Numerical methods for piecewise Chebyshevian splines and applications

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CALCOLO SCIENTIFICO E MODELLI MATEMATICI:
alla ricerca delle cose nascoste attraverso le cose manifeste
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outline

1. From polynomial splines to. . .
2. . . . piecewise Chebyshevian splines
3. Existence of B-spline bases and refinability
4. Evaluation and computational aspects
5. Applications: design, interpolation and isogeometric analysis

Polynomial splines

- Given $I = [a, b] \subset \mathbb{R}$, $\mathbb{P}_n$ polynomials of deg. n, choose:
  - $t_1 < \cdots < t_q \in [a, b]$: **interior knots**
  - positive integers $m_1, \ldots, m_q$: **knot multiplicities**, with $1 \leqslant m_k \leqslant n$ for $1 \leqslant k \leqslant q$

- **Spline space** $\mathbb{S}$:= set of all functions $S : [t_0, t_{q+1}] := [a, b] \to \mathbb{R}$ s.t.
  - for $0 \leqslant k \leqslant q$, there exists $F_k \in \mathbb{P}_n$ such that $S(x) = F_k(x)$ for all $x \in [t_k, t_{k+1}]$
  - S is C^{n-m_k} at t_k

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  - S is C^{n-m_k} at t_k
- $\dim \mathbb{S} = n + 1 + m$, with $m := \sum_{k=1}^{q} m_k$

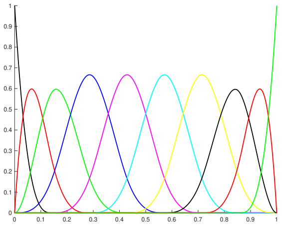
B-spline bases

- For any $S \in \mathbb{S}$: $S(x) = \sum_{i=-n}^{m} N_i^n(x) P_i, \quad x \in [a,b]$

  where $(N_{-n}, \ldots, N_m)$ is the **B-spline basis**. For simple knots it satisfies :

  - Support property: $Supp(N_i^n) = [t_i, t_{i+n+1}]$
  - Positivity property: $N_i^n(x) > 0$ for $x \in]t_i, t_{i+n+1}[$
  - Normalisation property: $\sum_{i=-n}^{q} N_i^n = \mathbb{1}$

  $\longrightarrow$ optimal basis for shape preservation and numerical behavior



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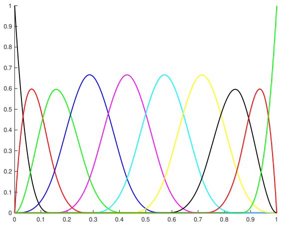

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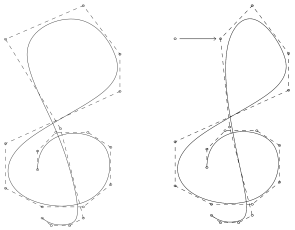
- When $q = 0$ (*no interior knot*) $\Rightarrow$ **Bernstein basis**

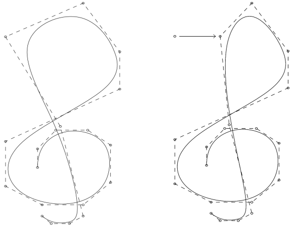
Polynomial splines and design



cubic splines with 19 control points

- ☺ Excellent mimicking, local control
- ☹ Need to change control points to improve the curve

Polynomial splines and design/interpolation



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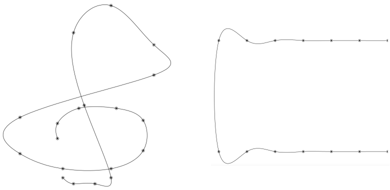

interpolating cubic splines

- Unique solution under interlacing conditions [Schoenberg-Whitney 53]
- Undesired oscillations

A possible solution:

introduce shape parameters

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  - $1, x, \dots, x^{n-2}, \cosh x, \sinh x$ (on $I = \mathbb{R}$) "hyperbolic spaces"
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Theorem [Mazure 2005, 2006]

Design in $\mathbb{E}$ i.e. $\exists$ Bernstein bases in $\mathbb{E}$ $\Longleftrightarrow$ Interpolation in $D\mathbb{E}$ i.e. $D\mathbb{E}$ is an EC-space on I *

* $D\mathbb{E}$ EC-space on $I \Rightarrow \mathbb{E}$ EC-space on I

Piecewise Chebyshevian splines

- Choose:
  - interior knots $t_1 < \cdots < t_q$ knots interior to $I := [t_0, t_{q+1}]$ and their multiplicities $m_1, \dots, m_q$, $0 \leqslant m_k \leqslant n$
  - a sequence $\mathbb{E}_k$, $k = 0, \dots, q$ of **section spaces**: $\forall k$, $\mathbb{E}_k$ is $(n+1)$-dimensional, contains constants, and $D\mathbb{E}_k$ is a EC on $[t_k, t_{k+1}]$
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- $\dim\mathbb{S} = (n+1) + m$, with $m := \sum_{k=1}^{q} m_k$

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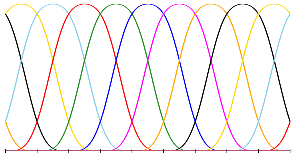

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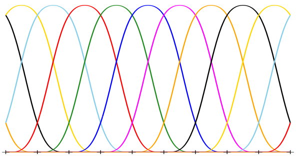

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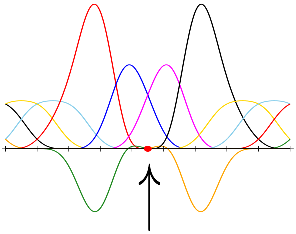

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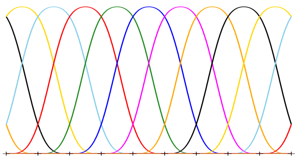

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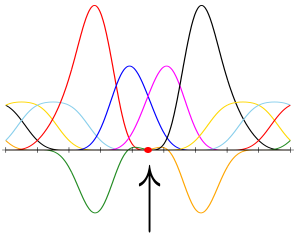

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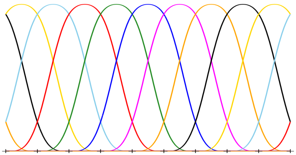

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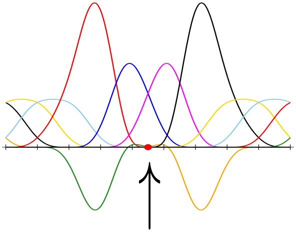

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- Joining a sequence of section EC-spaces $\mathbb{E}_k$ $\not\Rightarrow$ Existence of a B-spline basis.
- One B-spline basis $\not\Rightarrow$ Refinable B-spline basis

How to determine the existence of a

REFINABLE B-SPLINE BASIS?

Characterization of existence

- **piecewise weight functions**
  $(w_1, \ldots, w_n)$:
  w_i is defined, positive, C^{n-i}
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Equivalence of:

(i) $\exists$ Refinable B-spline basis

(ii) $\exists$ piecewise generalized derivatives $L_1, \ldots, L_n$ such that, for each $S \in \mathbb{S}$ and each $k = 1, \ldots, q$:

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leads to
- explicit necessary and sufficient conditions for **(i)** if $n \leqslant 4$
- numerical test for any n

Numerical test of existence

and evaluation algorithm

Key ideas

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Let $U = \sum_{j=-n}^{m} u_j N_j$, $w_1 = DU$. Are equivalent:

(a) (u_j) strictly increasing sequence

(b) $w_1 > 0$ on each $[t_k^+, t_{k+1}^-]$ + $L_1\mathbb{S} = \frac{1}{w_1} D\mathbb{S}$ is a space of continuous piecewise Cheb. splines with a refinable B-spline basis

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  ☆ $$w_1 = \sum_{j=-n+1}^{m} (u_j - u_{j-1}) Q_j$$

  $Q_j := DN_j^\star$ B-spline like basis (non normalized)

  $N_j^\star := \sum_{p \geqslant j} N_p$ transition functions

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1) **attempt** to compute candidate transition functions $N_\ell^\star \in \mathbb{S}$ satisfying (Hermite) interpolation conditions;

2) **if** such a sequence **exists**, compute $Q_\ell := DN_\ell^\star \in D\mathbb{S}$

3) **if**, for all ℓ, "**Q_ℓ is positive + ...**", compute $w_1 := \sum_\ell Q_\ell$ and $\bar{N}_\ell := \frac{Q_\ell}{w_1} \in L_1\mathbb{S}$

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☺ If we can get to the end (i.e. local dimension 2) $\Rightarrow$ $\mathbb{S}$ has ref. basis

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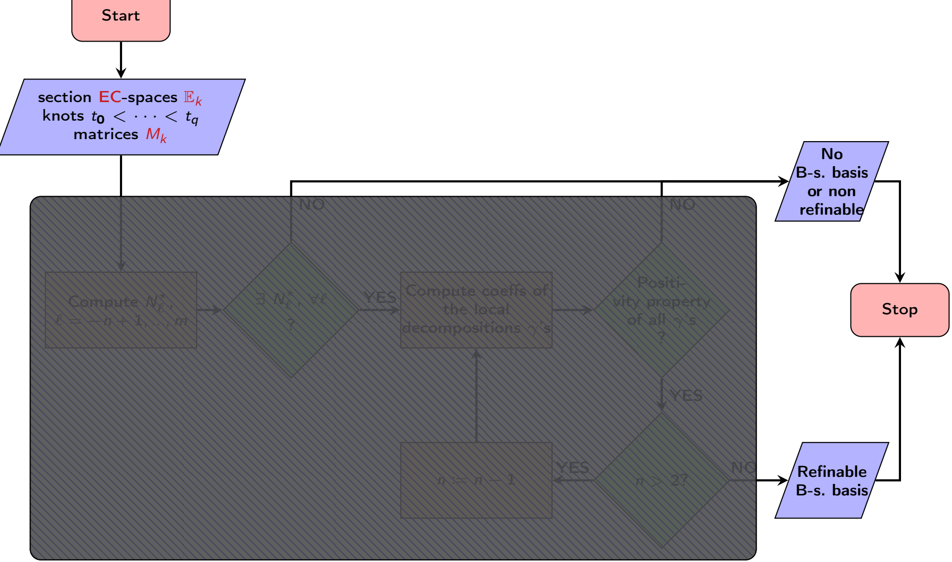
If we can get to the end (i.e. local dimension 2) $\Rightarrow \mathbb{S}$ has ref. basis

In case any of the above **if** statements has a **negative answer**:
STOP, $\mathbb{S}$ does not have a refinable B-spline basis.

A numerical test and an evaluation method

[B., Casciola, Mazure, 2017-19-20]



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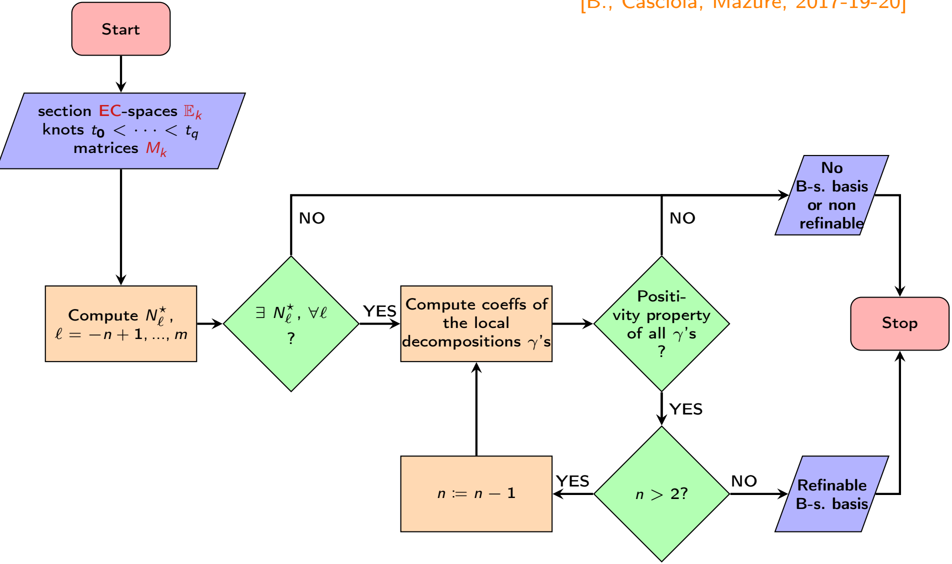
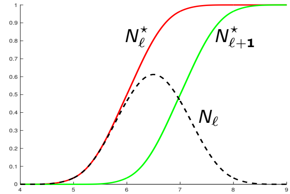


More on the evaluation of the B-spline basis

& computational aspects

Evaluation by transition functions

- Recall: $$N_\ell^\star := \sum_{p \geqslant \ell} N_p \quad \longleftrightarrow \quad N_\ell = N_\ell^\star - N_{\ell+1}^\star$$



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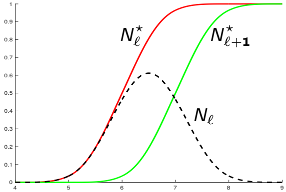
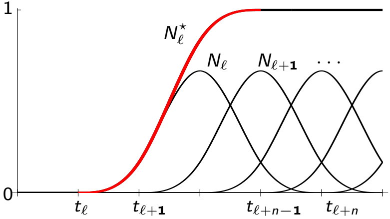


- From the properties of the B-spline basis (simple knots):

(i) $N_\ell^\star = \begin{cases} 0 & x \leqslant t_\ell \\ 1 & x \geqslant t_{\ell+n} \end{cases}$

(ii) N_ℓ^* vanishes exactly n times at t_ℓ and $1 - N_\ell^\star$ vanishes exactly n times at $t_{\ell+n}$



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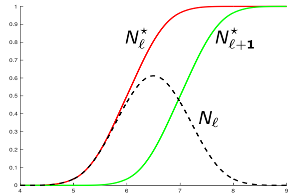

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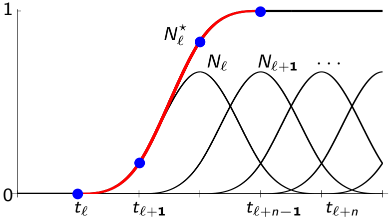
- (i) + (ii) $\rightsquigarrow$ each $N_\ell^\star$ is the *unique solution* of an Hermite interpolation problem of size $n^2 + n$ at most

★ $$\begin{cases} D^r N_\ell^\star(t_\ell) = 0 \\ D_-^r N_\ell^\star(t_j) = D_+^r N_\ell^\star(t_j), & j = \ell+1, \ldots, \ell+n-1 \\ & r = 0, \ldots, n-1 \\ D^r N_\ell^\star(t_\ell) = \delta_{r,0} \end{cases}$$



A comparison of computational procedures

Evaluation by Transition Functions B., Casciola, Romani, 2022

TF1 for each $k = 0, \ldots, q$, evaluate the Wronskian matrices of the given basis in $\mathbb{E}_k$ at t_k and t_{k+1};

TF2 solve as many linear systems ☆ as $\dim(\mathbb{S}) - 1$, to determine all the $N_\ell^\star$.

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Evaluation by Extraction Operator
Hiemstra, Hughes, Manni, Speleers, Toshniwal, 2020; Speleers, 2022

EO1 = **TF1**;

EO2 for each k, compute the BB in $\mathbb{E}_k$ by solving (n+1) linear systems of size $(n+1)$;

EO3 for each k, evaluate the Wronskian matrices of the BB at t_k and t_{k+1} and construct the matrix that stores the continuity conditions of $\mathbb{S}$;

EO4 compute the Extraction Operator $\rightsquigarrow$ determine (for each break-point and each continuity order) the null space of a suitable vector.

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Qualitative comparison of step 2:

$n = 6$, 4 intervals, simple knots $\Rightarrow \dim(\mathbb{S}) = 10$ $\rightsquigarrow$

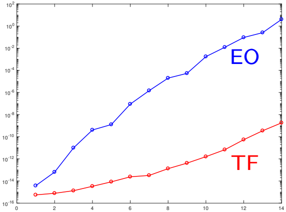
	# systems	size
TF2	9	$\leqslant 28 \times 28$
EO2	28	7×7

Numerical experiments I

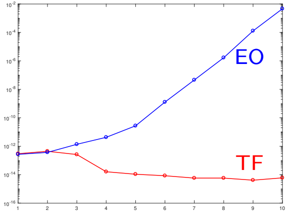
Apply TF and EO to the computation of the Bernstein basis in $\mathbb{E}$ spanned by

$$1, \ldots, x^n, \cosh(10x), \sinh(10x), \quad \text{on} \quad [a, b]$$

Symbolic error test: max abs err on Bernstein basis functions (16-digits decimal precision) w.r.t. symbolic computation



$n = 1, \ldots, 14,\ b - a = 4$



$n = 5$ (dim($\mathbb{E}$)=8) on $[0, b]$, $b \in \{10^{-3}, \ldots, 2^6\}$

Numerical experiments !

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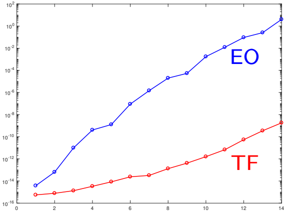

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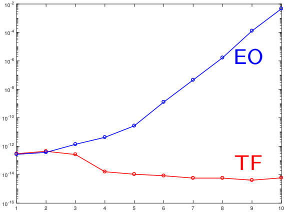

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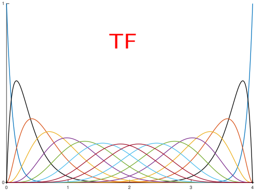
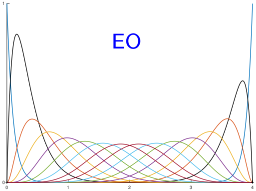




Symmetry check:
max abs err distance of symmetric functions

$\longleftarrow\ n = 13$

Numerical experiments I

Apply TF and EO to the computation of the Bernstein basis in $\mathbb{E}$ spanned by

$$1, \ldots, x^n, \cosh(10x), \sinh(10x), \quad n = 13$$

Symbolic error test

B_i	**TF**	**EO**
0	6.66134e-16	9.63072e-07
1	3.89433e-12	**2.60750e-01**
2	2.63505e-11	1.82676e-03
3	1.68028e-10	1.06772e-03
4	3.09308e-10	5.95284e-05
5	**3.49708e-10**	7.39515e-06
6	2.91161e-10	7.17741e-07
7	1.49372e-10	8.07312e-08
8	4.63247e-11	4.78432e-09
9	6.89226e-12	5.24595e-10
10	1.89137e-12	3.09941e-11
11	5.37847e-13	4.54567e-12
12	1.27044e-13	1.51379e-13
13	6.84730e-14	6.10623e-15
14	3.21965e-15	9.99201e-16
15	5.55111e-17	4.13590e-25

Symmetry check

$(B_i,\ B_{15-i})$	**TF**	**EO**
0,15	2.10942e-15	9.62666e-07
1,14	3.89667e-12	**2.60741e-01**
2,13	2.64075e-11	1.82279e-03
3,12	1.68133e-10	1.06728e-03
4,11	3.09746e-10	5.94964e-05
5,10	**3.49886e-10**	7.39387e-06
6,9	2.94925e-10	7.17135e-07
7,8	1.82598e-10	8.54473e-08

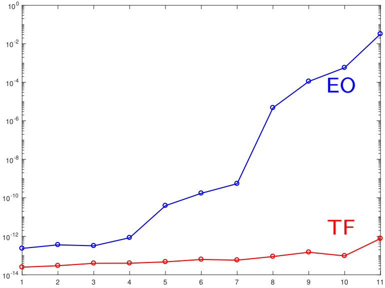
Numerical experiments II

C^6 spline with $n = 7$ and:

$$\mathbb{E}_0 = \mathbb{E}_3 \quad \text{spanned by} \quad 1, x, \ldots, x^5, \cos x, \sin x$$
$$\mathbb{E}_1 = \mathbb{E}_2 \quad \text{spanned by} \quad 1, x, \ldots, x^5, \cosh x, \sinh x$$

$$t_1 - t_0 = 1 - w, \quad t_2 - t_1 = w, \quad t_3 - t_2 = w, \quad t_4 - t_3 = 1 - w$$

Symbolic error test



$w = 0.5, \ldots, 0.998$

Symmetry check, $w = 0.998$

$N_{i,11}$	TF	EO
1,11	**2.73836e-13**	1.50357e-12
2,10	2.73337e-13	1.91716e-12
3,9	2.20102e-14	9.11726e-12
4,8	5.15490e-14	**3.23910e-02**
5,7	6.73439e-14	3.12476e-02
6,6	3.02536e-14	2.54093e-06

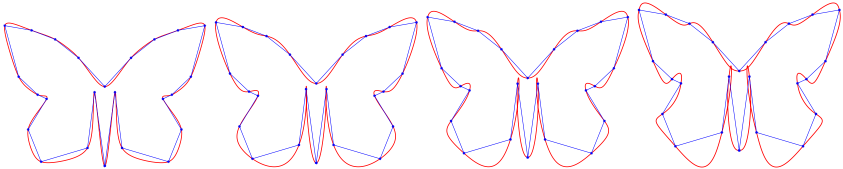
Applications and examples

Cardinal, symmetric, C^2 quintic splines

C^2 quintic splines, equispaced knots, everywhere $M_k = M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & b & 1 & 0 \\ 0 & \frac{bc}{2} & c & 1 \end{bmatrix}$

Numerical characterization [B., Casciola, Mazure, 2019]

$\mathbb{S}$ has a refinable B-spline basis $\iff b > -6$ and $c > -4$



Interpolation in $\mathbb{S}$ with $c = 0$ and, left to right, $b = -5.99; 0$ (ordinary C^4 splines);$10; 100$

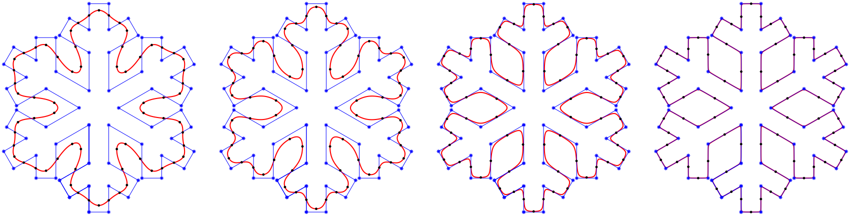
Cardinal mixed trigonometric-hyperbolic splines

C^5 splines with $t_{k+1} - t_k = h$ and all $\mathbb{E}_k$ spanned by:

$$1, \cosh ax \cos x, \cosh ax \sin x, \sinh ax \cos x, \sinh ax \sin x, \cosh ax, \sinh ax$$

Numerical characterization [B., Casciola, Mazure, 2019]

$\mathbb{S}$ has a refinable B-spline basis $\iff h < \pi$



$h = 3$, left to right: $a = 0.01; 0.7; 1.5; 3.$

Isogeometric analysis with piecewise Chebyshevian splines

- NURBS [Hughes et al. 2005] and generalized splines [Manni et al. 2011] (classical tools in IgA) are examples of Piecewise Chebyshevian splines with refinable B-spline bases

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- Now we have many more such PC spaces . . .

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- Now we have many more such PC spaces . . .
- **Ex:**

$$\begin{cases} -u''(x) + -2u'(x) + e^x u(x) = f(x), & x \in [-\pi, \pi] \\ u(-\pi) = -e^{-\pi}, \qquad u(\pi) = -e^{\pi} \end{cases}$$

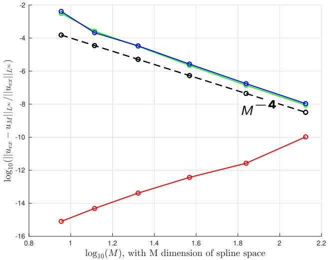
with solution $u_{ex}(x) = e^x \sin(x) + e^{-x} \cos(x)$

- IGA collocation by spline space $\mathbb{S}$ with C^4 cont nuity, uniform knots and all sections $\mathbb{E}_k$ in:

A) $1, x, x^2, x^3, x^4, x^5$

B) $1, x, x^2, x^3, \cos x, \sin x$, [Manni et al. 2015]
ref. basis for $h < \pi$

C) $1, x, \cosh x \cos x, \cosh x \sin x, \sinh x \cos x, \sinh x \sin x$ ref. basis for $h < \pi$

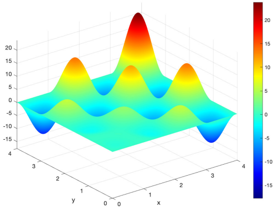


Isogeometric analysis with piecewise Chebyshevian splines

$$\begin{cases} -\Delta u + u = f, & (x,y) \in \Omega := [0,4]^2 \\ u|_{\delta\Omega} = 0 \end{cases}$$

with exact solution

$$u_{ex}(x,y) = (x^2 + y^2 - 1)\sin(\pi x)\sin(\pi y)$$



- IGA collocation in local dimension 6 with C^4 splines, $t_{k+1} - t_k = h$ and all $\mathbb{E}_k$ spanned by:

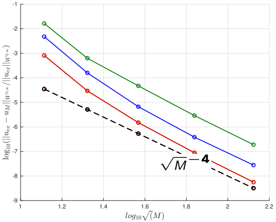
  A) $1, x, \ldots, x^5$

  B) $1, x, x^2, x^3, \cos \pi x, \sin \pi x$, ref. basis for $h < 1$

  C) $1, x, \cos \pi x, \sin \pi x, x \cos \pi x, x \sin \pi x$, ref. basis for $h < 1$

Relative errors in $W^{2,\infty}$ norm for $h = 2^{-j}$:

j	1	2	3	4	5
A)	1.65e-02	6.29e-04	4.70e-05	2.90e-06	1.89e-07
B)	4.77e-03	1.60e-04	6.67e-06	3.83e-07	2.79e-08
C)	**8.17e-04**	**2.95e-05**	**1.50e-06**	**8.88e-08**	**5.64e-09**

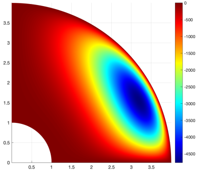


Advection–diffusion problem on a quarter of annulus

$$\begin{cases} -\Delta u + \frac{\delta u}{\delta x} + \frac{\delta u}{\delta y} = f, & \text{in } \Omega \\ u|_{\delta\Omega} = 0 \end{cases}$$

with exact solution

$$u_{ex}(x,y) = e^x xy(x^2+y^2-1)(x^2+y^2-16)$$



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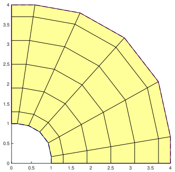
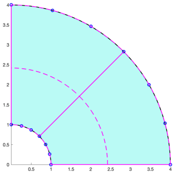
  - **A)** $1, x, \dots, x^5$
  - **B)** $1, x, x^2, x^3, \cos\frac{\pi}{2}x, \sin\frac{\pi}{2}x$, ref. basis for $h < 2$
  - **C)** $1, x, \cosh x, \sinh x, x\cosh x, x\sinh x$, ref. basis $\forall h$
  - **D)** $1, x, \sin\frac{\pi}{2}x, \cos\frac{\pi}{2}x, e^x, xe^x$, ref. basis for $h < 2$



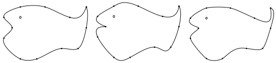
Relative errors in $W^{2,\infty}$ norm for $h = 2^{-j}$:

j	1	2	3	4	5
A)	1.36e-01	1.66e-02	1.18e-03	1.08e-04	8.46e-05
B)	1.46e-01	2.13e-02	1.90e-03	1.25e-04	7.49e-06
C)	1.29e-01	1.33e-02	6.93e-04	1.37e-04	1.67e-04
D)	**1.09e-01**	**8.54e-03**	**4.15e-04**	**3.25e-05**	**1.74e-06**

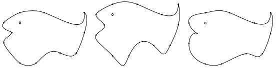


Concluding remarks. . .

- Piecewise Chebyshevian splines offer efficient shape parameters
- The main difficulty lies in the fact that they do not always have refinable B-spline bases
- . . . but we can numerically answer this question with high accuracy and efficiency and with effective evaluation of the basis
- When such bases exist, we can use them just as we use polynomial splines
- Not only for design and interpolation, but also multiresolution analysis and subdivision, approximation by Schoenberg-type operators, image processing, isogeometric analysis . . .



Thank you!



* Examples of mixed hyperbolic-trigometric interpolating splines

Bibliography

- I.J. Schoenberg, A. Whitney, On Pólya frequency functions III. The positivity of translation determinants with applications to the interpolation problem by spline curves, Trans. Amer. Math. Soc., 74 (1953), 246–259
- H. Spath, Exponential spline interpolation, Computing, 4 (1969), 225–233
- L.L. Schumaker, Spline Functions, Wiley Interscience, N.Y., 1981
- P. de Casteljau, Formes à Pôles, Mathématiques et CAO, Volume 2. Hermès, Paris, Londres, Lausanne.
- P. Costantini, On monotone and convex spline interpolation, Math. Comp., 46 (1986), 203–214
- T.N.T. Goodman, Properties of *beta*-splines, J. Approx. Theory, 44 (1985), 132–153
- N. Dyn, C.A. Micchelli, Piecewise polynomial spaces and geometric continuity of curves, Num. Math., 54 (1988), 319–337
- L. Ramshaw, Blossoms are polar forms, Comput. Aided Geom. Design, 6 (1989), 323–358
- P.D. Kaklis, D.G. Pandelis, Convexity preserving polynomial splines of non-uniform degree, IMA J. Numerical Analysis, 10 (1990), 223–234
- H.-P. Seidel, New algorithms and techniques for computing with geometrically continuous spline curves of arbitrary degree, Math. Model. Num. Anal., 26 (1992), 149–176
- M.-L. Mazure, P.-J. Laurent, Affine and Non-affine Blossoms, in Computational Geometry, World Scientific Pub. Singapore, (1993), 201–230
- H. Pottmann, The geometry of Tchebycheffian splines, Comput. Aided Geom. Design, 10 (1993), 181–210
- P.J. Barry, R.N. Goldman, C.A. Micchelli, Knot insertion algorithms for piecewise polynomial spaces determined by connection matrices, Adv. Comp. Math., 1 (1993), 139–171
- P.J. Barry, de Boor-Fix dual functionals and algorithms for Tchebycheffian B-splines curves, Constr. Approx., 12 (1996), 385–408
- M.-L. Mazure, H. Pottmann, Tchebycheff splines, in Total positivity and its applications, 1996, 187–218
- T. Lyche, L.L. Schumaker, Total positivity properties of LB-splines, in Total Positivity and its Applications, 1996, 35–46
- M.-L. Mazure, Blossoming: a geometrical approach, Constr. Approx., 15 (1999), 33–68
- M.-L. Mazure, P.-J. Laurent, Piecewise smooth spaces in duality: application to blossoming, J. Approx. Theory, 98 (1999), 316–353
- M.-L. Mazure, Chebyshev splines beyond total positivity, Adv. Comp. Math. 14 (2001), 129–156

Bibliography

- T.N.T. Goodman, M.-L. Mazure, Blossoming beyond extended Chebyshev spaces, J. Approx. Theory, 109 (2001), 48–81
- M.-L. Mazure, Quasi-Chebyshev splines with connection matrices. Application to variable degree polynomial splines, Comput. Aided Geom. Design, 18 (2001), 287–298
- B. Buchwald, G. Mühlbach, Construction of B-splines for generalized spline spaces generated from local ECT-systems, J. Comput. Applied Math., 159 (2003), 249–267
- M.-L. Mazure, Blossoms and optimal bases, Adv. Comp. Math., 20 (2004), 177–203
- M.-L. Mazure, On the equivalence between existence of B-spline bases and existence of blossoms, Constr. Approx., 20 (2004), 603–624
- P. Costantini, T. Lyche, C. Manni, On a class of weak Tchebycheff systems, Numer. Math., 101 (2005), 333–354
- M.-L. Mazure, Chebyshev spaces and Bernstein bases, Constr. Approx., 22 (2005), 347–363
- T.J.R. Hughes, J.A. Cottrell, Y. Bazilevs, Isogeometric analysis: CAD, finite elements, NURBS, exact geometry and mesh refinement, Comp. Methods Appl. Mech. Engrg., 194 (2005), 4135–4195
- M.-L. Mazure, Ready-to-blossom bases in Chebyshev spaces, in Topics Multivariate Approx. Interpol., Elsevier, 12, 2006, 109–148
- T. Bosner, Knot insertion algorithms for Chebyshev splines, PhD thesis, Univ. Zagreb, 2006
- M.-L. Mazure, Choosing spline spaces for interpolation, Proc. Transgressive Computing Conference, Granada, 2006, 311–326
- G. Mühlbach, One sided Hermite interpolation by piecewise different generalized polynomials, J. Comput. Applied Math., 196 (2006), 285–298
- M.-L. Mazure, Extended Chebyshev Piecewise spaces characterised via weight functions, J. Approx. Theory, 145 (2007), 33–54
- A. Kayumov, M.-L. Mazure, Chebyshevian splines: interpolation and blossoms, CRAS, 344 (2007), 65–70, 2007
- M.-L. Mazure, Which spaces for design?, Num. Math., 110 (2008), 357–392
- M.-L. Mazure, On differentiation formulæ for Chebyshevian Bernstein and B-spline bases, Jaén J. Approx., 1 (2009), 111–143
- M.-L. Mazure, Ready-to-blossom bases and the existence of geometrically continuous piecewise Chebyshevian B-splines, CRAS, 347 (2009), 829–834

Bibliography

- M.-L. Mazure, On a general new class of quasi-Chebyshevian splines, Num. Algorithms, 58 (2011), 399–438
- M.-L. Mazure, Finding all systems of weight functions associated with a given Extended Chebyshev space, J. Approx. Theory, 163 (2011), 363–376
- M.-L. Mazure, How to build all Chebyshevian spline spaces good for Geometric Design, Num. Math., 119 (2011), 517–556
- C. Manni, F. Pelosi, M.-L. Sampoli, Generalized B-splines as a tool in isogeometric analysis, Comp. Methods Appl. Mech. Engrg., 200 (2011), 867–881
- C. Manni, F. Pelosi, M.-L. Sampoli, Isogeometric analysis in advection-diffusion problems: Tension splines approximation, J. Comp. Appl. Math., 236 (2011), 511–528
- T. Lyche, M.-L. Mazure, Piecewise Chebyshevian Multiresolution Analysis, East J. Approx., 17 (2012), 419–435
- M.-L. Mazure, Polynomial splines as examples of Chebyshevian splines, Num. Algorithms, 60 (2012), 241–262
- M.-L. Mazure, Piecewise Chebyshev-Schoenberg operators: shape preservation, approximation and space embedding, J. Approx. Theory, 166 (2013), 106–135
- R. Ait-Haddou, Y. Sakane, T. Nomura, Chebyshev blossoming in Müntz spaces: Toward shaping with Young diagrams, J. Comp. Appl. Math., 247 (2013), 172–208
- M. Brilleaud, M.-L. Mazure, Design with L-splines, Num. Algorithms, 65 (2014), 91–124
- M.-L. Mazure, Which spline spaces for design?, CRAS, 353 (2015), 761–765
- M.-L. Mazure, Lagrange interpolatory subdivision schemes in Chebyshev spaces, J. Found. Comp. Math., 15 (2015), 1035–1068.
- R. Ait-Haddou, M.-L. Mazure, Approximation by Chebyshevian Bernstein Operators versus Convergence of Dimension Elevation, Constr. Approx., 43 (2016), 425-461
- M.-L. Mazure, Design with Quasi Extended Chebyshev Piecewise Spaces, Comp. Aided Geom. Design, 47 (2016), 3–28
- C.-V. Beccari, G. Casciola, M.-L. Mazure, Piecewise Extended Chebyshev Spaces: a numerical test for design, Applied Math. Comp., 296 (2017), 239–256
- T. Bosner, M. Rogina, Quadratic convergence for CCC-Schoenberg operators, Num. Math., 135 (2017), 1253–1287

Bibliography

- M.-L. Mazure, Piecewise Chebyshevian Splines: Interpolation versus Design, Num. Algorithms, 77 (2018), 1213–1247
- M.-L. Mazure, Constructing totally positive piecewise Chebyhevian B-splines, J. Comp. Appl. Math., 342 (2018), 550–586
- C.-V. Beccari, G. Casciola, M.-L. Mazure, Design or not design? A numerical characterisation for piecewise Chebyshevian splines, Num. Algorithms, 81 (2019), 1–31
- M.-L. Mazure, Geometrically continuous Piecewise Chebyshevian NU(R)BS, BIT, 60 (2020), 687–714
- C.-V. Beccari, G. Casciola, M.-L. Mazure, Critical length: an alternative approach, J. Comp. Appl. Math., 370 (2020)
- T. Bosner, B. Crnkovic, J. Skific, Application of CCC-Schoenberg operators on image resampling, BIT, 60 (2020), 129–155
- C.-V. Beccari, G. Casciola, M.-L. Mazure, Dimension elevation is not always corner-cutting, Applied Math. Letters, 109 (2020), article 106529.
- C.-V. Beccari, G. Casciola, L. Romani, A practical method for computing with piecewise Chebyshevian splines, J. Comp. Appl. Math., 406 (2022) article 114051.