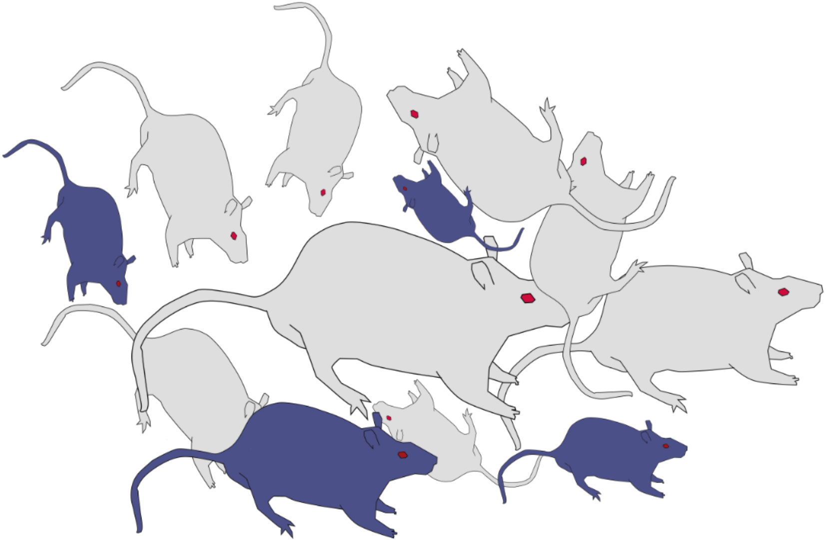
Invariant Shape function

- A function: $f : X = \{\text{Objects}\} \to \mathbb{R}$, which does not change by a transformation of the object: $f(g.A) = f(A)$.
- Factorization i.e., commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\;f\;} & \mathbb{R} \\ {\scriptstyle q}\big\downarrow & \nearrow_{\tilde{f}} & \\ X/G & & \end{array}$$

- Is it easy to construct? In general: no, depends on what we want...
- The structure of the set of objects orbits X/G: (*quotient set*) may turn to be *complicated!*
- Depends on the group.

An $SE(2)$-orbit



Some standard ways

- Quotient construction. Finite dimension.
  - D. J. Kendall, *Shape Manifolds, Procrustean Metrics and Complex Projective Spaces*, Bull. London Math. Soc., 1984.
- Special Orbit representatives.
  - M. E. Leventon, W. E. Grimson, O. Faugeras, *Statistical Shape Influence in Geodesic Active Contours*, CVPR 2000. (**Normalisation/denormalisation**, PCA).
  - D. Cremers, et al. *op. cit.*, 2006. **Normalisation** prior to comparison.
  - J. Wang, S.-K. Yeung, K. L. Chan, *Matching-consrained active contours with affine-invariant shape prior.*, CVIU 2015. **Denormalisation**, Point distributions.
- Optimisation over orbits.
  - M. Fussenegger, R. Deriche, A. Pinz, *A Multiphase Level Set Based Segmentation Framework with Pose Invariant Shape Priors*, ACCV 2006. Training shape is **normalised**.
- Comparison of invariant features.
  - A. Foulonneau, P. Charbonnier, F. Heitz, *Affine-Invariant Geometric Shape Priors for Region-Based Active Contours*, PAMI 2006. Features are invariant via **normalisation**.

In this talk: explore invariance via equivariant features to reduce orbit search.

Outline

Normalisation and Denormalisation

- In most approaches, normalisation and denormalisations to and from "canonical forms" are used.
- For objects or features.
- Essential to build shape densities.
- Is it always possible?

A few generalities, facts...

G-actions, equivariance, orbits stabilizers

- A mapping $\varphi : G \times X \to X$, $(g, x) \to \varphi(g, x) = g.x$ is a **(left)** G-**action** if

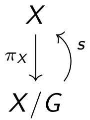
$$i)\, e_G.x = x, \quad ii)\, g.(h.x) = (gh).x$$

  X is called a G-space.
- $G.x = \{g.x, g \in G\}$ is the G-**orbit** of x.
- $G_x = \{g \in G, g.x = x\}$ is the **stabilizer** of x.
- One-liner: $G_{g.x} = gG_xg^{-1}$.
- If X and Y are G-spaces, a mapping $f : X \to Y$ is G-**equivariant** if

$$f(g.x) = g.f(x)$$

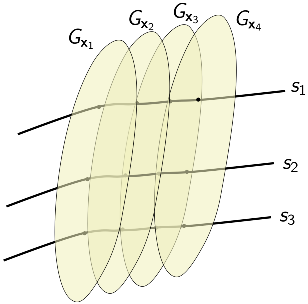
  f is called a G-**map**.
- One liner: $G_x \subset G_{f(x)}$: extracting features may add, but not remove symmetries.

- The map $\pi_X : x \to G.x$ is the orbit/quotient map $X \to X/G$.
- A **section** of π_X is a mapping $s : X/G \to X$ such that



$$\pi_X(s(G.x)) = G.x$$

- s **chooses a unique orbit representative** : $s(G.x) = \bar{x}$ is the **canonical representation** of **any** element of orbit $G.x$.



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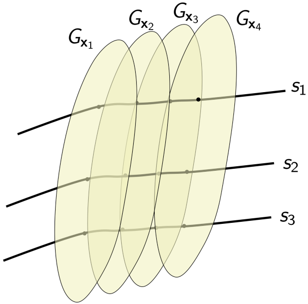

- As a simple mapping: always exists, and many are possible (axiom of choice). G continuous/Lie group, s continuous/smooth? No in general, but with caution...

Invariance via Equivariance

- Assume given $f : X \to Y$ a G-equivariant map and canonical representatives for elements of Y:
  - a section s_Y of $\pi_Y : Y \to Y/G$ and
  - $\bar{y}$ the **canonical** form of y, $\bar{y} = s_Y(\pi_Y(y))$
- For each $x \in X$, set

$$N(x) = \{g \in G, f(g.x) = g.f(x) = \overline{f(x)}\}$$

$N(x)$ transformations g which normalize $f(x)$: $N(x) = M(f(x))$.

Proposition

- *$N(x)$ satisfies the relation $N(g.x) = N(x)g^{-1}$: one liner.*
- *Consequently, if $L : X \to \mathbb{R}$ is any function, then*

$$\mathcal{L}(x) = \inf_{g \in N(x)} L(g.x)$$

*is G-**invariant**: one liner too.*

Sort of kind of in words...

- Given a "complicated set" X where G acts, extract some G-equivariant features $f(x)$ in feature space Y, which would be simpler.
- Assume that one can "easily" **normalise** features (for G) in Y, i.e., find a unique "good" representative of each orbit.

- For an object x in X, $N(x)$ will be the set of transformations g such that the **features** of $g.x$ are normalised.
- There may be more that one of these g, which provide feature normalisation. I can have $f(g_1.x) = f(g_2.x)$ while $g_1.x \neq g_2.x$!

- For any function $L : X \to \mathbb{R}$, choose $\mathcal{L}(x) = min\{f(g.x), g \in N(x)\}$.

G-invariant shape priors for segmentation

- One training shape B, feature-normalised.

$$\mathcal{L}_B(A) = d(A, B) = \inf_{g \in N(A)} d(\chi_{g.A}, \chi_B)^p$$

- Multiple training shapes $B_1, \ldots, B_n$ all feature-normalised.

$$\mathcal{L}(A; B_1, \ldots, B_n) = \sum_{i=1}^{n} \mathcal{L}_{B_i}(A)$$

$$\mathcal{L}(A; B_1, \ldots, B_n) = -\log \left(\sum_{i=1}^{n} e^{-\frac{\mathcal{L}_{B_i}(A)}{2\rho^2}} \right)$$

- ...

Back to the optimisation problem

Given: equivariant feature map $f : X \to Y$ to a feature manifold.
Objective function/optimisation problem of the form

$$\min_{A \in X} \mathcal{F}(A) = \mathcal{E}(A) + \inf_{g \in N(A)} \mathcal{L}(g.A)$$

- To optimise it:
- Differential structure on X, on Y, on the group G. But not enough:

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- To optimise it:
- Differential structure on X, on Y, on the group G. But not enough:
  - Existence of good feature normalisation?
  - Even if it exists, $N(A)$ depends on A.
- General problem $f : X \to Y$ equivariant, normalisation in Y, good way to deal with

$$\inf_{g \in N(x)} L(g.x)$$

Outline

Denormalisation and Normalisation

$S \in Y$ image of a (local) section. $G.S = \{g.\bar{y}, g \in G, \bar{y} \in S\}$.
Note that $G.S \subseteq Y$, not necessarily equal.

$$\textit{denormalisation:} \quad \begin{array}{ccc} G \times S & & (g, \bar{y}) \\ \downarrow p & & \downarrow \\ G.S & & g^{-1}.\bar{y} \end{array}$$

Normalisation of $y \in G.S$: **fibre** $p^{-1}(y)$

$$p^{-1}(y) = \{(g, \bar{y}),\ g.y = \bar{y}\} = M(b) \times \{\bar{y}\}$$

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$M(y)$: set of transformations which **normalise** y.

$$\forall g \in M(y), M(y) = G_{\bar{y}}\, g, \quad N(x) = M(f(x))$$

Favourable situation: G locally compact Lie group, Y manifold with proper G-action. $S \in Y$:

- smooth submanifold
- a subgroup H of G, such that $G_{\bar{y}} = H$ for each $\bar{y}$ in S.

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If it exists:

- $M(y) = Hg$ for any $g \in M(y)$
- H is a compact subgroup of G
- local map $y \mapsto g_y \in M(y)$ smooth.

$$\inf_{g \in N(x)} L(g.x) = \inf_{h \in H} L(hg_{f(x)}.x)$$

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Does it exists? Not always...

Slices – representing Y/G in Y

$S \subset Y$ is a H-slice if

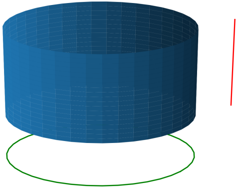
- S is H-invariant: $H.S = S$,
- S is closed in $G.S$,
- If $g \in G \backslash H$, $gS \cap S = \emptyset$.
- $G.S$ is open in Y.

- Palais 1960: Existence of Slices.
- In good case, at and $y \in Y$, they provide sorts of local charts in Y to Y/G...
- They should be "transverse to orbits"
- Depends on G_y as $G.y \simeq G/G_y$. In my cases local is very large with "small stabilisers".
- Think of $SO(2)$ acting on $\mathbb{R}^2$ for instance, orbits and stabilisers at $y = \vec{0}$ and at $y \neq \vec{0}$.
- In favourable case of a "good slice", denormalisation map has a classical structure: **principal H-bundle**.

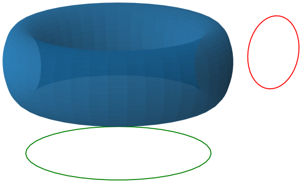
- R. Palais, "The classification of G-spaces", Memoirs of the AMS, 36, 1960.
- R. Palais, "On the existence of slices for actions of non-compact Lie groups", Ann. Math. 73 (1961).
- M. Audin, "Torus Action on symplectic manifolds", Springer, 2004.
- A. Antonyan, "Characterizing slices for proper actions", arXiv 2017.

Fibre bundles

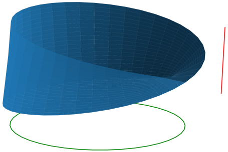
A manifold which looks locally like a Cartesian product basis×fibre, but maybe not globally.



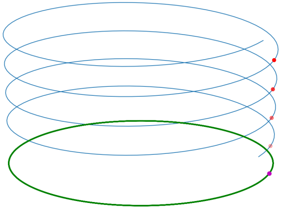
Cylinder: $\mathbb{S}^1 \times [-1, 1]$, trivial



Torus: $\mathbb{S}^1 \times \mathbb{S}^1$, trivial



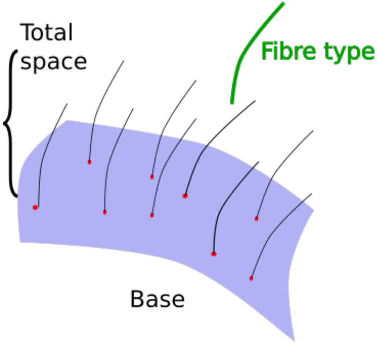
Möbius strip, non trivial



Covering map (helicoid), non trivial

A fibre bundle is a n-uple (E, F, π, B) (or just $\pi : E \to B$)

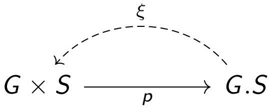
- E is the **total space**
- B is the **base space**
- $\pi : E \to B$ is the projection
- F is the **fiber type**: each **fibre** $E_b := \pi^{-1}(b) \simeq F$
- Each point b has a neighbourhood U such that $E_U = \pi^{-1}(U) \simeq U \times F$.
- **Local section**: A smooth mapping $\sigma : V \subset B \to E$, $\pi(\sigma(b)) = b$. **Always exists.**



- A H-principal bundle: fibres are copies of group H, H acts on the bundle via its fibres.

Sections – same but different ones!

$S \subset \mathcal{M}$ "good" H-slice: denormalisation is a H-principal bundle



- ξ is a local section of p: $p(\xi(y)) = y$,
- ξ defined on an open set of $G.S$: $\xi(y) = (g_y, \bar{y}) \subset M(y) \times \{\bar{y}\}$ and

$$M(y) = Hg_y \implies M(y)g_y^{-1} = H, \quad g_y.y = \bar{y} \implies \xi(y) = (g_y, g_y.y)$$

- Always possible to find one.
- Only globally defined over $G.S$ if $G \times S \simeq G.S \times H$ (**trivial** bundle)

For optimisation:

$$\inf_{g \in N(x)} L(g.x) = \inf_{h \in H} L(hg_{f(x)}.x)$$

Outline

An example for scaling and translations

- $G = \mathbb{R}_+^* \rtimes \mathbb{R}^n$: group of scalings and translations.
- Action of $g = (\lambda \, \mathrm{Id}_n, \vec{t})$ on object space:

$$g.A = \{\lambda x + \vec{t}, x \in A\}.$$

- Feature manifold: scale and position, $\mathcal{M} = \mathbb{R}_+^* \times \mathbb{R}^n$.
- Action of g on an element of the feature manifold:

$$g.(\sigma, \tau) = (\lambda\sigma, \lambda\tau + \vec{t})$$

- **Transitive** and **free action**: a **unique** g such that $g.(\sigma, \tau) = (\sigma', \tau')$
- $\mathcal{M}$ is a **principal homogeneous space** of G.
- Only one orbit: any point can be chosen as "nice representation". For instance: unit scale and centred $(\sigma, \tau) = (1, \boldsymbol{O})$

Feature map built from object *moments*.

- volume , barycentre, covariance

$$|A| = \int_A dx, \quad \mu(A) = \frac{1}{|A|}\int_A x\, dx$$

$$\Sigma(A) = \frac{1}{|A|}\int_A (x-\mu(A))\,(x-\mu(A))^T\ dx$$

Feature map: $F(A)$: position $\mu(A)$ and scale $\sigma(A) = \sqrt{\mathrm{Tr}\,\Sigma(A)}$
Equivariance: $F(g.A) = g.F(A)$ by simple check.

- $N(A)$: Transformations g which normalise the scale and position of A: only one such transformation $g_A = (\sigma(A)^{-1}, -\sigma(A)^{-1}\mu(A))$,

$$g_A.A = \frac{A - \mu(A)}{\sigma(A)}$$

- $A \mapsto L\left(\frac{A-\mu(A)}{\sigma(A)}\right)$ is G-invariant. One gets [Cremers et al. 2006].
- Here $A \mapsto g_A$ normalises the features and the object too.
- There is a simple relation between the group and the feature space.

- $E = \{(\sigma^{-1}, -\sigma^{-1}\tau), (\sigma, \tau) \in \mathcal{M}\} \to \mathcal{M}$:
  - very trivial principal bundle
  - Fiber type $H \approx \{\mathrm{id}\}$.

Similarities: scalings, rotations and translations

- Same features, augmented group $G = \mathbb{R}_+^* \times SO(n) \rtimes \mathbb{R}^n$
- Action on features:
$$(s, R, \vec{t}).(\sigma, \tau) = (s\sigma, sR\tau + \vec{t})$$
- One orbit, but the action is not free: many transformations send (σ, τ) to $(1, O)$:
$$\{(\sigma^{-1}, R, -\sigma^{-1}R\tau),\ R \in SO(n)\} \simeq SO(n)$$
- Feature normaliser
$$M(\sigma, \tau) = \{(\sigma^{-1}, R, -\sigma^{-1}R\tau),\ R \in SO(n)\} \simeq SO(n)$$
Computing $\inf_{g \in N(A)} L(g.a)$ is more complicated!
- Scale and position features do not convey orientation information.

Add covariance $\Sigma(A)$ features: $\mathcal{M} = \mathbb{R}^n \times SPD(n)$

- Action: $g = (s, R, \vec{t})$ acts on (τ, Σ) as

$$g.(\tau, \Sigma) = (g\tau, s^2 R\Sigma R^T) = (sR\tau + \vec{t}, s^2 R\sigma R^T)$$

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- Normalised representations: the set $\mathcal{S}$ of all the (O, Λ), with Λ
  - diagonal
  - $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n > 0$
  - $\sum_i \lambda_i = 1$

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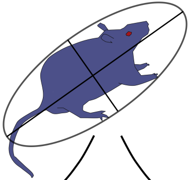
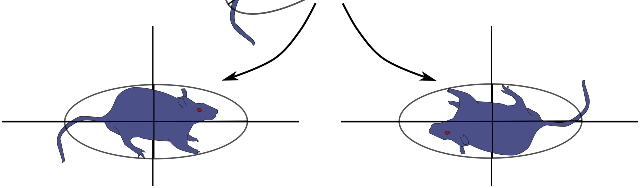
- good: each orbit $G.(0, \Lambda)$ contains exactly one such $(0, \Lambda)$,
- but the stabilisers $G_{(0,\Lambda)}$ **are not all identical**.

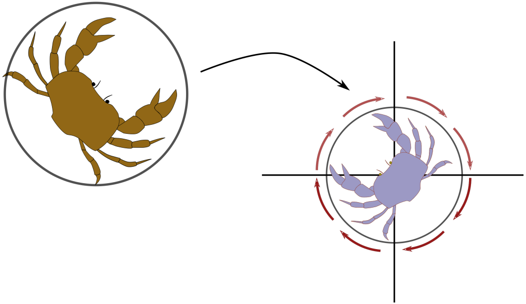
$$g.(0, \Lambda) = (0, \Lambda) \iff s^2 R\Lambda R^T = \Lambda \implies s = 1$$

- reduced to $\tilde{G} = SO(n)$, computing the stabiliser $\tilde{G}_\Lambda$
- $\tilde{G}_\Lambda$ depends on the pattern of repeated eigenvalues in Λ



Minimal symmetries: $N(A)$ is a translated of $\{\pm \mathrm{id}\}$





Circular symmetries: $N(A)$ is a translated of $SO(2)$

The fibre bundle structure for similarities

- Restrict $\mathcal{L}$ to $SO(n) \times D^*$, No repetition patterns in D^*
  - $SPD(n)^*$ matrices with distinct eigenvalues.
  - Non trivial H-principal bundle.

$$\begin{array}{ccc} SO(n) \times D^* & & R, \Lambda \\ \downarrow & & \downarrow \\ SPD^*(n) & & R^T \Lambda R \end{array}$$

$$H \ni P = \begin{pmatrix} \alpha_1 & \dots & 0 \\ 0 & \dots & 0 \\ \vdots & & \vdots \\ 0 & \vdots & \alpha_n \end{pmatrix}$$

$$\alpha_i \in \{\pm 1\}, \prod_i \alpha_i = 1$$

- Finite fibres, translations of H: Galois covering map.

Optimisation for Similarities

- Priors of the form

$$\mathcal{L}(A) = \min_{P \in H} L\left(\frac{(PR_A)\,(A - \mu(A))}{\sigma(A)}\right)$$

- R_A diagonalises Σ_A
- if L is continuous and $t \mapsto A(t)$ a continuous shape trajectory

$$P(t) = \left\{\arg.\min_{P \in H} L\left(\frac{\left(PR_{A(t)}\right)\,(A(t) - \mu(A(t)))}{\sigma(A(t))}\right)\right\} \equiv P(0)$$

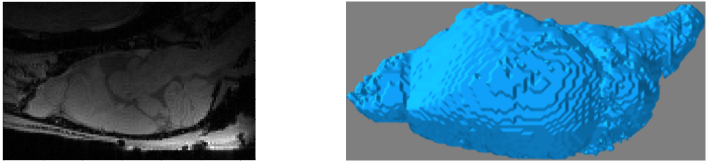
for $t \in [0, T]$ small enough

- Always the case when the denormalisation bundle has finite fibres.

A segmentation functional

$$\mathcal{E}_{D_I}(A, c_1, c_2) = \frac{1}{2}\int_\Omega g * \left[(u - c_1(x))^2\chi_A + (u - c_2(x))^2\chi_{\Omega\setminus A}\right](x)\,dx$$

$$\mathcal{E}_S(A) = -\log F(A, B_1, \ldots, B_N) = -\log\left(\sum_{i=1}^{N} e^{-\frac{\mathcal{L}_{B_i}(A)}{2\rho^2}}\right).$$



(a) **(b)**

Figure: Slice of an MRI scan of a rat cranium (a), 3D brain segmentation (b).

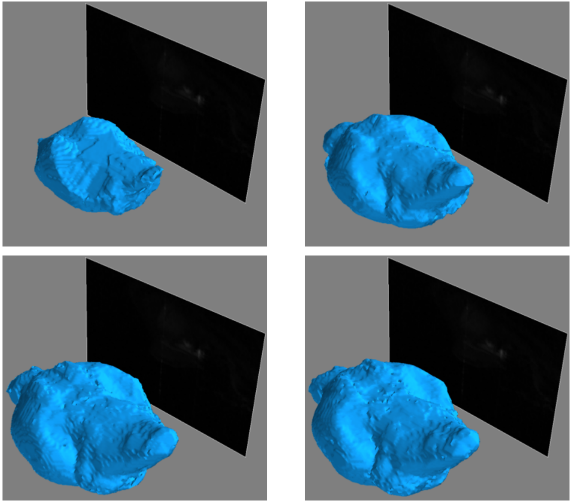


Figure: Shape evolution snapshots

Faithful features

- Scaling position situation: faithful action of $G = \mathbb{R}_+^* \rtimes \mathbb{R}^n$:

$$G_y = \mathrm{Id}$$

- Similarities, (τ, Σ): we probe a shape with Gaussian type features: inner symmetries

$$G_{\bar{y}} = H \neq \{\mathrm{Id}\}$$

- But with H discrete, simplifications are possible.
- Modify features? Probe with non symmetric: Calculation complexity?
- If $Y = Y_1 \times \cdots \times Y_n$, each Y_i a G-space:

$$G_y = (y_1, \ldots, y_n), G_y = \cap_{i=1}^n G_{y_i}$$

- Suggest: add features to simplify the action?
- But can too increase calculations complexity.

Conclusion

- An attempt to understand some of the principles used when designing invariant shape priors,
- A "recipe" for generating invariant functions,
- Link with classical objects of differential geometry.
- Limited use in 3D printing until now apart from femurs (and rat brains)...
- Other areas where locally compact groups actions would be of interest in CV / Medical imaging?
- For infinite dimensional groups? Theory of Moduli spaces. Slices may not even locally exist.