3D@UniPV
Virtual Modeling and Additive Manufacturing for Advanced Materials

INdAM WORKSHOP
"Mathematical Methods for Objects Reconstruction: from 3D Vision to 3D Printing"
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UNIVERSITÀ DI PAVIA

Recent results on Additive Manufacturing Graded-material Design based on Phase-field and Topology Optimization

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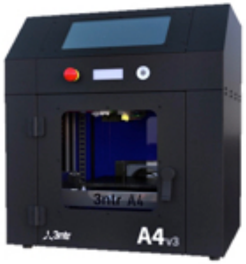
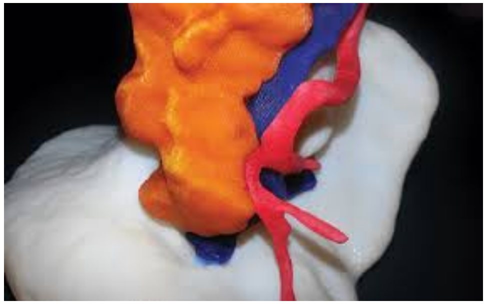



Outline of the talk

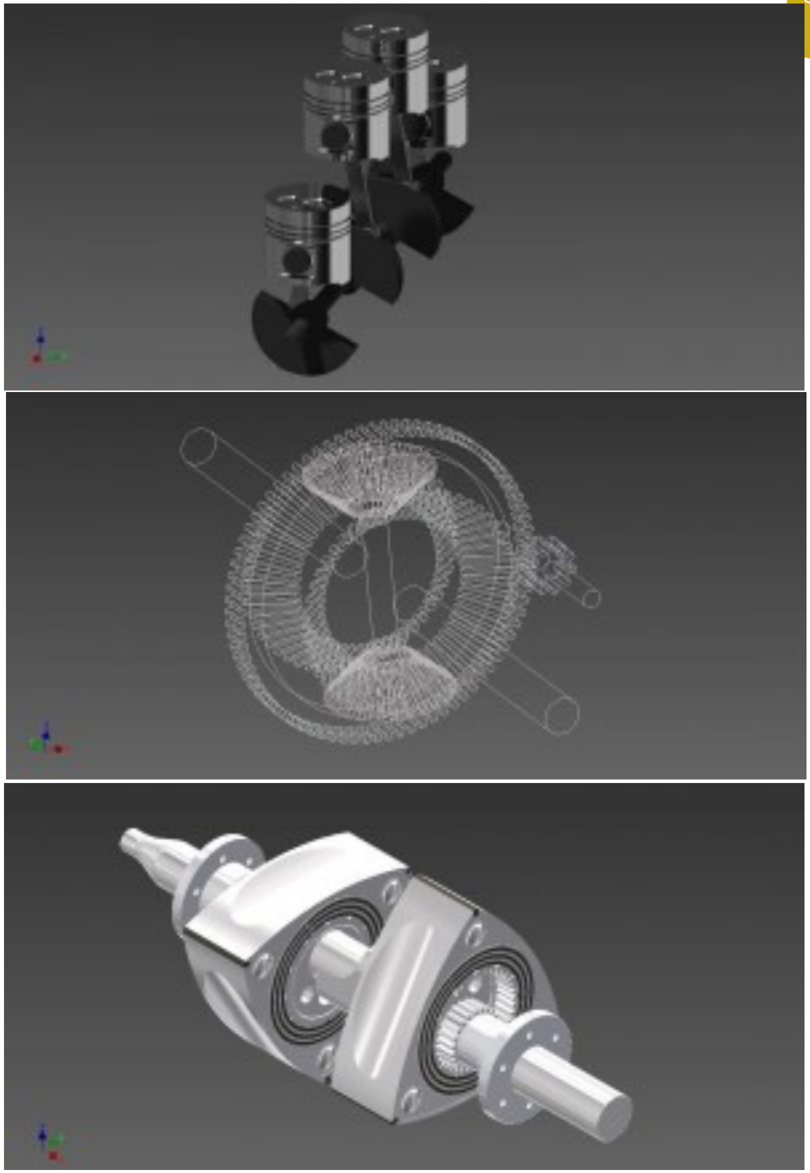
- ➢ Graded material design in AM
  - 2 papers: [M. Carraturo, E. Rocca, E. Bonetti, D. Hömberg, A. Reali, F. Auricchio, Computational Mechanics and M3AS, 2019-20]
- ➢ Identification of elastic inclusions and cavities
  - Work in progress with A. Aspri, E. Beretta, C. Cavaterra and M. Verani
- ➢ Mixed variational formulations for structural topology optimization based on phase-field
  - With M. Marino, F. Auricchio, A. Reali and U. Stefanelli, submitted

Additive Manufacturing

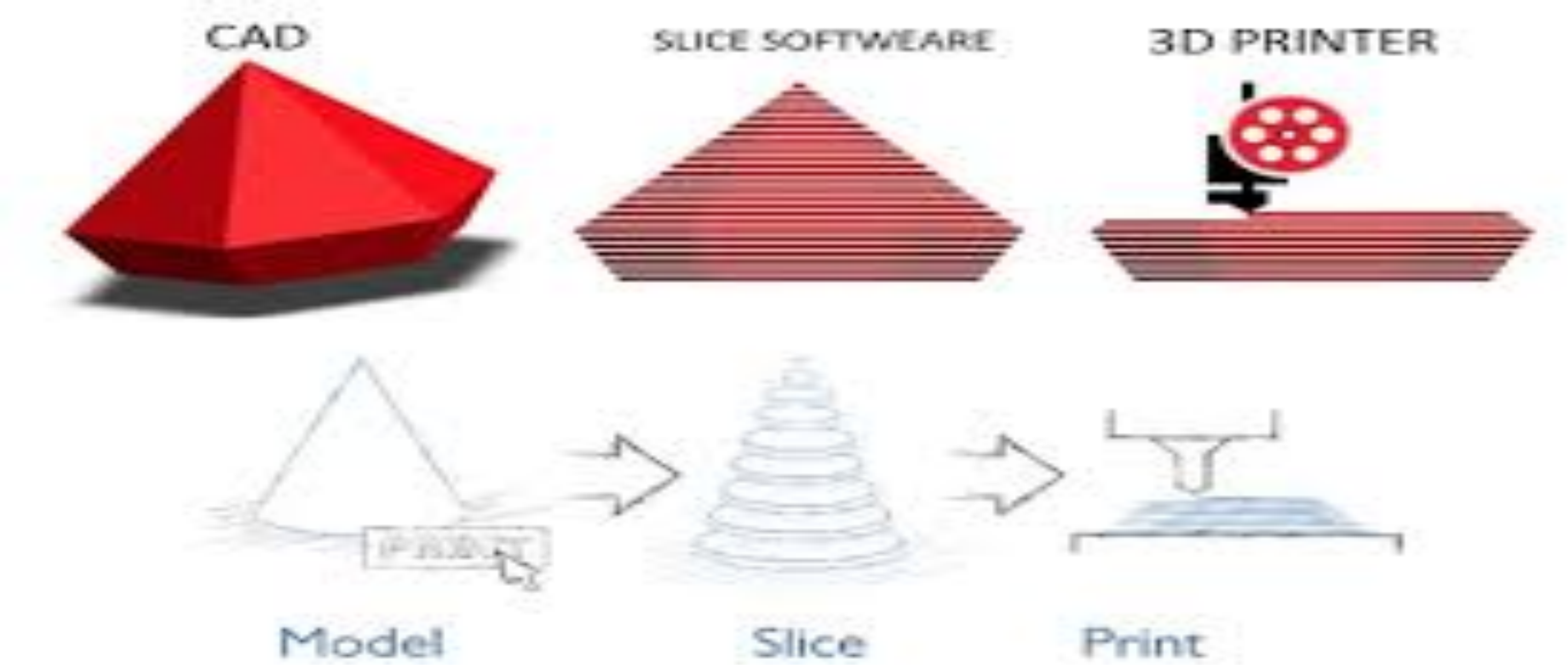
- AM is **deeply changing** paradigms in design and industrial production in comparison with more traditional technologies, like casting, stamping, and milling.

- Based on the fact that components or complete structures are constructed through sequences of **material layers deposition** and/or curing

- Through deposition of fused material (FDM technology) or by melting/sintering of powders (SLS and SLM technologies).



Additive Manufacturing



- To create the object from 3D model, the corresponding STL file must be imported in a “**slicing**” software.

- Slicing software generate the 3D printer machine code, which contains the necessary instructions to make the object.

- Finally, the object can be subjected to post-processing operations, to remove any support structure and improve mechanical and chromatic features.

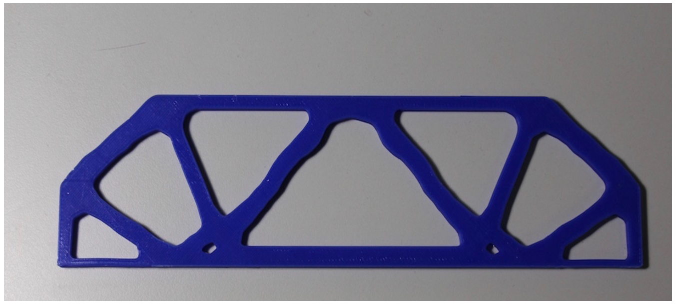
AM at large scales

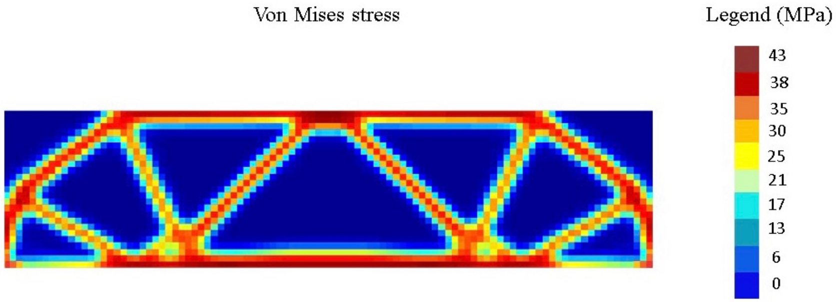
- ✓ AM was used to design small objects, prototypes and in turn is now gradually being used for large-scale achievements, such as building houses or bridges or building restoration:

The first 3D printed pedestrian bridge in the world opened to the public on December 14 in Madrid

Additive manufacturing at the service of architecture in New York

Our main interest





- ✧ We focus on the problem of **structural optimization:** to find the best way to distribute a material in order to minimize an objective functional.
- ✧ The **shape of the domain is a-priori unknown**, while known quantities are the applied loads as well as regions where we want to have holes or material.
- ✧ Our main interest is to find regions which should be filled by material in order to optimize some properties of the sample.
- ✧ As the boundaries of these regions are unknown this is a free boundary problem we use here the **phase-field method**.
- ✧ This topic is of particular interest in the industrial field: used to predict and maximize the performance of a structure at the design stage.

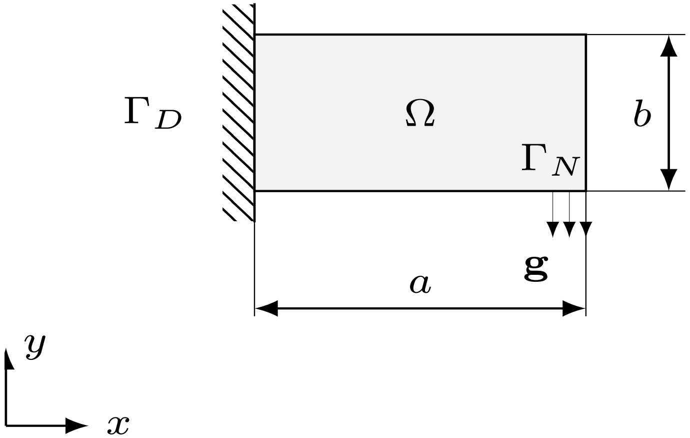
Outline

- Additive Manufacturing: single-material Design
- Additive Manufacturing: graded-material Design
- Phase-field method for the topology optimization problem
- Optimality conditions
- Numerical results

Phase-field and Topology optimization

- Consider a domain, $\Omega \subset R^n$, with n = 3 or 2
- Denote by **u** the displacement and $\varepsilon(\mathbf{u})$ the symmetric strain
- Introduce a **scalar phase field variable** $\Phi \in [0,1]$ describing material presence
  - $\Phi \equiv 0$ corresponds to no material
  - $\Phi \equiv 1$ indicates material

$$
\begin{aligned}
-\mathrm{div}\,[\mathbb{C}\varepsilon(\mathbf{u})] &= \mathbf{0} && \text{in} \quad \Omega \\
\mathbf{u} &= \mathbf{0} && \text{on} \quad \Gamma_D \\
[\mathbb{C}\varepsilon(\mathbf{u})] &= \mathbf{g} && \text{on} \quad \Gamma_N
\end{aligned}
$$



$$\mathbb{C}(\phi) = \mathbb{C}_{bulk}\phi^p + \mathbb{C}_{void}(1-\phi)^p$$

Single material: the functional

- **Optimized topology** aims at minimizing compliance, complemented with a measure of the perimeter which is here regularized by a Ginzburg-Landau type potential and subject to the volume constraint

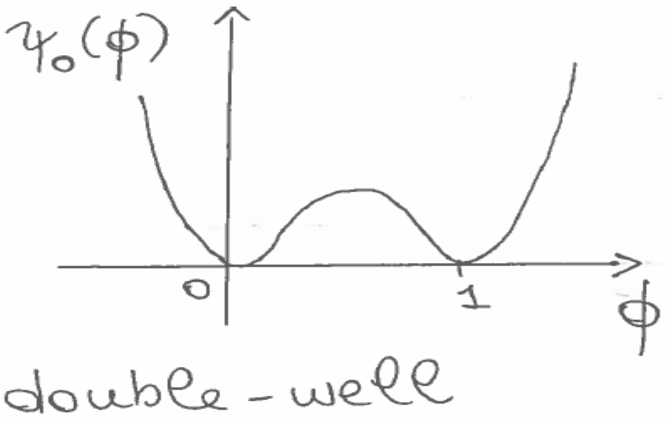
$$\int_\Omega \phi \mathrm{d}\Omega = m \mid \Omega \mid$$

$$\mathcal{J}(\phi, \mathbf{u}(\phi)) =$$

$$\int_{\Gamma_N} \mathbf{g} \cdot \mathbf{u}(\phi) \mathrm{d}\Gamma + \kappa \int_\Omega \left[\frac{\gamma}{2} \parallel \nabla \phi \parallel^2 + \frac{1}{\gamma} \psi_0(\phi) \right] \mathrm{d}\Omega$$

$$\psi_0(\phi) = (\phi - \phi^2)^2$$

ϒ represents the tickness of the interface between ϕ=0 and ϕ=1



Single Material: optimization problem

Problem $(\mathcal{P})$:

$$\min_{\phi} \quad \mathcal{J}(\phi, \mathbf{u}(\phi))$$

such that the following constraints are satisfied:

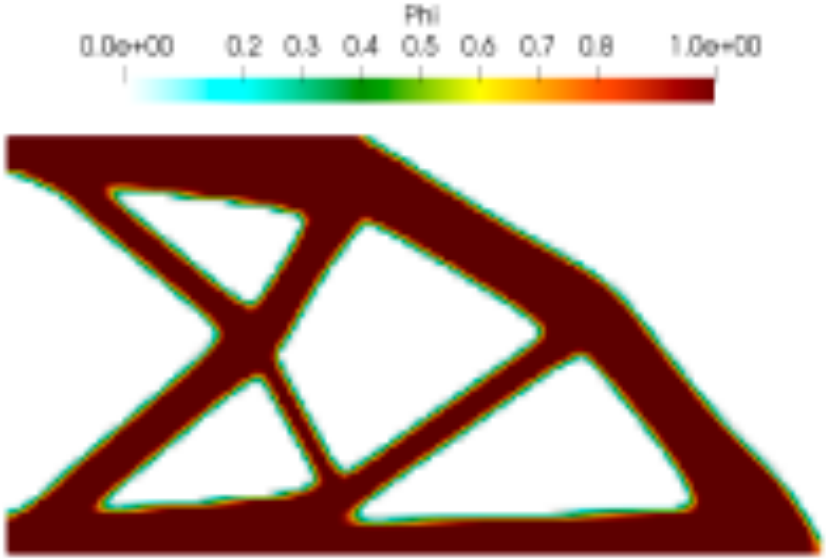
$$\int_{\Omega} \boldsymbol{\varepsilon}(\mathbf{u}) : \mathbb{C}(\phi)\boldsymbol{\varepsilon}(\mathbf{v})\mathrm{d}\Omega = \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{v}\mathrm{d}\Gamma.$$

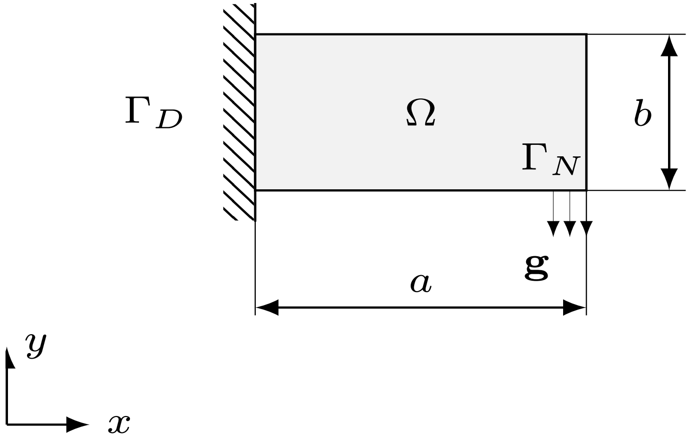
$$\mathcal{M}(\phi) = \int_{\Omega} \phi \mathrm{d}\Omega - m \mid \Omega \mid = 0,$$

with $\phi \in H^1(\Omega)$ satisfying the constraint:

$$0 \leq \phi \leq 1 \qquad \text{a.e. in } \Omega.$$

Single material: cantilever optimized beam





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- Optimality conditions

- Numerical results

The Graded Material Design

- ✓ A single material is gradually distributed through the body
- ✓ The result is a structure with with graded stiffness values alternating regions of soft material with other with stiffer material
- ✓ AM technologies allow us to grade density of a body almost in a continuous way varying the amount of distributed material point by point during the printing process

Modeling of Graded material

- Collaboration with F. Auricchio, E. Bonetti, M. Carraturo, D. Hömberg, A.Reali. Computational Mechanics (2019) and M3AS (2020)

- **IDEA:** Introduce a new grading **scalar phase field variable**

$$\chi \in [0,\phi]$$

Material stiffness can continuously vary from a **stiff material $\chi = \phi$** to a **soft material $\chi = 0$** and let the material tensor be:

$$\mathbb{C}(\phi,\chi) = \mathbb{C}(\chi)\phi^p + \gamma_\phi^2 \mathbb{C}(\chi)(1-\phi)^p,$$

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Optimization: graded material

The new graded-material optimization functional is

$$\mathcal{J}^M(\phi, \chi, \mathbf{u}(\phi, \chi)) = \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{u}(\phi, \chi) \mathrm{d}\Gamma +$$

$$\kappa_\phi \int_\Omega \left[\frac{\gamma_\phi}{2} \mid \nabla \phi \mid^2 + \frac{1}{\gamma_\phi} \psi_0(\phi) \right] \mathrm{d}\Omega + \kappa_\chi \int_\Omega \frac{\gamma_\chi}{2} \mid \nabla \chi \mid^2 \mathrm{d}\Omega$$

- Minimization problem is solved employing Allen-Cahn gradient flow, i.e. a steepest descent pseudo-time stepping method, with a time-step increment
- Alternate solution of gradient flow and of equilibrium problem

The new optimization problem

$$\min_{\phi,\chi} \quad \mathcal{J}^M(\phi,\chi,\mathbf{u}(\phi,\chi)),$$

$$\int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}) : \mathbb{C}(\phi,\chi)\boldsymbol{\varepsilon}(\mathbf{v})\mathrm{d}\Omega = \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{v}\mathrm{d}\Gamma,$$

$$\mathcal{M}(\phi) = \int_\Omega \phi \mathrm{d}\Omega - m \mid \Omega \mid = 0,$$

where $\phi,\chi \in H^1(\Omega)$, under the constraint

$$0 \leq \phi \leq 1 \qquad \text{a.e. in } \Omega, \qquad \Phi_{ad}$$

and the additional constraint on χ:

$$0 \leq \chi \leq \phi \quad \text{a.e. in } \Omega. \qquad \Xi_{ad}$$

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Lagrangean/optimality conditions

$$\mathcal{L}^M(\phi, \chi, \mathbf{u}, \lambda, \mathbf{p}) =$$

$$\mathcal{J}^M(\phi, \chi, \mathbf{u}) + \lambda \mathcal{M}(\phi) + \mathcal{S}^M(\phi, \chi, \mathbf{u}, \mathbf{p}),$$

$$\mathcal{S}^M(\phi, \chi, \mathbf{u}, \mathbf{p}) = \int_\Omega \varepsilon(\mathbf{u}) : \mathbb{C}(\phi, \chi)\varepsilon(\mathbf{p}) \mathrm{d}\Omega - \int_{\Gamma_N} \mathbf{g} \cdot \mathbf{p} \mathrm{d}\Gamma.$$

$$D_\phi \mathcal{L}^M(\bar{\phi}, \bar{\chi}, \bar{\mathbf{u}}, \bar{\lambda}, \bar{\mathbf{p}})\left(\phi - \bar{\phi}\right) \geq 0 \quad \forall \phi \in \Phi_{ad}$$

and $\bar{\mathbf{p}} = \bar{\mathbf{u}}.$

$$D_\chi \mathcal{L}^M\left(\bar{\phi}, \bar{\chi}, \bar{\mathbf{u}}, \bar{\lambda}, \bar{\mathbf{p}}\right)(\chi - \bar{\chi}) \geq 0 \quad \forall \chi \in \Xi_{ad},$$

Allen-Cahn gradient flow method

The optimal control problem can be solved as in the single-material case by means of the Allen-Cahn gradient flow, leading to the following set of equations:

$$\begin{multline}\frac{\gamma_\phi}{\tau}\int_\Omega (\phi_{n+1}-\phi_n)v_\phi \mathrm{d}\Omega + \kappa_\phi\gamma_\phi\int_\Omega \nabla\phi\cdot\nabla v_\phi \mathrm{d}\Omega + \\ \int_\Omega v_\phi \lambda \mathrm{d}\Omega - \int_\Omega v_\phi \frac{\partial\mathcal{E}^M(\phi_n,\chi_n,\mathbf{u}_n)}{\partial\phi}\mathrm{d}\Omega \\ \frac{\kappa_\phi}{\gamma_\phi}\int_\Omega \frac{\partial\psi_0(\phi_n)}{\partial\phi}v_\phi \mathrm{d}\Omega = 0, \qquad (21)\end{multline}$$

$$\begin{multline}\frac{\gamma_\chi}{\tau}\int_\Omega (\chi_{n+1}-\chi_n)v_\chi \mathrm{d}\Omega + \kappa_\chi\gamma_\chi\int_\Omega \nabla\chi\cdot\nabla v_\chi \mathrm{d}\Omega - \\ \int_\Omega v_\chi \frac{\partial\mathcal{E}^M(\phi_n,\chi_n,\mathbf{u}_n)}{\partial\chi}\mathrm{d}\Omega = 0, \qquad (22)\end{multline}$$

Under the volume constraint

$$\int_\Omega v_\lambda(\phi - m)\mathrm{d}\Omega = 0.$$

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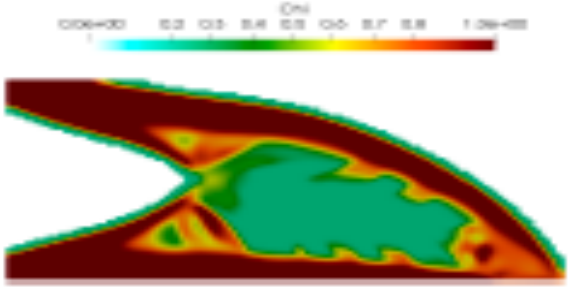
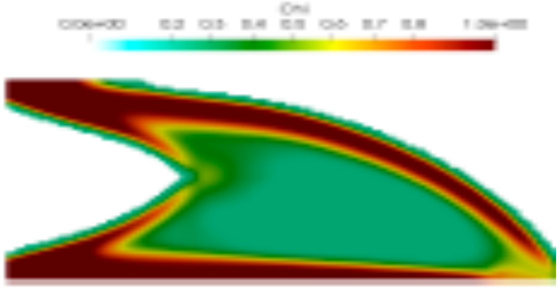
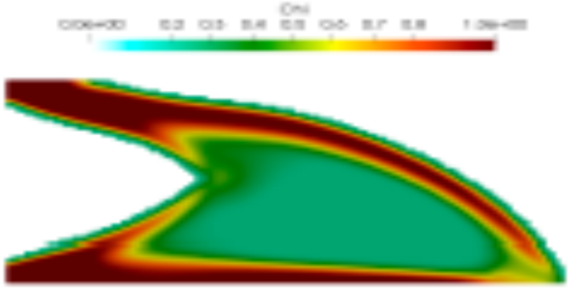
The cantilever beam



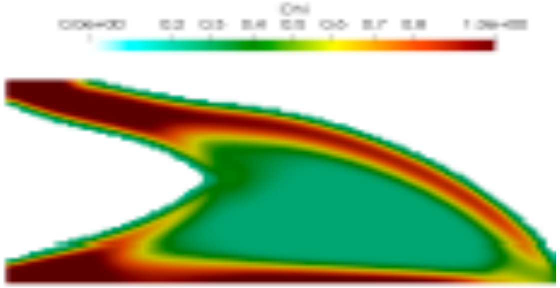
(a) $\gamma_\chi = 0.001$



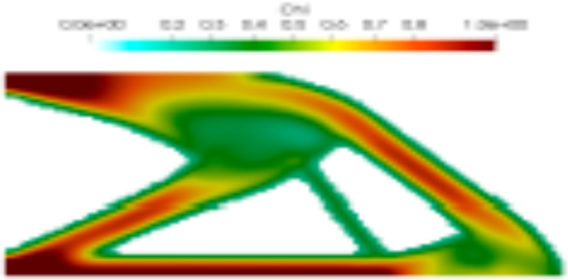
(b) $\gamma_\chi = 0.005$



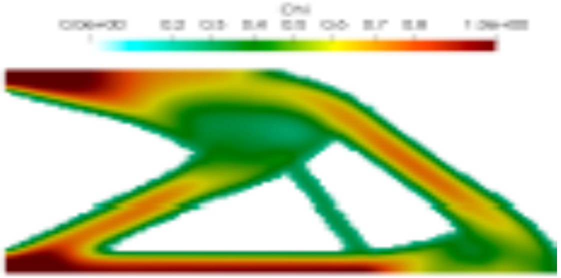
(c) $\gamma_\chi = 0.010$



(d) $\gamma_\chi = 0.020$



(e) $\gamma_\chi = 0.050$



(f) $\gamma_\chi = 0.100$

$\gamma_\chi \leq \gamma_\phi = 0.02$

The functional and the elasticity tensor

$$\mathcal{J}^M(\phi,\chi,\mathbf{u}(\phi,\chi)) = \int_{\Gamma_N} \mathbf{g}\cdot\mathbf{u}(\phi,\chi)\mathrm{d}\Gamma +$$

$$\kappa_\phi \int_\Omega \left[\frac{\gamma_\phi}{2} \mid \nabla\phi \mid^2 + \frac{1}{\gamma_\phi}\psi_0(\phi)\right] \mathrm{d}\Omega + \kappa_\chi \int_\Omega \frac{\gamma_\chi}{2} \mid \nabla\chi \mid^2 \mathrm{d}\Omega,$$

Simply supported beam: sensitivity w.r.t. the softening parameter

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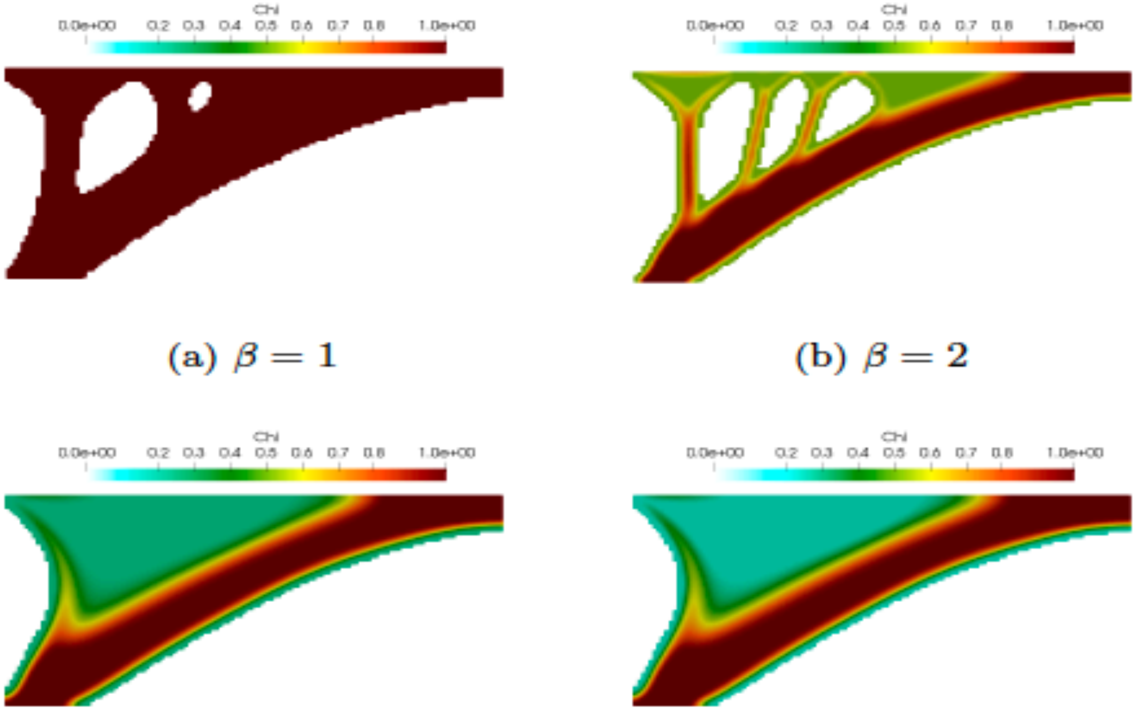
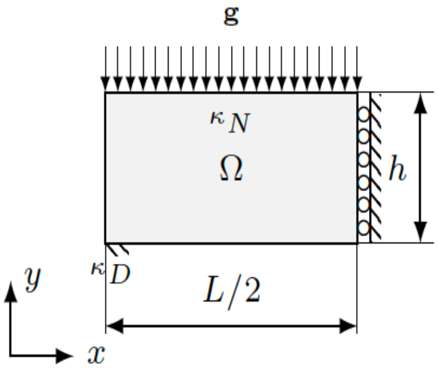


(a) $\beta = 1$ (b) $\beta = 2$

(c) $\beta = 3$ (d) $\beta = 4$

Fig. 6: Simply-supported beam: Sensitivity study of the softening factor β. Increasing the values of the softening factor, i.e., employing a softer material, the optimized structure does not present anymore the typical holes resulting from a single-material optimization 6a. Voids are now replaced by a region of soft material.



Compliance and material fraction index

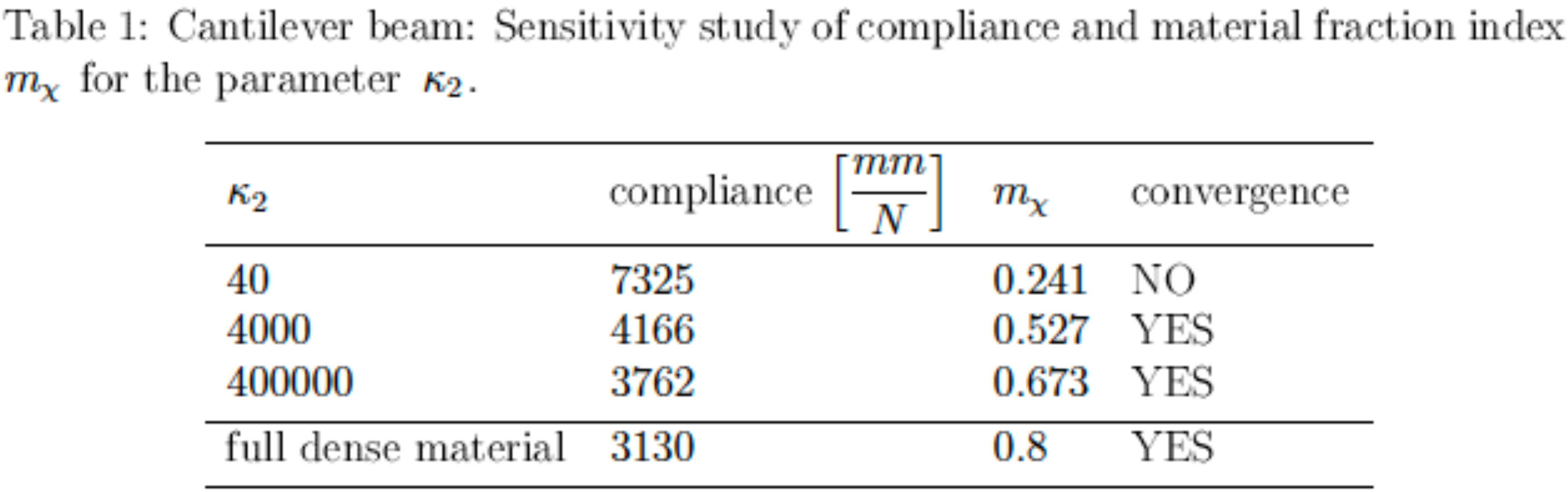


Table 1: Cantilever beam: Sensitivity study of compliance and material fraction index m_χ for the parameter κ_2.

κ_2	compliance $\left[\frac{mm}{N}\right]$	m_χ	convergence
40	7325	0.241	NO
4000	4166	0.527	YES
400000	3762	0.673	YES
full dense material	3130	0.8	YES

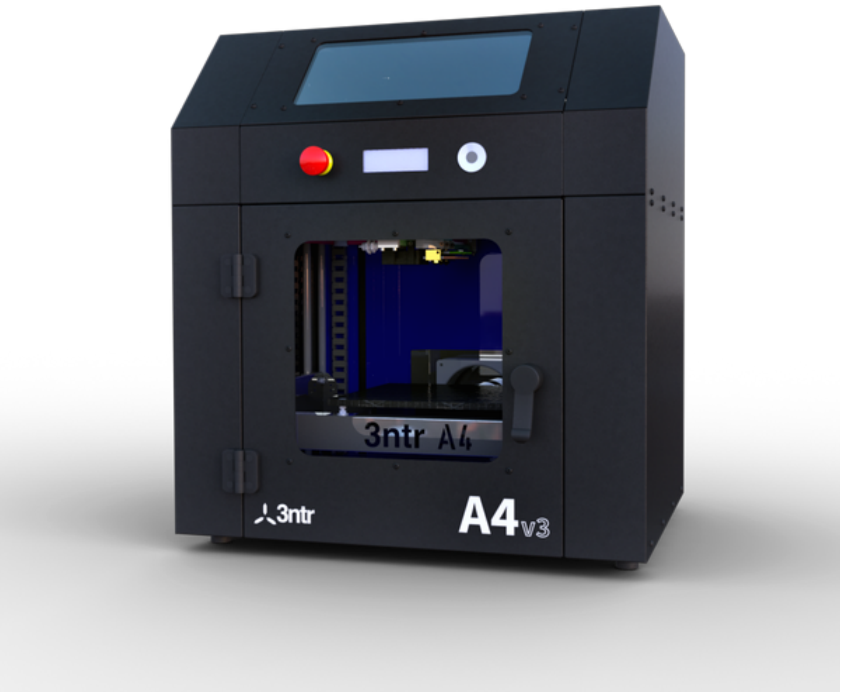
$$m_\chi = \frac{1}{|\Omega|}\int_\Omega \chi d\Omega,$$

- ✓ The lowest compliance is achieved with the single stiffer material
- ✓ Using the graded material we can obtain FGM structure with relatively low compliance using considerably LESS MATERIAL

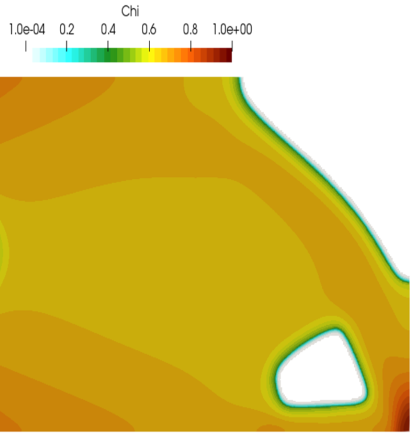
Printed cantilever beam: FDM 3D printer at the ProtoLab in Pavia

- ✓ This machine prints a filament of thermoplastic polymer which is first heated and then extruded through a printing nozzle

- ✓ Then it is deposited layer by layer until the desired object is obtained.



A 3D printing workflow for optimized FGM structures

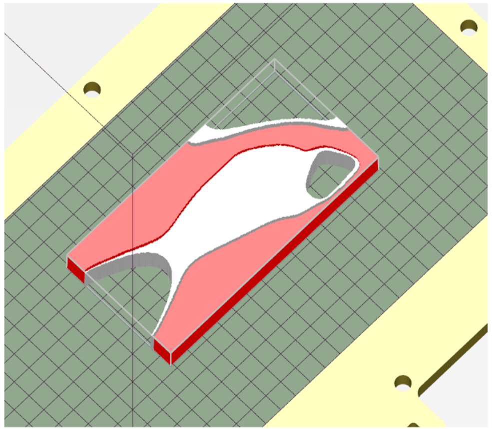


(a)

Extraction of STL files

(b)

Extrusion

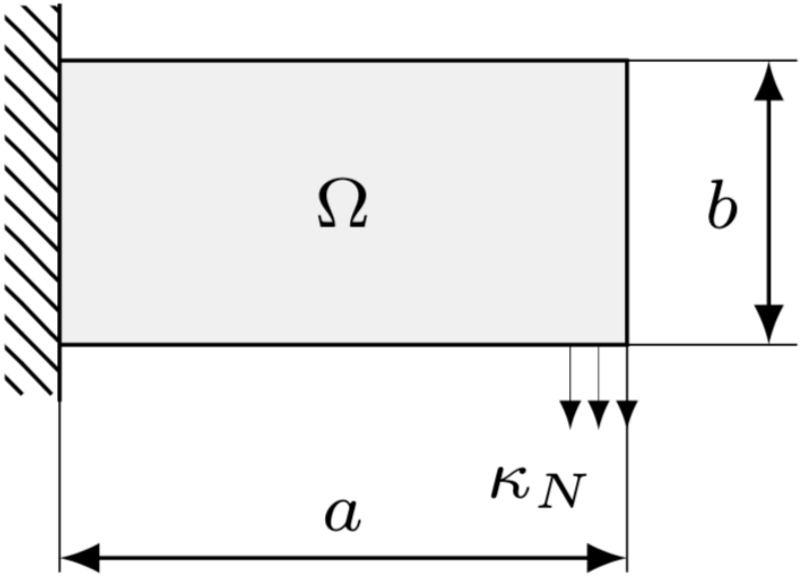


(c)

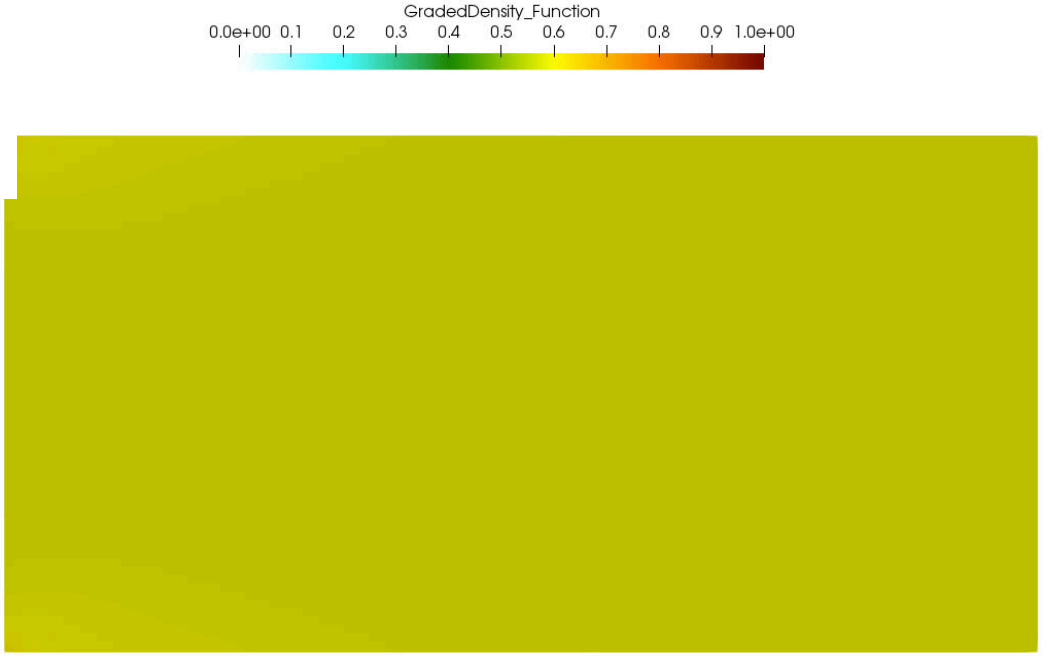
Printing

(d)

Optimization: graded material



Grey = no material
Color = material
Different color = different materials



OPEN PROBLEMS:

- Introduce constraint on stress and displacement
- The stiffness is strongly influenced by the micro-structure of the partially filled regions

The stress constraint

In the mathematical part of the paper we handle a more general functional including stress constraint:

$$(2.5)\quad \mathcal{J}(\mathbf{u},\boldsymbol{\sigma},\varphi,\chi)=\kappa_1\int_\Omega\left(\frac{\mathcal{W}(\varphi)}{\gamma}+\gamma\frac{|\nabla\varphi|^2}{2}\right)\mathrm{d}x+\kappa_2\int_\Omega\left(I_C(\varphi,\chi)+\frac{|\nabla\chi|^2}{2}\right)\mathrm{d}x$$
$$+\kappa_3\int_\Omega\varphi\,(\mathbf{f}\cdot\mathbf{u})\,\mathrm{d}x+\kappa_3\int_{\Gamma_g}\mathbf{g}\cdot\mathbf{u}\,\mathrm{d}x+\kappa_5\int_\Omega F(\boldsymbol{\sigma})\,\mathrm{d}x$$

over $(\mathbf{u},\boldsymbol{\sigma},\varphi,\chi)\in\mathcal{U}_{ad}$, and subject to the stress-strain state relation

$$(2.6)\qquad -\operatorname{div}\boldsymbol{\sigma}=\varphi\,\mathbf{f}\quad\text{in }\Omega$$
$$(2.7)\qquad \boldsymbol{\sigma}\cdot\mathbf{n}=\mathbf{g}\quad\text{on }\boldsymbol{\Gamma}_{\mathbf{g}}$$
$$(2.8)\qquad \boldsymbol{\sigma}=\mathbb{K}(\varphi,\chi)\varepsilon(\mathbf{u})\quad\text{in }\Omega$$
$$F(\boldsymbol{\sigma})=\left(\Phi(\boldsymbol{\sigma})^2-\Phi^2_{max}\right)_+$$

where $(\cdot)_+$ denotes the positive part function and we can choose, for example, $\Phi(\boldsymbol{\sigma})=\sqrt{\frac{\sum_{i,j}(\lambda_i-\lambda_j)^2}{2}}$, being $\{\lambda_j\}$ the eigenvalues of the stress $\boldsymbol{\sigma}$

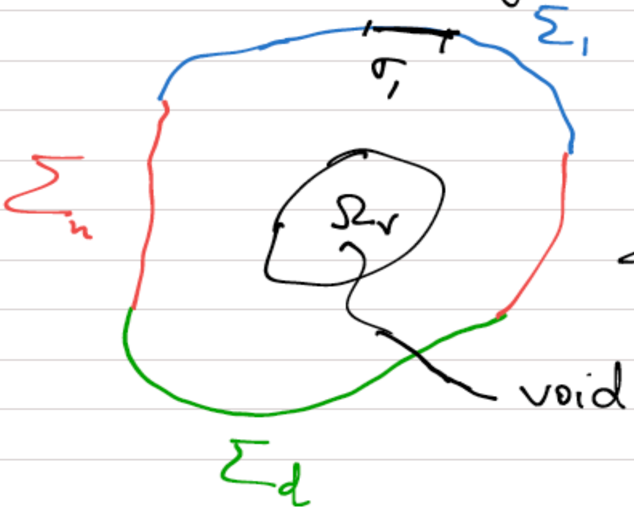
Open problem: include it in the simulations

A different approach: inverse problem

Joint with
A. Aspri, E. Beretta,
C. Cavaterra, M. Verani

We start from the boundary value problem

$$
(1.1)\quad \begin{cases}
\operatorname{div}(\mathbb{C}^M \hat{\nabla} u) = 0 & \text{in } \Omega_M \\
u = 0 & \text{on } \Sigma_d \\
(\mathbb{C}^M \hat{\nabla} u)\nu = 0 & \text{on } \Sigma_n \cup \Sigma \\
(\mathbb{C}^M \hat{\nabla} u)\nu = g & \text{on } \Sigma_1
\end{cases}
$$



where

$\Omega_M = \Omega \setminus \overline{\Omega}_V$

$\Sigma = \partial\Omega_V$

We know that there exists a unique variational solution

The inverse problem

Assume that we can measure the displacement $u|_{\sigma_1} = \bar{u}$ where $\sigma_1 \subset \Sigma_1$

goal

Minimize

$$F(u, \Omega_v) = k_1 \int_{\sigma_1} |u - \bar{u}|^2 ds + k_2 \, \mathrm{vol}(\Omega_M)$$

$$+ k_3 \underbrace{\mathrm{Per}(\Sigma)}_{\text{perimeter of } \Sigma} + k_4 \underbrace{\int_{\Sigma_1} g \cdot u}_{\text{compliance}}$$

Phase-field approach

Let now for $\varphi \in H^1(\Omega, [0,1])$

$$\mathbb{C}_\varepsilon(\varphi) = \mathbb{C}^M \varphi + \varepsilon^2 \mathbb{C}^2 (1-\varphi)$$

(Setting $\mathbb{C}^M = \mathbb{C}^1$) and where $\mathbb{C}^2$ is an elast. tensor with same prop. as $\mathbb{C}^M$

Then,

$$\mathbb{C}_\varepsilon(1) = \mathbb{C}^M$$

$$\mathbb{C}_\varepsilon(0) = \varepsilon^2 \mathbb{C}^2$$

The new problem

Consider then $u_\varepsilon(\varphi) \in H^1(\Omega)$ the unique weak solution to

$$
1) \begin{cases}
\operatorname{div}\left(\mathbb{C}_\varepsilon(\varphi)\, \hat{\nabla} u \right) = 0 & \text{in } \Omega \\
u = 0 & \text{on } \Sigma_d \\
\left(\mathbb{C}_\varepsilon(\varphi)\, \hat{\nabla} u \right) \nu = 0 & \text{on } \Sigma_n \\
\left(\mathbb{C}_\varepsilon(\varphi)\, \hat{\nabla} u \right) \nu = g & \text{on } \Sigma_1
\end{cases}
$$

The new minimization problem

$$\min_{\varphi \in H^1(\Omega,[0,1])} F_\varepsilon(u,\varphi) \qquad (u = u(\varphi))$$

where

$$F_\varepsilon = k_1 \int_{\sigma_1} |u - \bar{u}|^2 \, ds + k_2 \int_\Omega \varphi \, dx + k_3 \int_\Omega \varepsilon \frac{|\nabla\varphi|^2}{2} + \frac{\Psi(\varphi)}{\varepsilon} + k_4 \int_{\Sigma_1} g \cdot u$$

$$\Psi(\varphi) = \varphi(1-\varphi)$$

Solved and Open problems

It is possible to get the optimality condition for F at fixed ε.

Open problems:

✓ Let ε go to zero: difficult problem

✓ Numerical results: elastic inclusions and cavities reconstructions (work in progress with A. Aspri, E. Beretta, C. Cavaterra, M. Verani)

Mixed variational formulations based on phase-field

Joint with
Michele Marino · Ferdinando Auricchio ·
Alessandro Reali · Elisabetta Rocca ·
Ulisse Stefanelli

- Introduce a mixed variational principle combining a phase-field functional for topology optimization **with a (three-field) Hu-Washizu functional**, then including directly in the formulation
  - equilibrium, constitutive
  - compatibility equation
- Investigate two different topology optimization principles:
  1. a formulation imposing a priori the amount of material to be distributed within the design domain (**formulation with volume constraint**)
  2. a formulation based on a **minimization of material to be distributed** given that a cost (i.e., a penalty parameter) is assigned to the material

The second approach avoids the introduction of a global constraint, respecting the convenient local nature of the finite element discretization

Numerical results

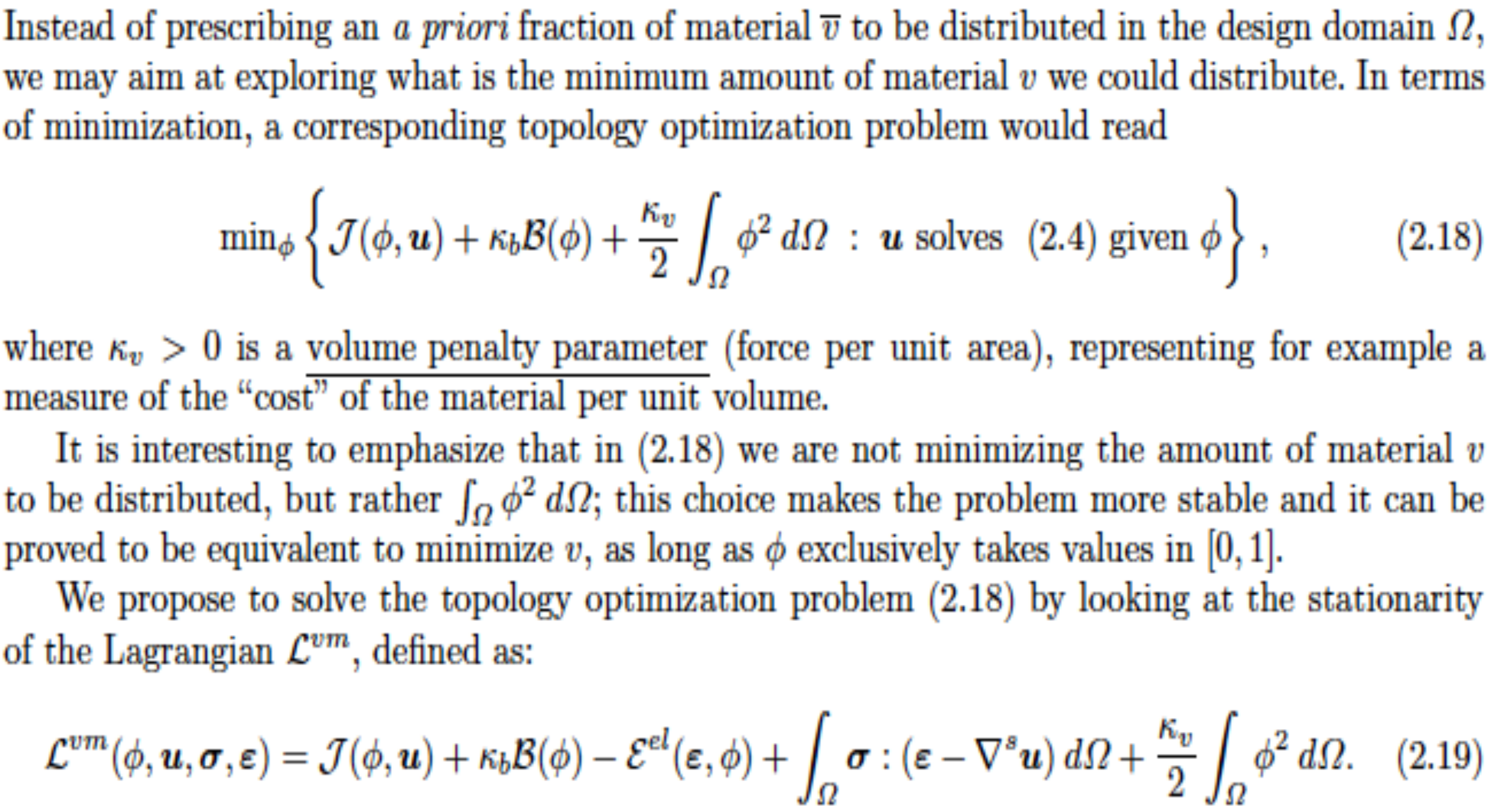
1. We introduce a **Simultaneous Analysis and Design (SAND)** monolithic solution strategy, thanks to the Hu-Washizu functional rationale and **based on an Allen-Cahn scheme**, where the phase-field variable evolves under the respect of mechanical equilibrium at each computational incremental step

2. We propose insightful investigations, analysing both numerical convergence behaviours and obtained final designs, based on comparative analyses between simulation strategies (monolithic SAND vs. staggered NAND) and between topology optimization principles (**volume constraint vs. minimization**)

3. We analyse the performance of the proposed variational formulation based on volume minimization **also on three-dimensional case studies**

Stationarity of the Lagrangean

Minimization of material to be distributed



Instead of prescribing an *a priori* fraction of material $\overline{v}$ to be distributed in the design domain Ω, we may aim at exploring what is the minimum amount of material v we could distribute. In terms of minimization, a corresponding topology optimization problem would read

$$\min_\phi \left\{ \mathcal{J}(\phi,\boldsymbol{u}) + \kappa_b \mathcal{B}(\phi) + \frac{\kappa_v}{2}\int_\Omega \phi^2\, d\Omega \ :\ \boldsymbol{u} \text{ solves } \ (2.4) \text{ given } \phi \right\}, \tag{2.18}$$

where $\kappa_v > 0$ is a volume penalty parameter (force per unit area), representing for example a measure of the "cost" of the material per unit volume.

It is interesting to emphasize that in (2.18) we are not minimizing the amount of material v to be distributed, but rather $\int_\Omega \phi^2\, d\Omega$; this choice makes the problem more stable and it can be proved to be equivalent to minimize v, as long as ϕ exclusively takes values in $[0, 1]$.

We propose to solve the topology optimization problem (2.18) by looking at the stationarity of the Lagrangian $\mathcal{L}^{vm}$, defined as:

$$\mathcal{L}^{vm}(\phi,\boldsymbol{u},\boldsymbol{\sigma},\boldsymbol{\varepsilon}) = \mathcal{J}(\phi,\boldsymbol{u}) + \kappa_b\mathcal{B}(\phi) - \mathcal{E}^{el}(\boldsymbol{\varepsilon},\phi) + \int_\Omega \boldsymbol{\sigma} : (\boldsymbol{\varepsilon} - \nabla^s \boldsymbol{u})\, d\Omega + \frac{\kappa_v}{2}\int_\Omega \phi^2\, d\Omega. \tag{2.19}$$

The functionals and the elasticity tensor

$$\mathcal{J}(\phi, \boldsymbol{u}) = \mathcal{C}(\phi, \boldsymbol{u}) + \kappa_\phi \mathcal{P}(\phi),$$

$$\mathcal{C}(\phi, \boldsymbol{u}) = \int_\Omega \phi \boldsymbol{b} \cdot \boldsymbol{u} \, d\Omega + \int_{\Gamma_N} \boldsymbol{t} \cdot \boldsymbol{u} \, d\Gamma \, ,$$

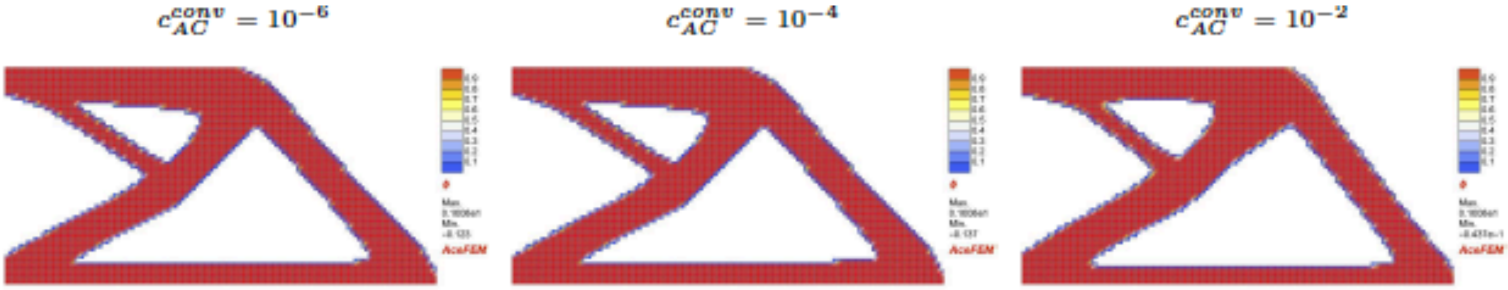
$$\mathcal{P}(\phi) = \int_\Omega \left[\gamma_\phi \|\nabla \phi\|^2 + \frac{1}{\gamma_\phi} \psi_0(\phi) \right] d\Omega \, .$$

$$\mathcal{B}(\phi) = \int_\Omega b(\phi) d\Omega \, , \quad \text{with} \;\; b(\phi) = \begin{cases} \dfrac{(\phi - 1)^2}{2} & \phi > 1 \\ 0 & 0 \leq \phi \leq 1 \\ \dfrac{\phi^2}{2} & \phi < 0 \end{cases} \, ,$$

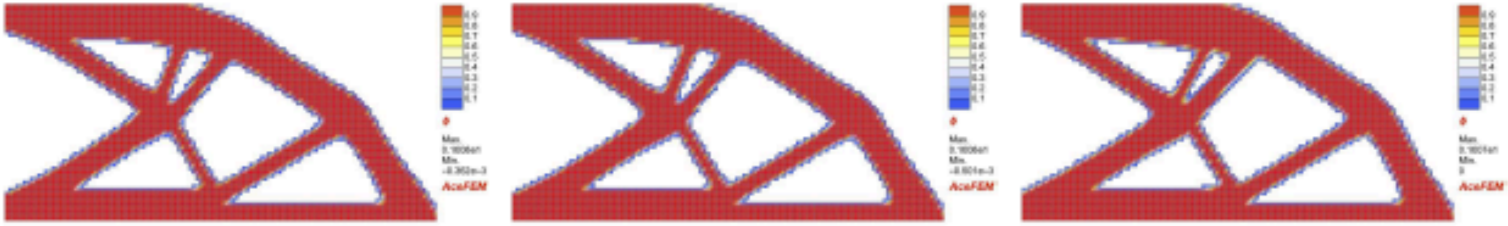
$$\mathcal{E}^{\text{el}}(\boldsymbol{\varepsilon}, \phi) = \frac{1}{2} \int_\Omega \mathbb{C}(\phi) \boldsymbol{\varepsilon} : \boldsymbol{\varepsilon} \, d\Omega.$$

$$\mathbb{C}(\phi) = \left[\delta + (1 - \delta) \frac{\text{Exp}(p\phi^p)}{\text{Exp}(p)} \right] \mathbb{C}_A \, ,$$

Material distribution with different initial conditions



(a) Volume constraint $\mathcal{L}^{vc}_{AC}$ – I.C. : $\phi_0 = 1$ and $\phi_0 = \bar{v}$

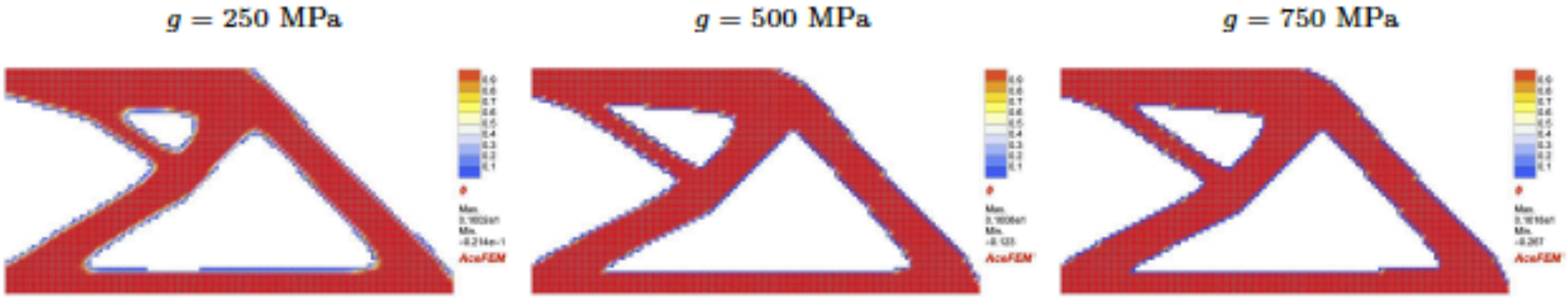


(b) Volume minimization $\mathcal{L}^{vm}_{AC}$ – I.C. : $\phi_0 = 1$

(c) Volume minimization $\mathcal{L}^{vm}_{AC}$ – I.C. : $\phi_0 = 0.5$

Fig. 6: Material distribution phase field ϕ_{sol} obtained with different values of the convergence parameter c^{conv}_{AC} and different initial conditions (I.C.) for: a) the volume constraint formulation (coinciding with $\phi_0 = \bar{v}$ and $\phi_0 = 1$); b) the volume minimization formulation with $\phi_0 = 1$; c) the volume minimization formulation with $\phi_0 = 0.5$.

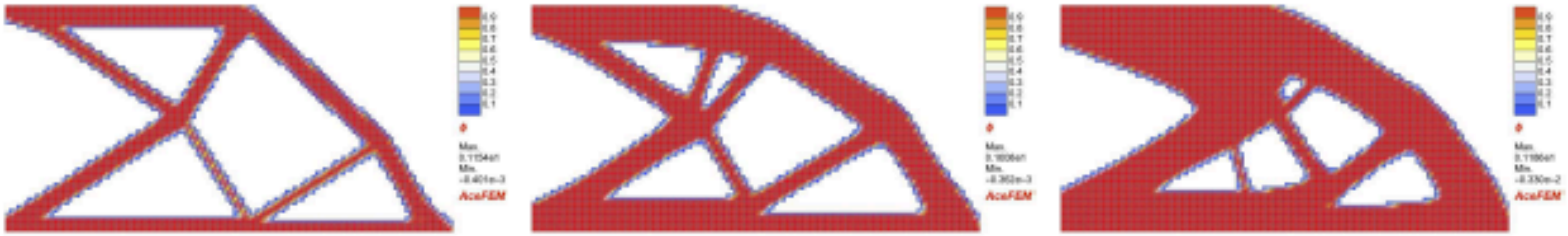
Volume constraint and volume minimization


$g = 250$ MPa

$g = 500$ MPa

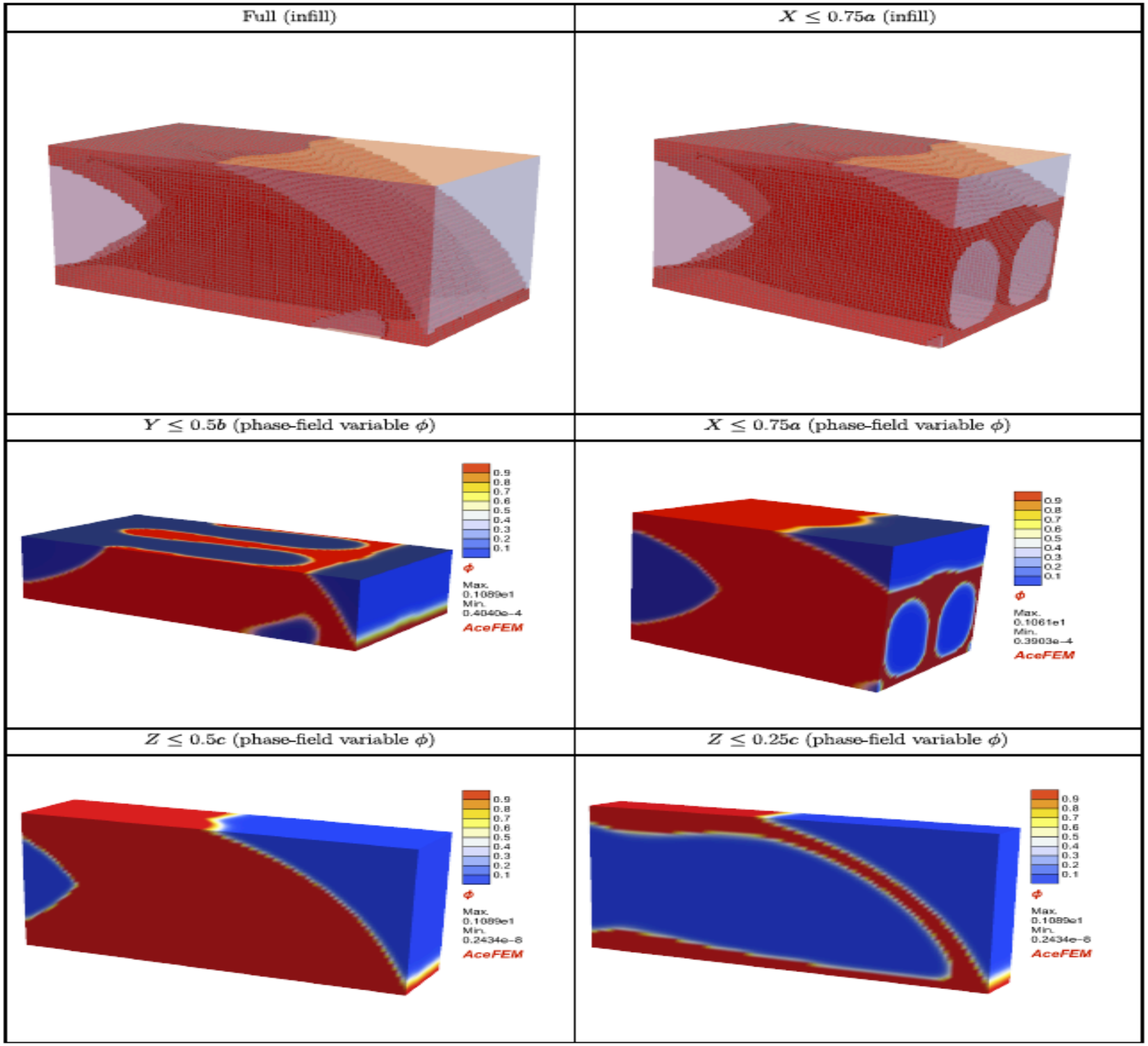
$g = 750$ MPa

(a) Volume constraint $\mathcal{L}^{vc}_{AC}$



(b) Volume minimization $\mathcal{L}^{vm}_{AC}$

Fig. 11: Material distribution phase field ϕ_{sol} obtained with different values of the applied load magnitude g for: a) the volume constraint formulation with $\bar{v} = 0.4$; b) the volume minimization formulation with $\kappa_v = 100$ MPa.



Full (infill)

$X \leq 0.75a$ (infill)

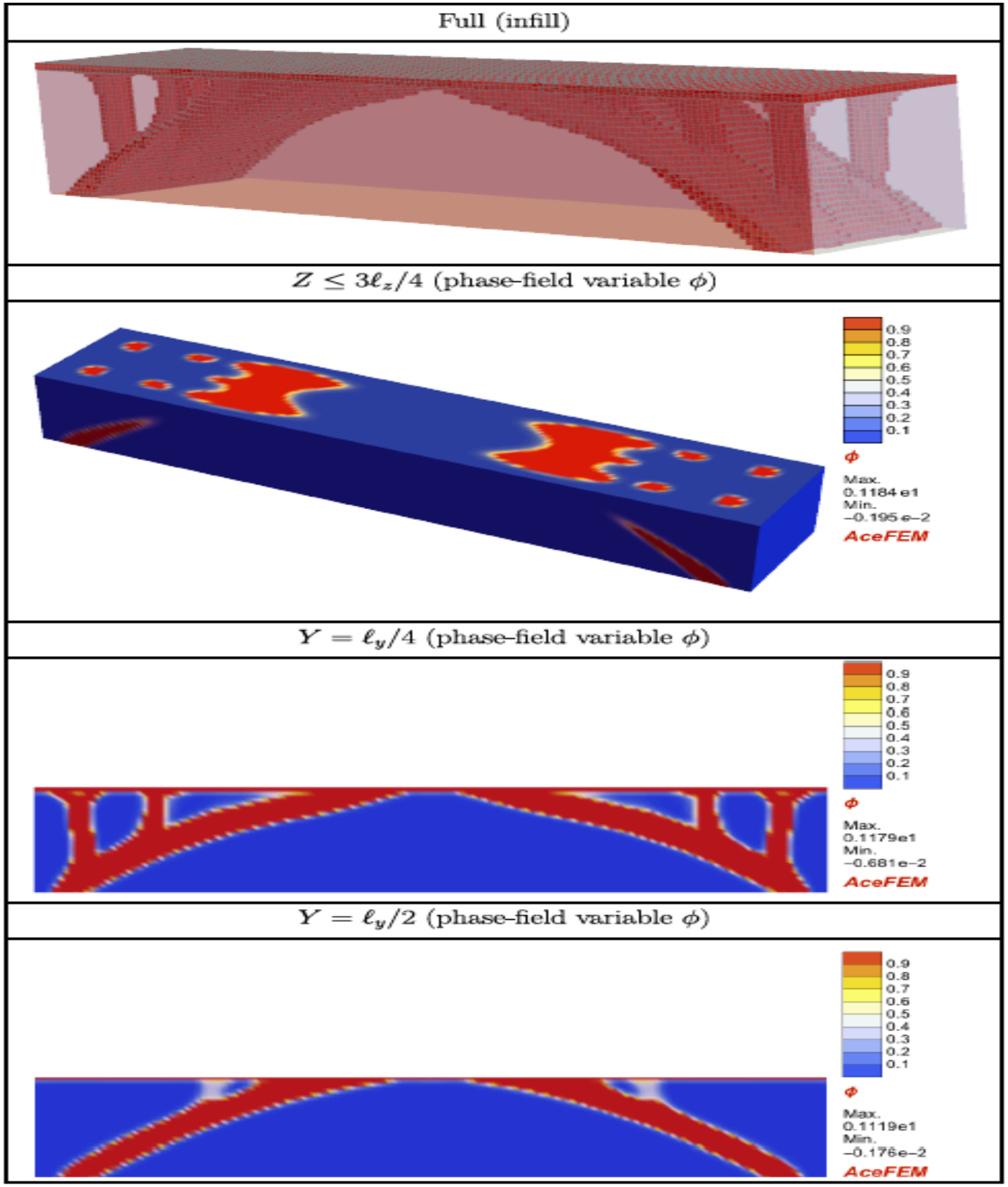
$Y \leq 0.5b$ (phase-field variable ϕ)

$X \leq 0.75a$ (phase-field variable ϕ)

$Z \leq 0.5c$ (phase-field variable ϕ)

$Z \leq 0.25c$ (phase-field variable ϕ)

Fig. 13: Topology optimization of the 3D cantilever.



Full (infill)

$Z \leq 3\ell_z/4$ (phase-field variable ϕ)

$Y = \ell_y/4$ (phase-field variable ϕ)

$Y = \ell_y/2$ (phase-field variable ϕ)

Fig. 14: Topology optimization of the 3D bridge.

Volume minimization vs Volume constraint

- The volume minimization formulation shows an Allen-Cahn **convergence behavior with better properties than the volume constraint formulation**, since the decrease of the Allen-Cahn error measure is highly oscillatory for the volume-contraint case but not for the volume-minimization case

- The **final design** is practically independent from the applied load with the volume constraint functional, while is **highly affected with the volume minimization one** which is seen as a significative advantage of this latter formulation

- **The number of Newton-Raphson iterations** required for solving the problem **with the the volume minimization functional is in most cases significantly lower** (> 50%) than the ones employed with the volume constraint functional

Conclusions and futher perspectives

These results can be a **starting point for more advanced developments** of phase-field topology optimization that considers

- ➢ loading uncertainties (cf. Dunning al.et al., 2011);
- ➢ multi-target strategies, e.g., controlling
  - ✓ both geometry and compliance (cf. Strömberg, 2010),
  - ✓ both geometry and stresses (Burger and Stainko, 2006),
  - ✓ the structure life-cycle cost (Sarma and Adeli, 2002),
  - ✓ manufacturing costs (Liu et al., 2019)

Thank you for the attention!

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