Calcolo Scientifico e Modelli Matematici

Alla ricerca della cose nascoste attraverso le cose manifeste

Smooth path planning for autonomous vehicles: perspectives from the theory of Pythagorean-hodograph curves

Carlotta Giannelli

(University of Florence)

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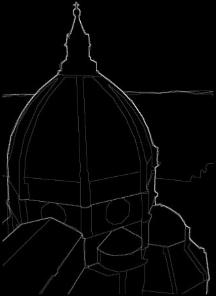
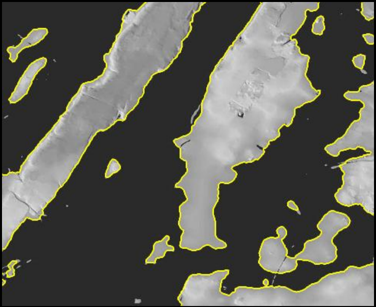


joint work with

Alessandra Sestini,
Lorenzo Sacco
(University of Florence)

Vincenzo Calabrò (MDM Team)

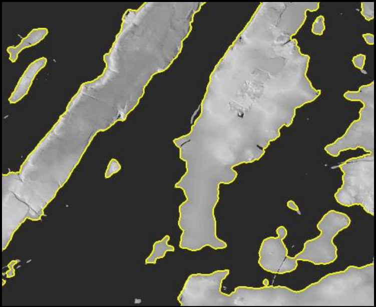
Planning curvilinear paths for autonomous vehicles

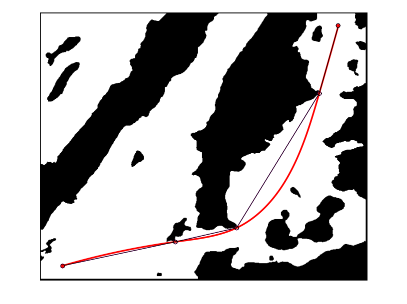


Path planning for autonomous or remotely operated vehicles

- Unmanned Aerial Vehicles (UAVs)
- Autonomous Underwater Vehicles (AUVs)
- ...

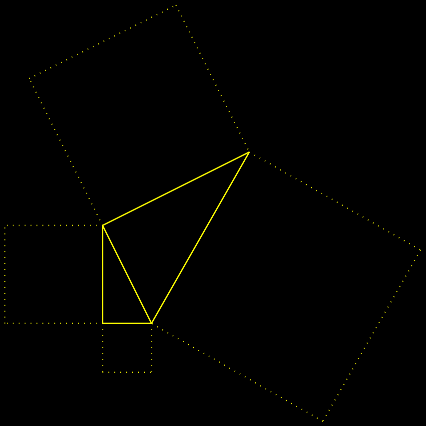
Planning curvilinear paths for autonomous vehicles





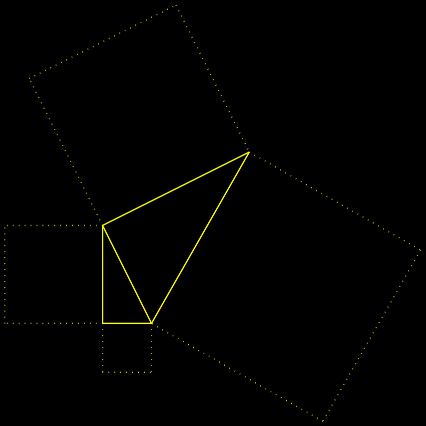
Feasible paths must satisfy various constraints

- bounds on the path curvature or climb angle
- avoidance of environmental obstacles
- maintenance of safe separations in vehicle swarms



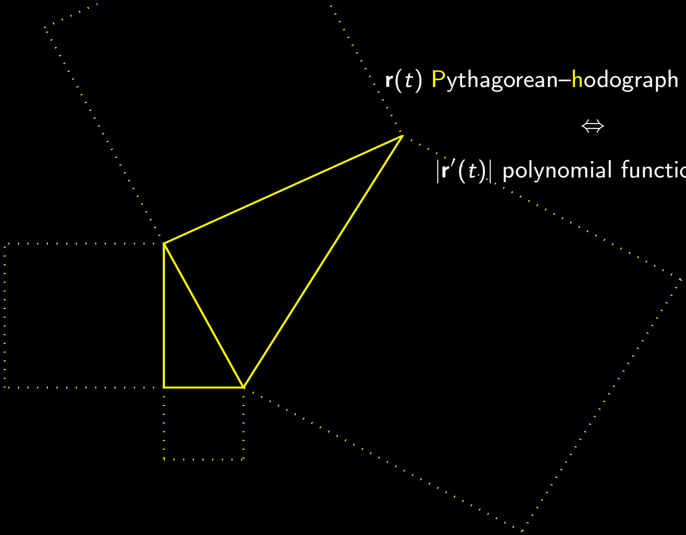
Pythagorean–hodograph curves

- smooth PH spline paths
- data stream interpolation
- real test case: Zeno AUV



Pythagorean–hodograph curves

Pythagorean–hodograph



$\mathbf{r}(t)$ Pythagorean–hodograph (PH) curve

$\Leftrightarrow$

$|\mathbf{r}'(t)|$ polynomial function of t

[Farouki and Sakkalis — IBMJRD, 1990]

PH space curves

$\mathbf{r}(t) = (x(t), y(t), z(t))$

$$|\mathbf{r}'(t)|^2 \;=\; x'^2(t) + y'^2(t) + z'^2(t)$$

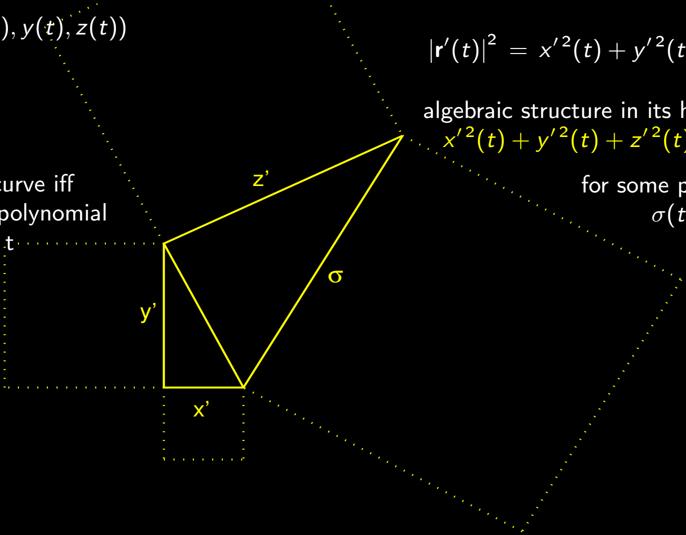
algebraic structure in its hodograph

$$x'^2(t) + y'^2(t) + z'^2(t) \;\equiv\; \sigma^2(t)$$

for some polynomial

$\sigma(t) = |\mathbf{r}'(t)|$

PH space curve iff
$|\mathbf{r}'(t)|$ is a polynomial
function of t



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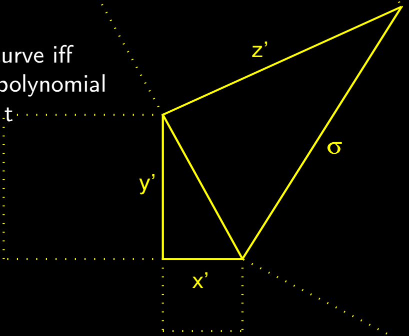

:: polynomial arc–length function

$$s(t) \; = \; \int_0^t |\mathbf{r}'(\tau)| d\tau$$

:: rational adapted frames

:: rational swept surfaces parametrization

PH space curves

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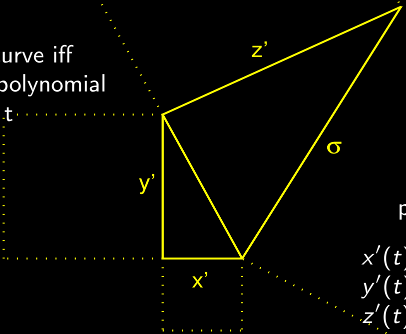

pythagorean quadruples of polynomials

$$\begin{aligned} x'(t) &= u^2(t) + v^2(t) - p^2(t) - q^2(t) \\ y'(t) &= 2\,[u(t)q(t) + v(t)p(t)] \\ z'(t) &= 2\,[v(t)q(t) - u(t)p(t)] \\ \sigma(t) &= u^2(t) + v^2(t) + p^2(t) + q^2(t) \end{aligned}$$

for some polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$

[Dietz, Hoschek and Jüttler — CAGD, 1993]

PH space curves

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$|\mathbf{r}'(t)|^2 \; = \; x'^2(t) + y'^2(t) + z'^2(t)$

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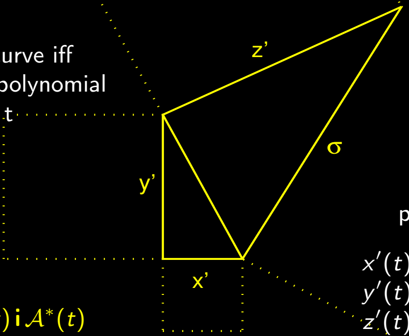

pythagorean quadruples of polynomials

$$
\begin{aligned}
x'(t) &= u^2(t) + v^2(t) - p^2(t) - q^2(t) \\
y'(t) &= 2\,[u(t)q(t) + v(t)p(t)] \\
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\sigma(t) &= u^2(t) + v^2(t) + p^2(t) + q^2(t)
\end{aligned}
$$

$\mathbf{r}'(t) \; = \; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$

quaternion polynomial
$\mathcal{A}(t) := u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$

for some polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$

[Choi, Lee and Moon — ACOM, 2002]

[Dietz, Hoschek and Jüttler — CAGD, 1993]

Quaternion algebra $\mathbb{H}$

$$\boxed{\mathcal{A} = a_0 + a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k} \qquad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\,\mathbf{j}\,\mathbf{k} = -1}$$

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- $\mathcal{A} = (a_0, \mathbf{a})$ where a_0 is the scalar part and $\mathbf{a} = (a_1, a_2, a_3)$ is the vector part;
- $\mathcal{A}\,\mathcal{B} = (a_0\,b_0 - \mathbf{a}\cdot\mathbf{b}, a_0\,\mathbf{b} + b_0\,\mathbf{a} + \mathbf{a}\times\mathbf{b})$ is the quaternion product;
- $\mathcal{A}^* = (a_0, -\mathbf{a})$ is the conjugate of $\mathcal{A}$ and $(\mathcal{A}\,\mathcal{B})^* = \mathcal{B}^*\,\mathcal{A}^*$;
- $|\mathcal{A}|^2 = \mathcal{A}\,\mathcal{A}^* = \mathcal{A}^*\mathcal{A} = a_0^2 + |\mathbf{a}|^2$ is the square module of $\mathcal{A}$ and $|\mathcal{A}\,\mathcal{B}| = |\mathcal{A}|\,|\mathcal{B}|$;

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If $v_0 = 0 \;\Rightarrow\; \mathcal{V} = (v_0, \mathbf{v})$ and also $\mathcal{A}\,\mathbf{v}\,\mathcal{A}^*$ are pure vectors

$\Rightarrow$ the general solution of the non–unilateral quadratic quaternion equation

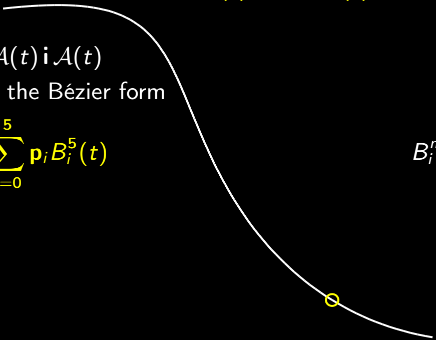
$$\mathcal{A}\,\mathbf{u}\,\mathcal{A}^* = \mathbf{v} \qquad \text{is given by} \qquad \mathcal{A} = \sqrt{|\mathbf{v}|}\,\frac{\mathbf{u} + \mathbf{v}/\mathrm{v}}{|\mathbf{u} + \mathbf{v}/\mathrm{v}|}\,(\cos\phi + \sin\phi\,\mathbf{u})$$

where ϕ is a free angular variable and $\mathbf{u}$ is a unit vector.

Bézier form of spatial PH quintics

Substituting

$$\mathcal{A}(t) = \mathcal{A}_0 B_0^2(t) + \mathcal{A}_1 B_1^2(t) + \mathcal{A}_2 B_2^2(t)$$



into

$$\mathbf{r}'(t) = \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}(t)$$

and integrating yields the Bézier form

$$\mathbf{r}(t) = \sum_{i=0}^{5} \mathbf{p}_i B_i^5(t)$$

Bernestein polynomials

$$B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i}$$

Bézier form of spatial PH quintics

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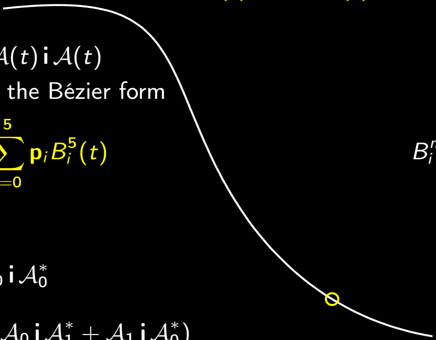


Bernestein polynomials

$$B_i^n(t) := \binom{n}{i} t^i (1-t)^{n-i}$$

with control points

$$\mathbf{p}_1 = \mathbf{p}_0 + \frac{1}{5}\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_0^*$$

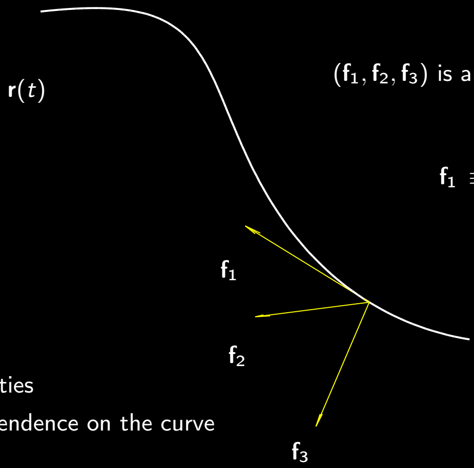
$$\mathbf{p}_2 = \mathbf{p}_1 + \frac{1}{10}(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_1^* + \mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_0^*)$$

$$\mathbf{p}_3 = \mathbf{p}_2 + \frac{1}{30}(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_2^* + 4\,\mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_1^* + \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^*)$$

$$\mathbf{p}_4 = \mathbf{p}_3 + \frac{1}{10}(\mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_2^* + \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_1^*)$$

$$\mathbf{p}_5 = \mathbf{p}_4 + \frac{1}{5}\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_2^*$$

Adapted frames



$(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ is an *adapted* frame on $\mathbf{r}(t)$

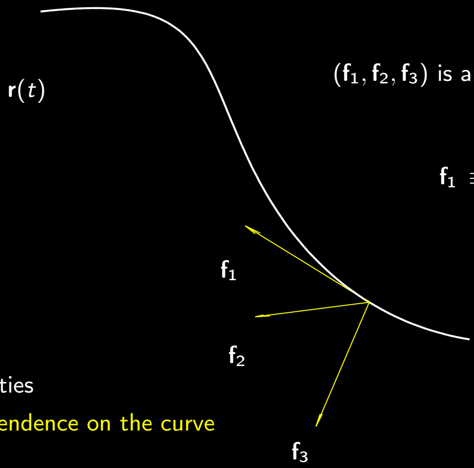
$$\Leftrightarrow$$

$$\mathbf{f}_1 \equiv \mathbf{t} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

2 desirable properties

- rational dependence on the curve parameter t
- rotation–minimizing property

Adapted frames



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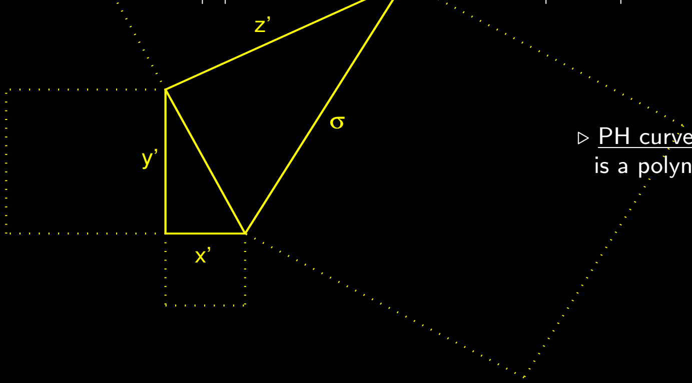
Curves with rational FF

$\mathbf{r}(t) = (x(t), y(t), z(t))$

Frenet frame (FF): $\mathbf{t} = \dfrac{\mathbf{r}'}{|\mathbf{r}'|}$ $\quad \mathbf{n} = \mathbf{t} \times \mathbf{b}$ $\quad \mathbf{b} = \dfrac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|}$



▷ PH curve $\Leftrightarrow |\mathbf{r}'(t)|$ is a polynomial in t

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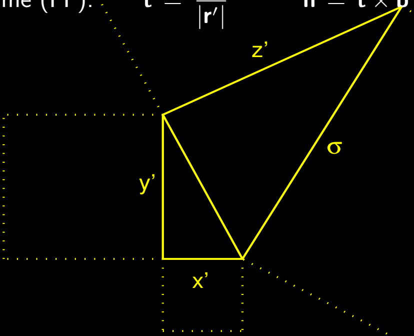


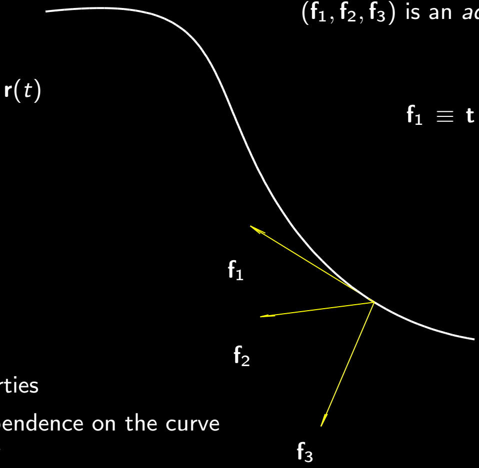
▷ PH curve ⇔ $|\mathbf{r}'(t)|$ is a polynomial in t

▷ double PH curve ⇔ $|\mathbf{r}'(t) \times \mathbf{r}''(t)|$ is also a polynomial in t

⇓

$\mathbf{r}(t)$ is a double PH (DPH) curve ⇔ $\mathbf{r}(t)$ has a rational FF

Rotation-minimizing property



$(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ is an *adapted* frame on $\mathbf{r}(t)$

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2 desirable properties

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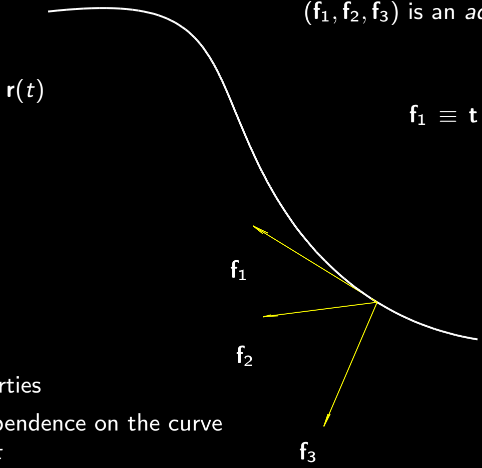
Rotation-minimizing property



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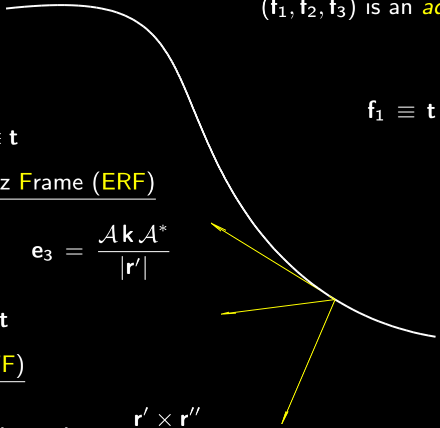


- $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, $\mathbf{e}_1 \equiv \mathbf{t}$

  Eulero–Rodriguez Frame (ERF)

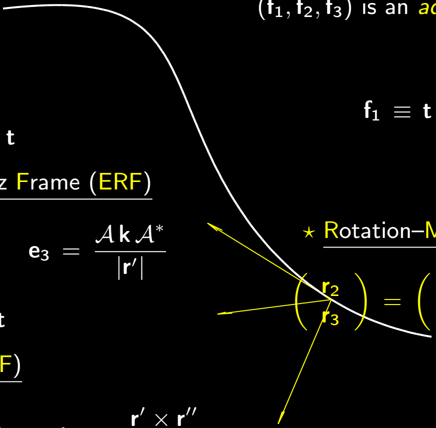
  $$\mathbf{e}_2 = \frac{\mathcal{A}\,\mathbf{j}\,\mathcal{A}^*}{|\mathbf{r}'|} \qquad \mathbf{e}_3 = \frac{\mathcal{A}\,\mathbf{k}\,\mathcal{A}^*}{|\mathbf{r}'|}$$

- $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{t}_3)$, $\mathbf{t}_1 \equiv \mathbf{t}$

  Frenet Frame (FF)

  $$\mathbf{t}_2 = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} \times \mathbf{t} \qquad \mathbf{t}_3 = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|}$$

Rotation-minimizing property



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$(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$, $\mathbf{r}_1 \equiv \mathbf{t}$

⋆ Rotation–Minimizing Frame (RMF)

$$\begin{pmatrix} \mathbf{r}_2 \\ \mathbf{r}_3 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \mathbf{t}_2 \\ \mathbf{t}_3 \end{pmatrix}$$

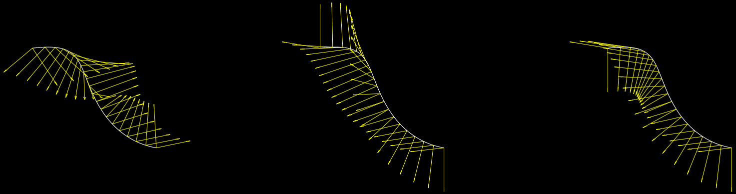
with $\theta = -\int \tau \mathrm{d}s$

[Bishop — AMM, 1975]

[Guggenheimer — CAGD, 1989]

[Klok — CAGD, 1986]

RMFs on space curves



- the angular velocity $\boldsymbol{\omega}(t)$ specifies the variation of $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ along $\mathbf{r}(t)$

$$\boldsymbol{\omega} \;=\; \omega_1\,\mathbf{f}_1 + \omega_2\,\mathbf{f}_2 + \omega_3\,\mathbf{f}_3$$

where $\mathbf{f}_j' = \boldsymbol{\omega} \times \mathbf{f}_j \,,\; j = 1, 2, 3,$

$$\omega_1 \;=\; \mathbf{f}_3 \cdot \mathbf{f}_2' \;=\; -\mathbf{f}_2 \cdot \mathbf{f}_3' \qquad \omega_2 \;=\; \mathbf{f}_1 \cdot \mathbf{f}_3' \;=\; -\mathbf{f}_3 \cdot \mathbf{f}_1' \qquad \omega_3 \;=\; \mathbf{f}_2 \cdot \mathbf{f}_1' \;=\; -\mathbf{f}_1 \cdot \mathbf{f}_2'$$

frame instantaneous angular speed: $\omega = |\boldsymbol{\omega}|$

frame instantaneous rotation axis: $\mathbf{a} = \boldsymbol{\omega}/|\boldsymbol{\omega}|$

RMFs on space curves

$(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ is an RMF $\Leftrightarrow$ $\boldsymbol{\omega}_1 = 0$ $\Leftrightarrow$ $\mathbf{f}_3 \cdot \mathbf{f}_2' = 0$
at every point of $\mathbf{r}(t)$, there is no instantaneous rotation of $\mathbf{f}_2$ and $\mathbf{f}_3$ about $\mathbf{f}_1$

$\Downarrow$

polynomial curves with rational RMFs (RRMFs)

$\mathbf{r}(t)$ is an RRMF curve $\Leftrightarrow$ $\mathbf{r}(t)$ has a rational RMF

- the angular velocity $\boldsymbol{\omega}(t)$ specifies the variation of $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ along $\mathbf{r}(t)$

$$\boldsymbol{\omega} = \omega_1 \, \mathbf{f}_1 + \omega_2 \, \mathbf{f}_2 + \omega_3 \, \mathbf{f}_3$$

where $\mathbf{f}_j' = \boldsymbol{\omega} \times \mathbf{f}_j$, $j = 1, 2, 3$,

$$\omega_1 = \mathbf{f}_3 \cdot \mathbf{f}_2' = -\mathbf{f}_2 \cdot \mathbf{f}_3' \qquad \omega_2 = \mathbf{f}_1 \cdot \mathbf{f}_3' = -\mathbf{f}_3 \cdot \mathbf{f}_1' \qquad \omega_3 = \mathbf{f}_2 \cdot \mathbf{f}_1' = -\mathbf{f}_1 \cdot \mathbf{f}_2'$$

frame instantaneous angular speed: $\omega = |\boldsymbol{\omega}|$

frame instantaneous rotation axis: $\mathbf{a} = \boldsymbol{\omega} / |\boldsymbol{\omega}|$

RMFs on space curves

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$\mathbf{r}(t)$ is an RRMF curve $\Leftrightarrow$ $\mathbf{r}(t)$ has a rational RMF

RMFs on space curves

$(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ is an RMF $\Leftrightarrow$ $\boldsymbol{\omega}_1 = 0$ $\Leftrightarrow$ $\mathbf{f}_3 \cdot \mathbf{f}_2' = 0$
at every point of $\mathbf{r}(t)$, there is no instantaneous rotation of $\mathbf{f}_2$ and $\mathbf{f}_3$ about $\mathbf{f}_1$

$\Downarrow$

polynomial curves with rational RMFs (RRMFs)

$\mathbf{r}(t)$ is an RRMF curve $\Leftrightarrow$ $\mathbf{r}(t)$ has a rational RMF

$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ ERF on PH curves defined by $\mathcal{A}(t)$

$$\mathbf{e}_1(t) = \frac{\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)}{\mathcal{A}(t)\,\mathcal{A}^*(t)} \qquad \mathbf{e}_2(t) = \frac{\mathcal{A}(t)\,\mathbf{j}\,\mathcal{A}^*(t)}{\mathcal{A}(t)\,\mathcal{A}^*(t)} \qquad \mathbf{e}_3(t) = \frac{\mathcal{A}(t)\,\mathbf{k}\,\mathcal{A}^*(t)}{\mathcal{A}(t)\,\mathcal{A}^*(t)}$$

RMFs on space curves

$(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ is an RMF $\Leftrightarrow$ $\boldsymbol{\omega}_1 = 0$ $\Leftrightarrow$ $\mathbf{f}_3 \cdot \mathbf{f}_2' = 0$
at every point of $\mathbf{r}(t)$, there is no instantaneous rotation of $\mathbf{f}_2$ and $\mathbf{f}_3$ about $\mathbf{f}_1$

$\Downarrow$

polynomial curves with rational RMFs (RRMFs)

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- ERF angular velocity component:

$$\omega_1(\mathrm{ERF}) = 2\,\frac{\mathrm{scal}\,(\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}'^*(t))}{|\mathcal{A}(t)|^2}$$

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$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ ERF on PH curves defined by $\mathcal{A}(t) := u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$

$$\mathbf{e}_1(t) = \frac{\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)}{\mathcal{A}(t)\,\mathcal{A}^*(t)} \qquad \mathbf{e}_2(t) = \frac{\mathcal{A}(t)\,\mathbf{j}\,\mathcal{A}^*(t)}{\mathcal{A}(t)\,\mathcal{A}^*(t)} \qquad \mathbf{e}_3(t) = \frac{\mathcal{A}(t)\,\mathbf{k}\,\mathcal{A}^*(t)}{\mathcal{A}(t)\,\mathcal{A}^*(t)}$$

- ERF angular velocity component:

$$\omega_1(\text{ERF}) = 2\,\frac{\text{scal}\,(\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}'^*(t))}{|\mathcal{A}(t)|^2} = \frac{2(uv' - u'v - pq' + p'q)}{u^2 + v^2 + p^2 + q^2}$$

$\Rightarrow$ the ERF is rational but not always RM ...

ERF vs. RMF

$$\mathbf{r}'(t) = \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$$

$$\frac{\text{scal}\,(\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}'^*(t))}{|\mathcal{A}(t)|^2} = \frac{\text{scal}\,(\mathcal{W}(t)\,\mathbf{i}\,\mathcal{W}'^*(t))}{|\mathcal{W}(t)|^2}$$

$$\mathcal{W}(t) = a(t) + \mathbf{i}\,b(t) \qquad \gcd(a(t), b(t)) = \text{const.}$$

$(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ ERF

$$\mathbf{e}_1(t) = \frac{\mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)}{\mathcal{A}(t)\,\mathcal{A}^*(t)} \qquad \mathbf{e}_2(t) = \frac{\mathcal{A}(t)\,\mathbf{j}\,\mathcal{A}^*(t)}{\mathcal{A}(t)\,\mathcal{A}^*(t)} \qquad \mathbf{e}_3(t) = \frac{\mathcal{A}(t)\,\mathbf{k}\,\mathcal{A}^*(t)}{\mathcal{A}(t)\,\mathcal{A}^*(t)}$$

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$(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$ RMF

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where

$$\mathcal{B}(t) = \mathcal{A}(t)\mathcal{W}^*(t)$$

A recent survey [Farouki, Giannelli, Sestini — in Springer INdAM Series, 2019]

Fundamentals, specializations & generalizations of polynomial PH curves

Rational orthonormal frames along PH curves

Algorithms for PH Curves

Surface constructions based on PH curves

Applications of PH curves

A recent survey [Farouki, Giannelli, Sestini — in Springer INdAM Series, 2019]

Fundamentals, specializations & generalizations of polynomial PH curves

- DPH curves, rational PH Curves
- ATPH curves, MPH curves
- Pythagorean-Normal and Linear Normal surfaces, ...

Rational orthonormal frames along PH curves

- rotation-minimizing adapted, directed & osculating frames
- RRMFs, RMTFs

Algorithms for PH Curves

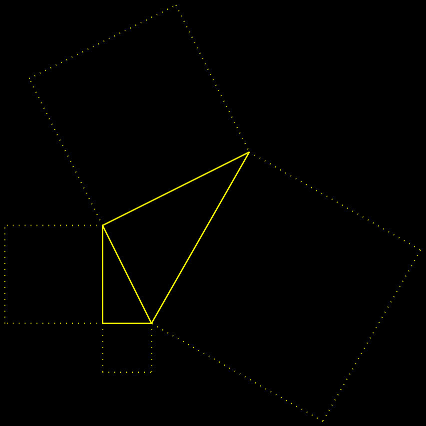
- construction algorithms (local and global interpolation schemes)
- PH Curves with prescribed arc lengths, reverse engineering of PH Curves

Surface constructions based on PH curves

- rational patches bounded by lines of curvature, rational swept surface constructions
- surface patches with PH isoparametric curves

Applications of PH curves

- real-time motion control, path planning applications



Pythagorean–hodograph curves

- smooth PH spline paths
- data stream interpolation
- real test case: Zeno AUV

- smooth PH spline paths

Path planning based on PH splines

Roadmap reconstruction		▶ visibility graph + dual graph
↓ *admissible piecewise linear paths*		
Path planning		▶ graph search algorithms
↓ *collision-free piecewise linear path*		
Path smoothing		▶ G^1/G^2 PH quintic splines
↓ *collision-free smooth path*		
Trajectory planning		▶ feedrate scheduling algorithm
↓ *suitable path traversal time*		

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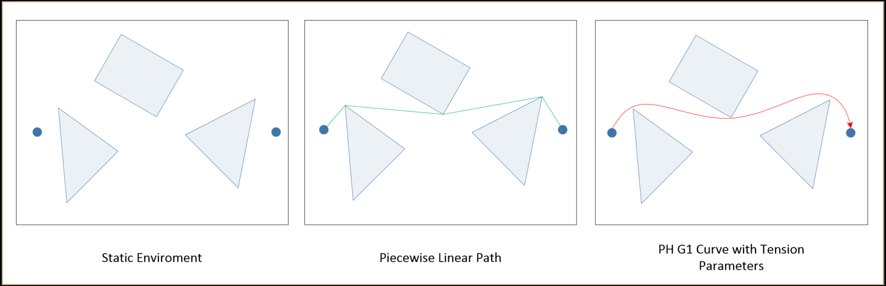
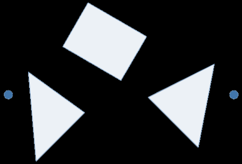
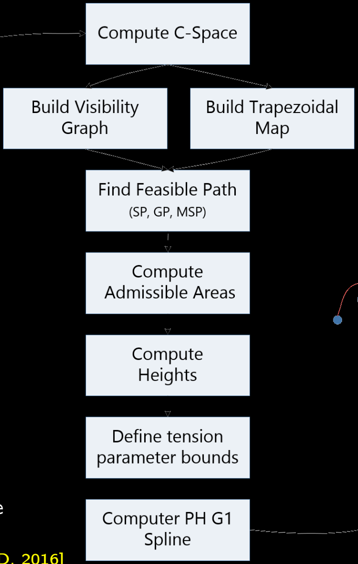
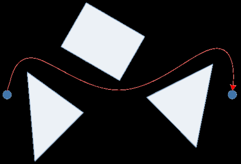
Path planning based on PH spline in tension



Path planning based on PH spline in tension







tension parameters

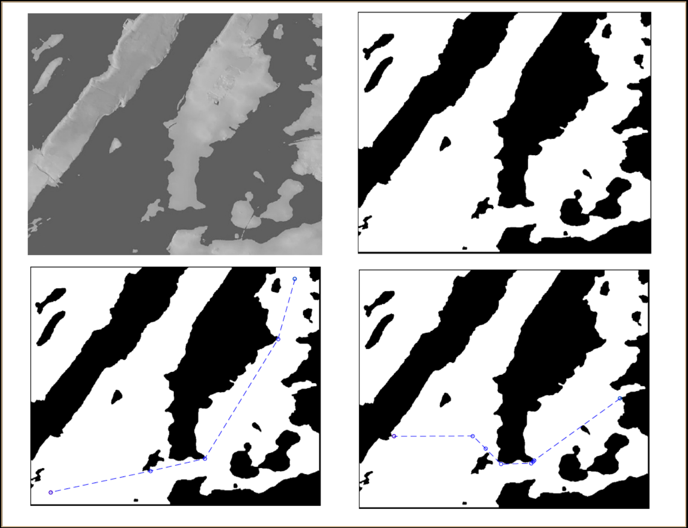
- collision avoidance

asymptotic analysis

- automatic choice of the tension parameters

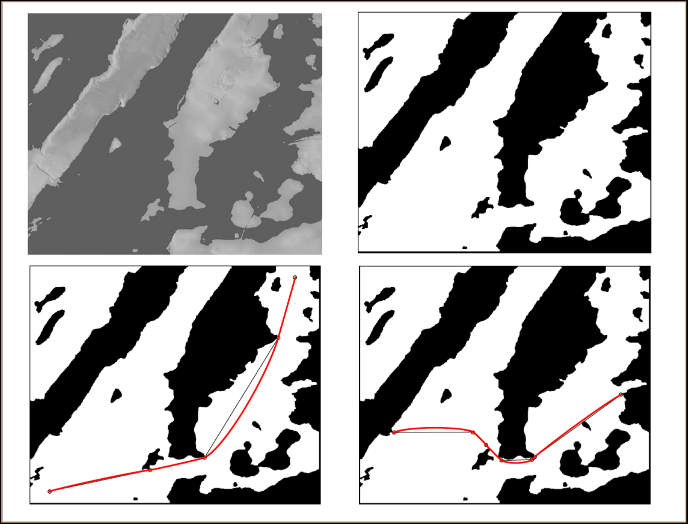
[Giannelli, Mugnaini, Sestini — CAD, 2016]

Path planning with scene reconstruction: C^0 path



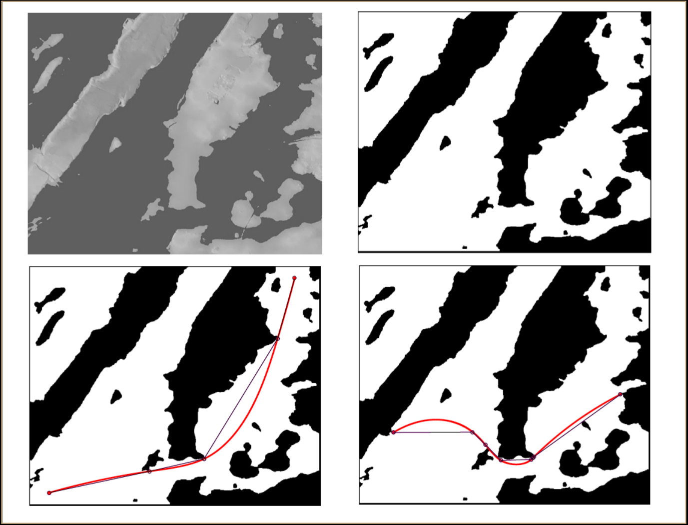
[Donatelli, Giannelli, Mugnaini, Sestini — CAD, 2017]

Path planning with scene reconstruction: G^1 path



[Donatelli, Giannelli, Mugnaini, Sestini — CAD, 2017]

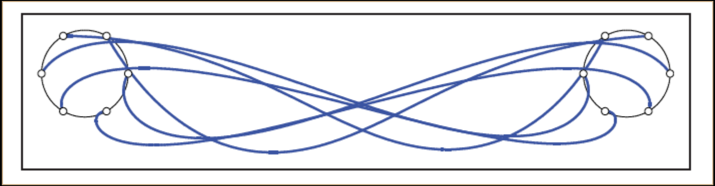
Path planning with scene reconstruction: G^2 path



[Donatelli, Giannelli, Mugnaini, Sestini — CAD, 2017]

Applications to unmanned or autonomous vehicles

- maintenance of minimum safe separations within vehicle swarms
- construction of paths of different shape but identical arc length, ensuring simultaneous arrival of vehicles travelling at a constant speed
- determination of the curvature extrema of PH paths, and their modification to satisfy a given curvature bound
- construction of curvature-continuous paths of bounded curvature



a family of simultaneous-arrival paths for a swarm of six unmanned constant speed vehicles, departing and arriving in different directions from a set of corresponding equidistant points on an initial and final target circle

[Farouki, Giannelli, Mugnaini, Sestini — JAE, 2018]

Curve vs. frame construction

The construction of an RMF on a pre–defined curve is an initial value problem. . .

. . . the orientation of the normal-plane vectors at any curve point determine their orientation at every other point

. . . it is not possible to construct RMFs along pre–defined curves with prescribed initial and final orientations

$\rightarrow$ the curve is an outcome of the construction algorithm

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To independently specify a curve and a rational frame along it, we consider a Minimal Twist Frame (MTF) associated with a pre–defined curve and initial/final orientations.
[Farouki and Moon — ACOM, 2018]

$\rightarrow$ the construction of an MTF on a pre–defined curve is a boundary value problem.

Minimal twist frames

- the angular velocity $\boldsymbol{\omega}(t)$ specifies the variation of $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ along $\mathbf{r}(t)$

$$\boldsymbol{\omega} = \omega_1 \, \mathbf{f}_1 + \omega_2 \, \mathbf{f}_2 + \omega_3 \, \mathbf{f}_3$$

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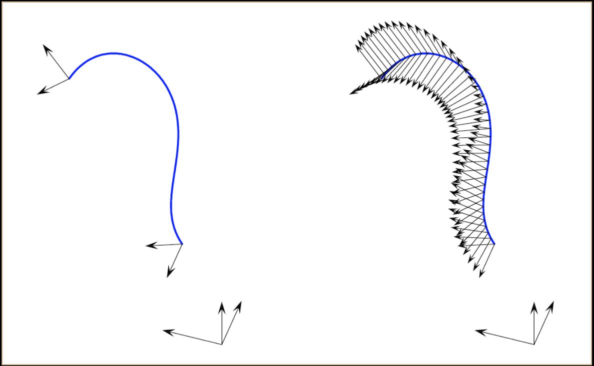


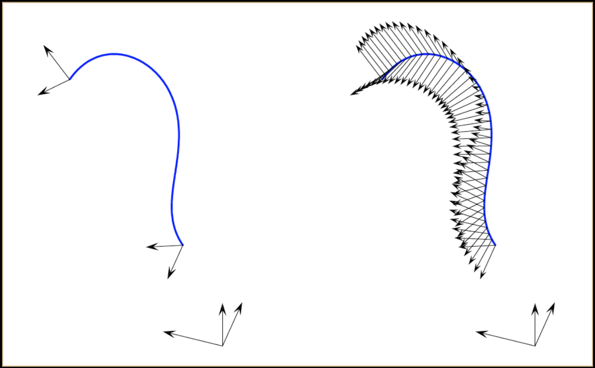
Definition:

- the twist of the framed curve is the integral of the component $\boldsymbol{\omega}_1$ with respect to arc length
- an MTF has the least possible twist value, subject to prescribed initial and final orientations

Minimal twist frames

- the angular velocity $\boldsymbol{\omega}(t)$ specifies the variation of $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ along $\mathbf{r}(t)$

$$\boldsymbol{\omega} = \omega_1 \mathbf{f}_1 + \omega_2 \mathbf{f}_2 + \omega_3 \mathbf{f}_3$$

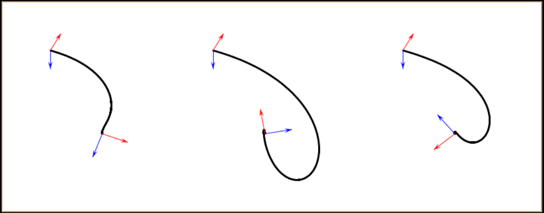


MTF:

- the angular velocity component $\boldsymbol{\omega} \cdot \mathbf{f}_1$ in the tangent direction does not change sign
- the total amount of rotation of the normal–plane vector about the tangent is minimized

(A constant $\boldsymbol{\omega} \cdot \mathbf{f}_1$ can only be approximately achieved for a rational MTF)

RMFs vs. MTFs



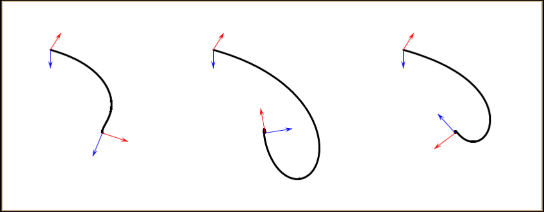
RMF

no end–frame interpolation on a fixed curve

the curve is an outcome of the algorithm

[Farouki, Giannelli, Sestini — JCAM, 2019]

RMFs vs. MTFs



RMF

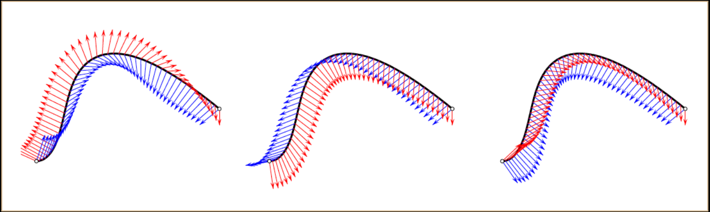
no end–frame interpolation on a fixed curve

the curve is an outcome of the algorithm

MTF

end–frame interpolation on a fixed curve

different MTFs on a fixed curve



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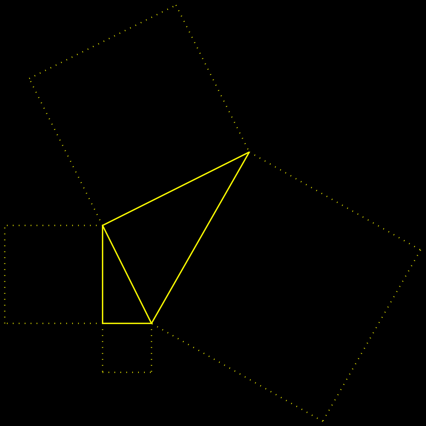


Pythagorean–hodograph curves

- smooth PH spline paths
- data stream interpolation
- real test case: Zeno AUV

- data stream interpolation

The local interpolation problem

Construction of

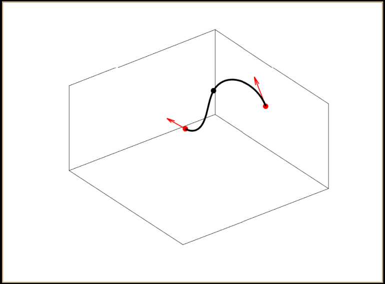
$$\mathbf{x}(u), u \in [u_i\ ,\ u_f]$$

so that

$$\mathbf{x}(u_i) = \mathbf{p}_i, \quad \mathbf{x}(u_f) = \mathbf{p}_f$$

$$\mathbf{x}'(u_i) = \mathbf{v}_i, \quad \mathbf{x}'(u_f) = \mathbf{v}_f$$

$$\mathbf{x}''(u_i) = \mathbf{w}_i.$$



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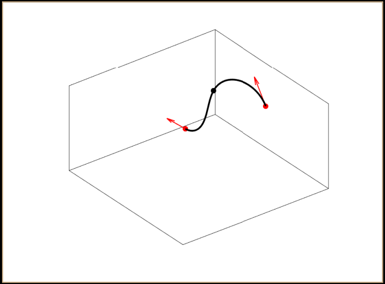


We consider the PH quintic biarc composed by 2 PH quintics joining at u_m

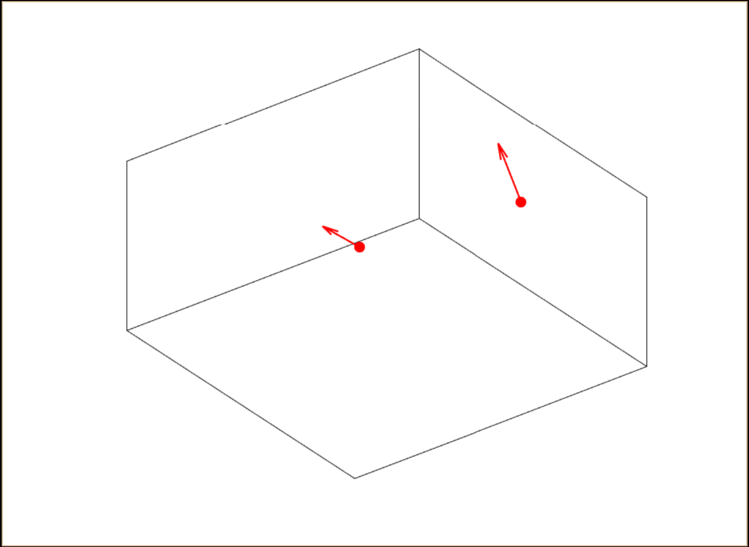
$$\mathbf{x}(u) = \begin{cases} \mathbf{x}_i(u) & \text{for } u \in [u_i\,,\, u_m], \\ \mathbf{x}_f(u) & \text{for } u \in [u_m\,,\, u_f]. \end{cases} \qquad \frac{d\mathbf{x}_i}{d\tau}(\tau) = \mathcal{A}(\tau)\,\mathbf{i}\,\mathcal{A}^*(\tau)\,, \quad \frac{d\mathbf{x}_f}{d\eta}(\eta) = \mathcal{B}(\eta)\,\mathbf{i}\,\mathcal{B}^*(\eta)\,,$$

where the quadratic quaternion polynomials

$$\mathcal{A}(\tau) := \sum_{j=0}^{2} \mathcal{A}_j B_j^2(\tau), \qquad \mathcal{B}(\eta) := \sum_{j=0}^{2} \mathcal{B}_j B_j^2(\eta)\,,$$

define the pre-image of $\mathbf{x}_i$ and $\mathbf{x}_f$, in the Bernstein basis,

Data stream interpolation: spline extension



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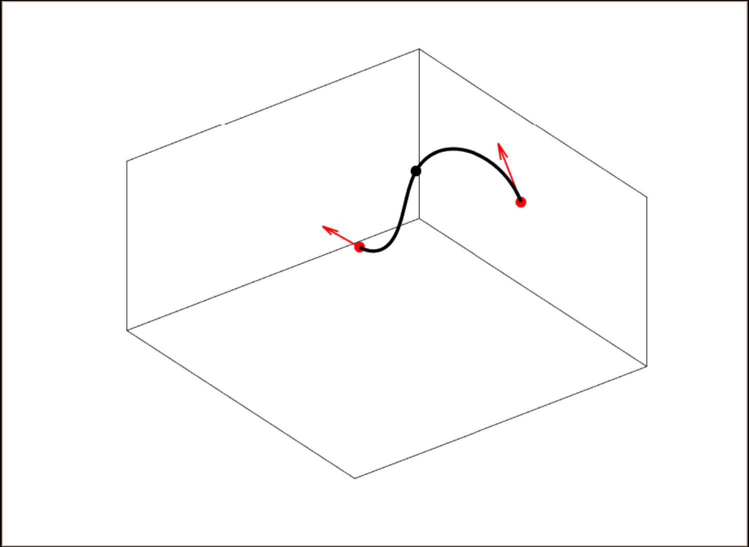

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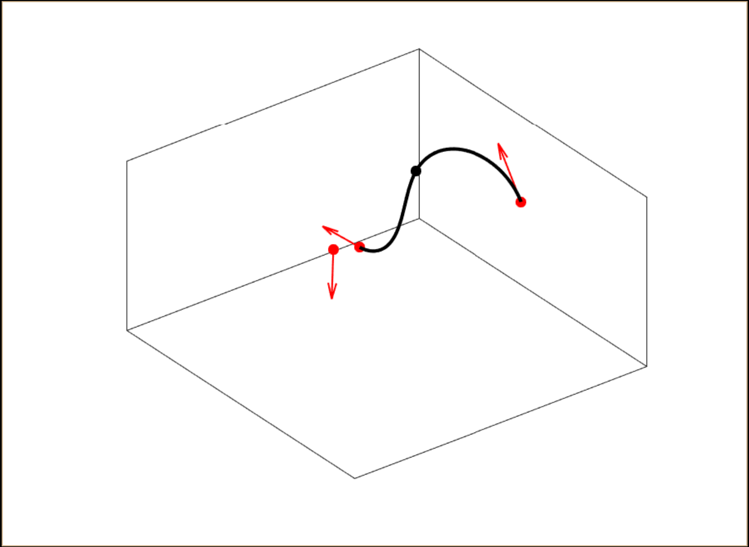

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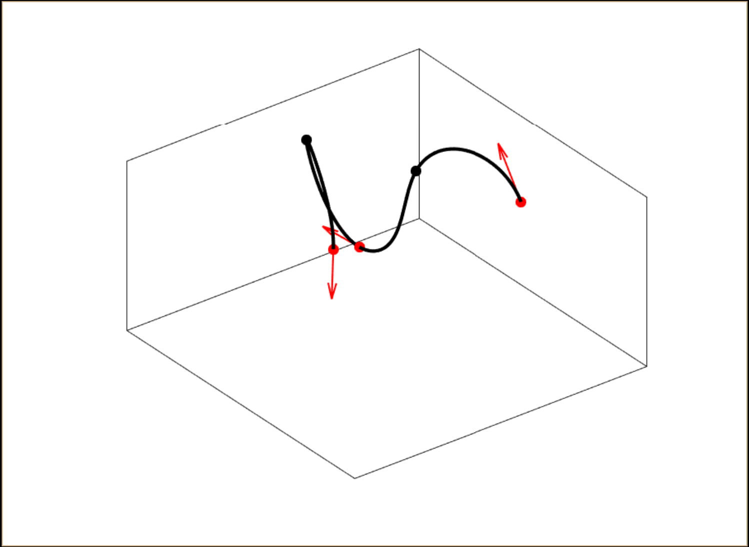

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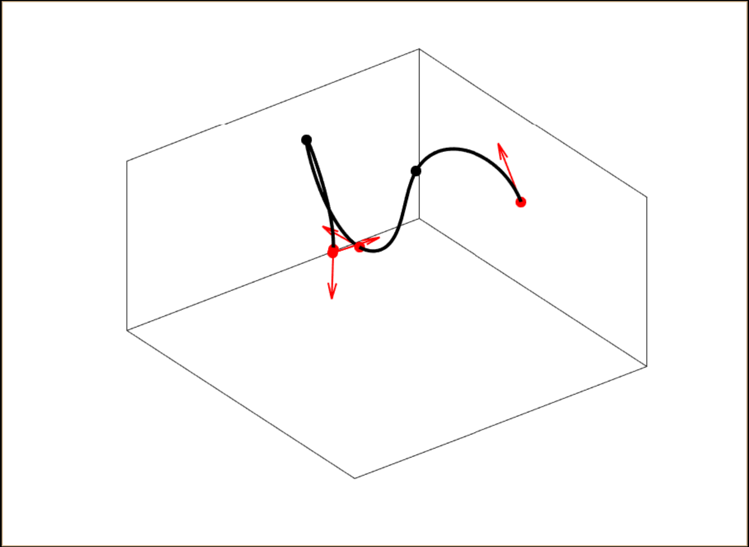

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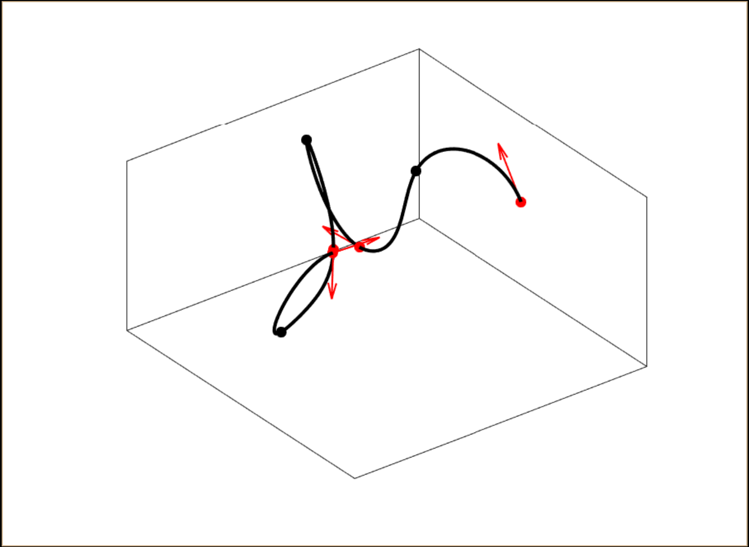

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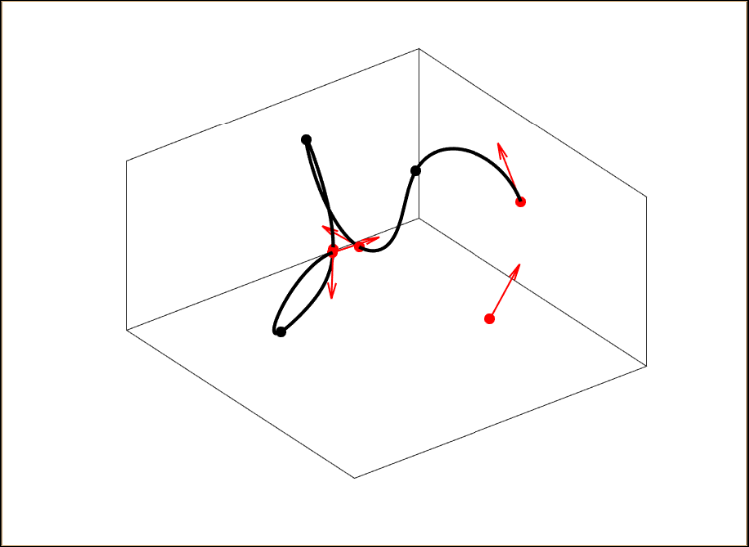

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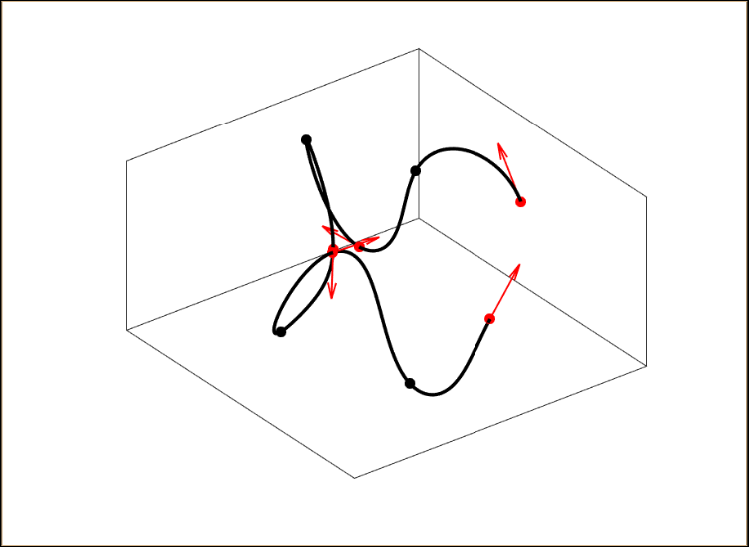
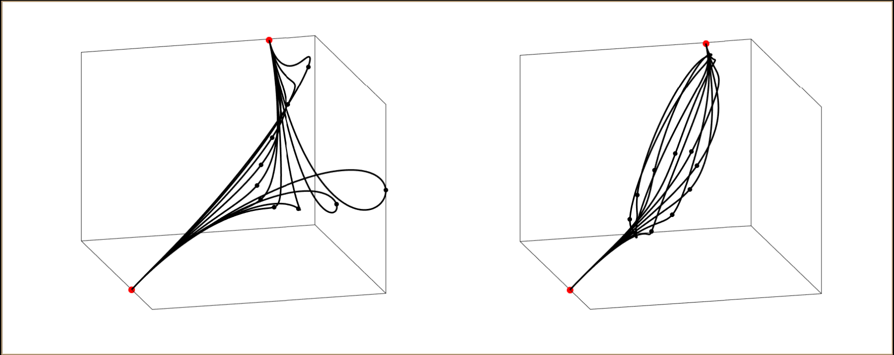


Local shape parameters



- 6 quaternion coefficients $\rightarrow$ 24 scalar degrees of freedom
- $6 \cdot 3 \rightarrow 18$ interpolation conditions
- 6 free parameters $\rightarrow$ reduced to 4 shape parameters by imposing C^1 joint between the quaternion pre-images of $\mathbf{x}_i$ and $\mathbf{x}_f$
  ($\rightarrow$ construction of just one PH quintic whenever possible)
  ($\rightarrow$ C^1 continuity of the ERF at the joint point)

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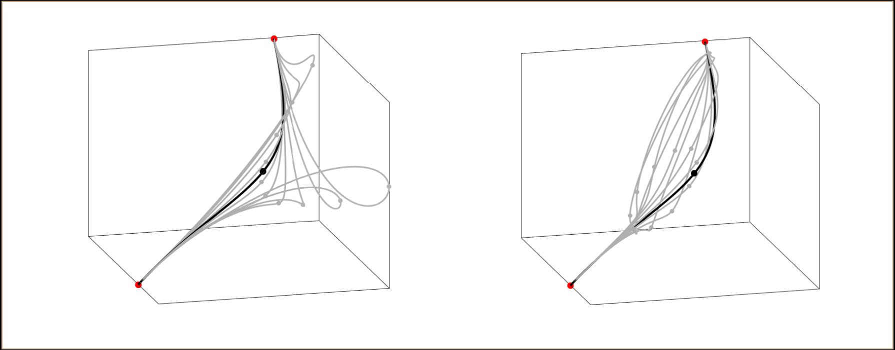

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Selection of free parameters

Biarc representation of CC C^1 PH quintic interpolant

[Farouki, Giannelli, Manni, Sestini — CAGD 2008]

$$\mathbf{x}^H(t) = \begin{cases} \mathbf{x}_i^H(t) & \text{for } t \in [0\,,\,\hat{t}] \\ \mathbf{x}_f^H(t) & \text{for } t \in [\hat{t}\,,\,1] \end{cases} \qquad \frac{d\mathbf{x}_i^H}{d\tau} = \mathcal{A}^H(\tau)\,\mathbf{i}\,\mathcal{A}^{H*}(\tau) \quad \frac{d\mathbf{x}_f^H}{d\eta} = \mathcal{B}^H(\eta)\,\mathbf{i}\,\mathcal{B}^{H*}(\eta)$$

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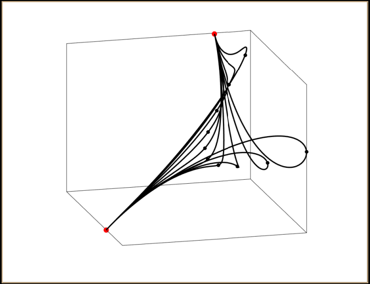
Parameter selection strategy for the biarc

- $\mathcal{A}_0 = \mathcal{A}_0^H \rightarrow$ angular parameter
- $\min |\mathcal{A}_1 - \mathcal{A}_1^H|^2 \rightarrow$ real parameter
- $\min |\mathcal{A}_2 - \mathcal{A}_2^H|^2 \rightarrow$ angular parameter
- $\mathcal{B}_2 = \mathcal{B}_2^H \rightarrow$ angular parameter



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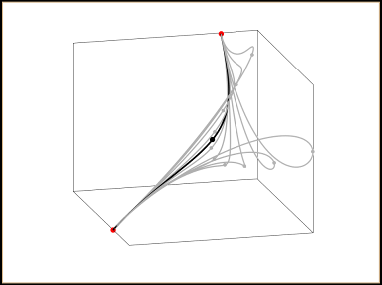
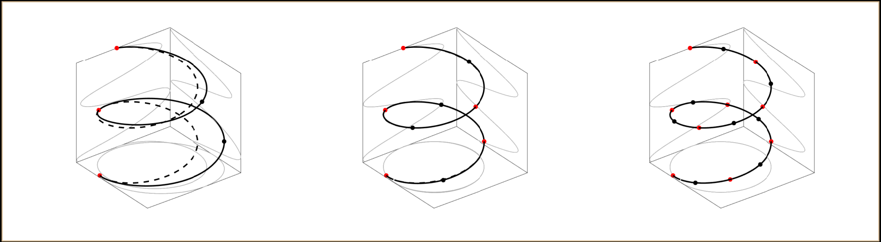
- $\min |\mathcal{A}_2 - \mathcal{A}_2^H|^2 \rightarrow$ angular parameter
- $\mathcal{B}_2 = \mathcal{B}_2^H \rightarrow$ angular parameter



Approximation order

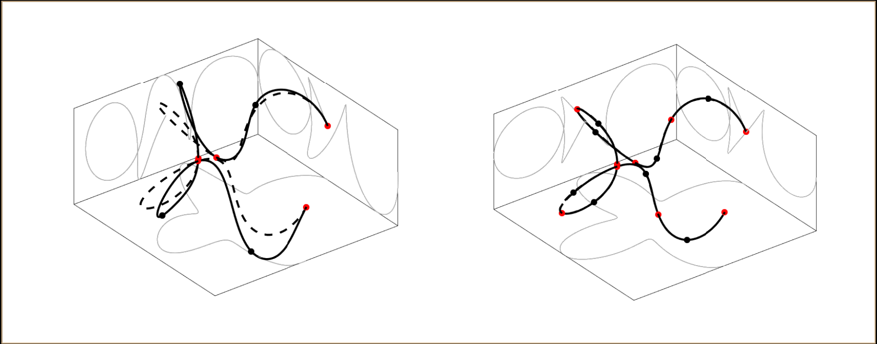
C^2 PH quintic spline reconstruction (solid line) of a circular helix (dashed line) from first order Hermite data — 3 (left), 5 (center), and 9 (right) sampled locations



$\mathbf{r}(s)$: sufficiently smooth arc length parameterized curve, $s \in [0, \Delta s]$
$\mathbf{x}(t)$: PH biarc interpolant

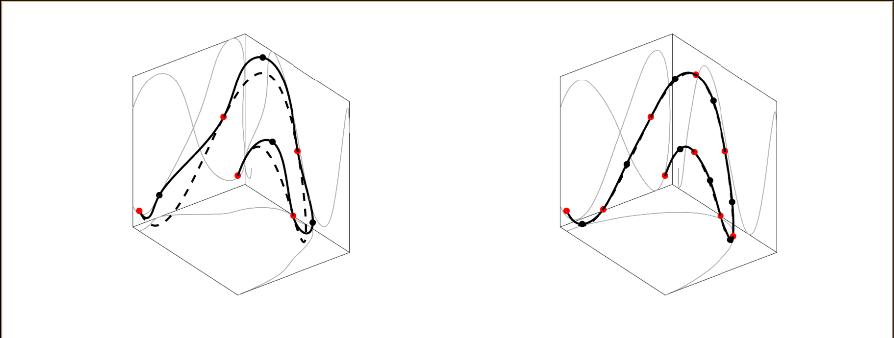
$$||\mathbf{x}(t) - \mathbf{r}(t\Delta s)|| = O(s^4) \qquad \forall\ t \in [0, 1]$$

Numerical approximation order



	curve #1		curve #2		curve #3	
k	E_k	p_k	E_k	p_k	E_k	p_k
2	1.6891e-01	5.00	1.1459e+01	3.76	2.7103e+00	2.78
3	1.2229e-02	3.79	2.1602e-01	2.77	3.9744e-01	5.73
4	1.0656e-03	3.52	2.3278e-02	3.32	3.9717e-02	3.21
5	8.0828e-05	3.72	2.6217e-03	4.42	1.8504e-03	3.15
6	5.5547e-06	3.86	2.1238e-04	3.01	2.3004e-04	3.63
7	3.6361e-07	3.93	1.4952e-05	3.78	1.6709e-05	3.83
8	2.3249e-08	3.97	9.8900e-07	3.94	1.0883e-06	3.92
9	1.4695e-09	3.98	6.3543e-08	3.98	6.8957e-08	3.96

Numerical approximation order



	curve #1		curve #2		curve #3	
k	E_k	p_k	E_k	p_k	E_k	p_k
2	1.6891e-01	5.00	1.1459e+01	3.76	2.7103e+00	2.78
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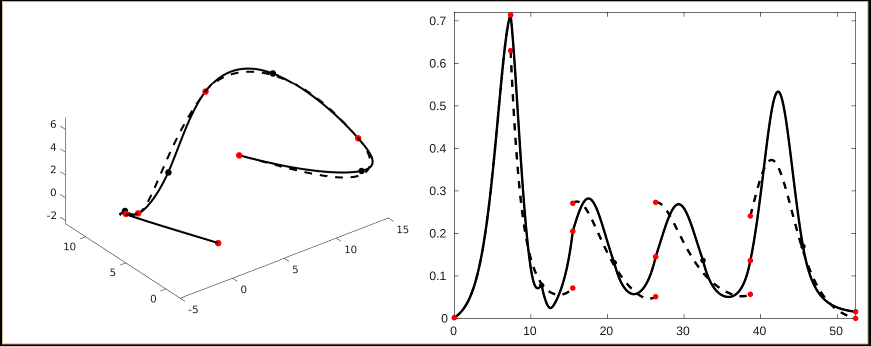
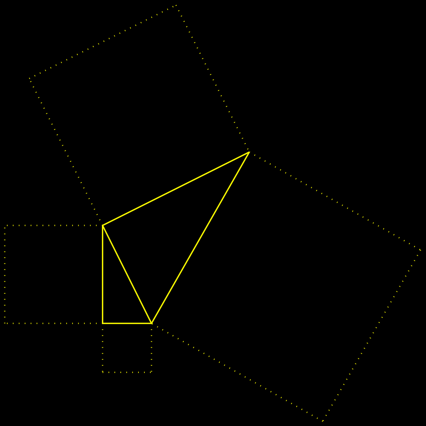
3D point stream interpolation

PH quintic splines curvature plots



C^2 PH biarc (solid line) vs. C^1 CC PH quintic (dashed line)



Pythagorean–hodograph curves

- smooth PH spline paths
- data stream interpolation
- real test case: Zeno AUV

- real test case: Zeno AUV

Guidance, navigation & control

Guidance

- responsible of providing kinematics reference to follow

  2 main approaches:

  trajectory traking (TT) & path following (PF)

  - TT: requires that the vehicle must track 42 both time and kinematics states: the vehicle should be at a certain time in a certain configuration (position/orientation)
  - PF: a software module is responsible of generating a suitable velocity profile to follow so that the vehicle moves along a desired geometric path without any particular time constraint

Navigation

- estimates the kinematic state of the vehicle
  (geodetic location and high order differential states)

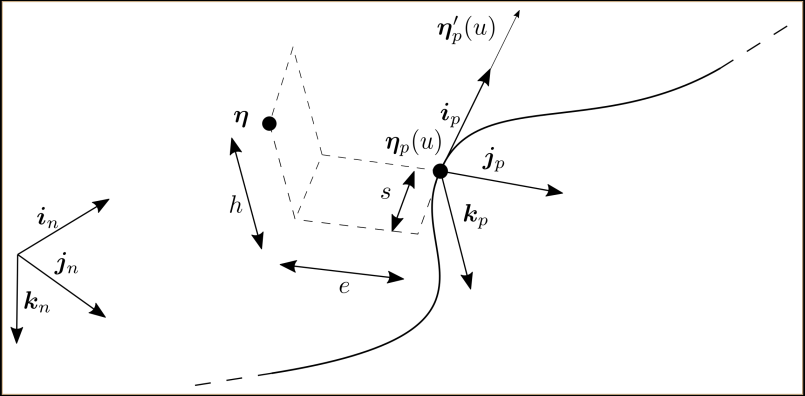
Control

- combine the results of the previous ones and allocate forces

Path following scheme

- Goal: prescribe the vehicle velocity commands to achieve motion control objectives

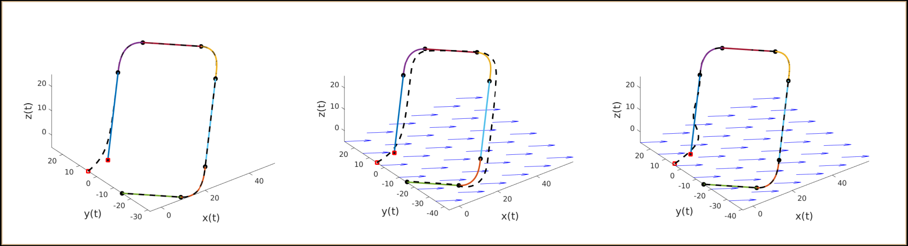
$\eta(t)$: vehicle position, $\eta_p(t)$: PH spline path, $\epsilon^p = (s, e, h)$: track error



$(\mathbf{i}_n, \mathbf{i}_n, \mathbf{i}_n)$: navigation frame, $(\mathbf{i}_n, \mathbf{i}_n, \mathbf{i}_n)$: path reference frame

Kinematic simulations: C^1 PH spline path

path to be followed (dashed line) & path of the vehicle solid line



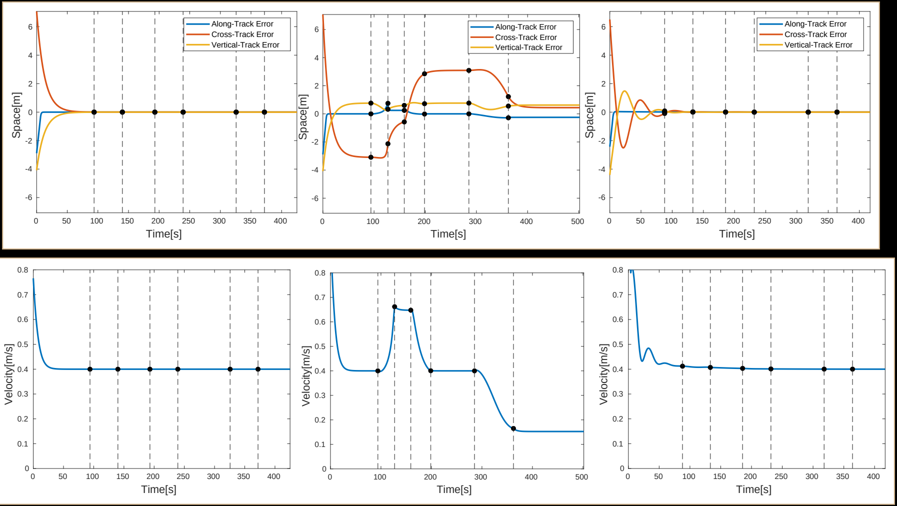
GL without current

GL with current

EGL with current

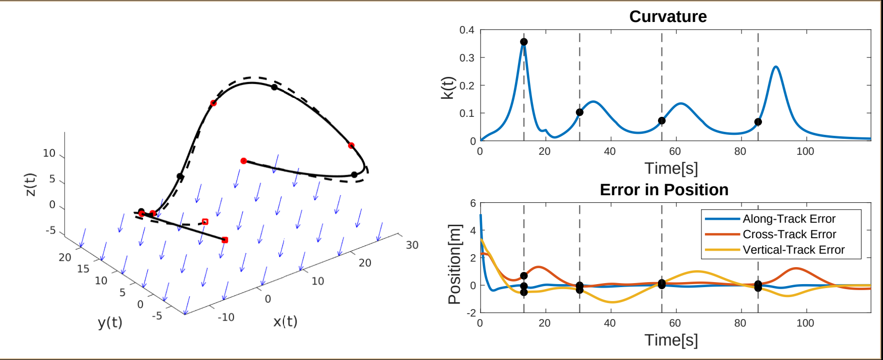
Kinematic simulations: C^1 PH spline path

track errors (top) & vehicle speed (bottom)

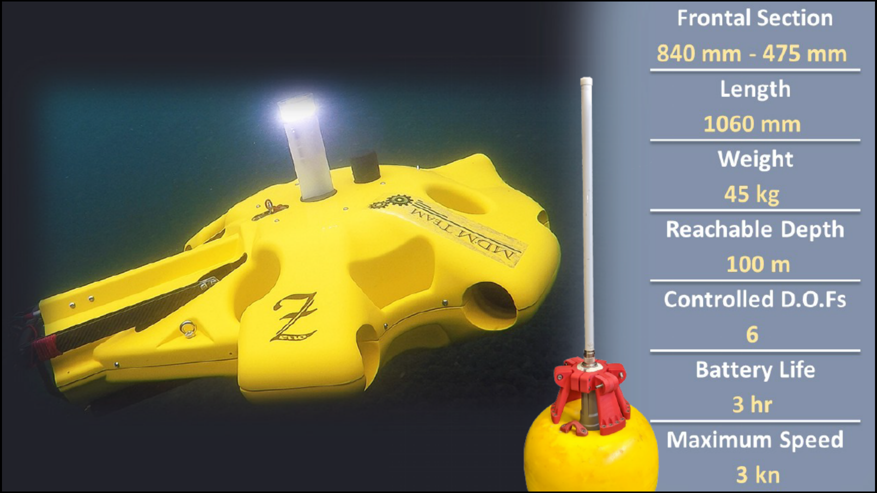


GL without current | GL with current | EGL with current

Kinematic simulations: C^2 PH spline path

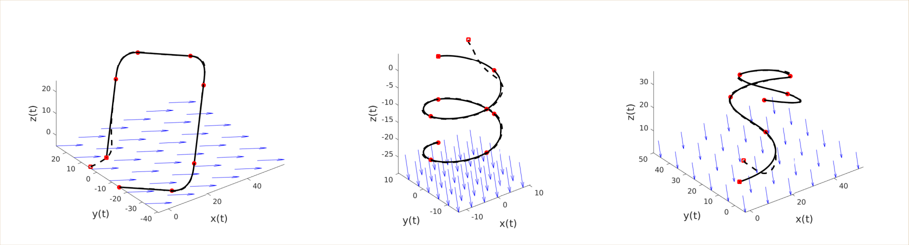


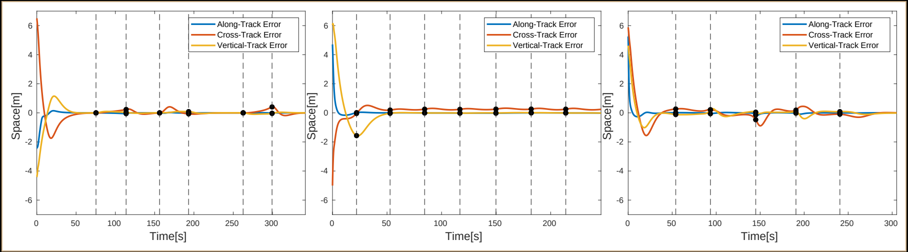
Zeno UAV

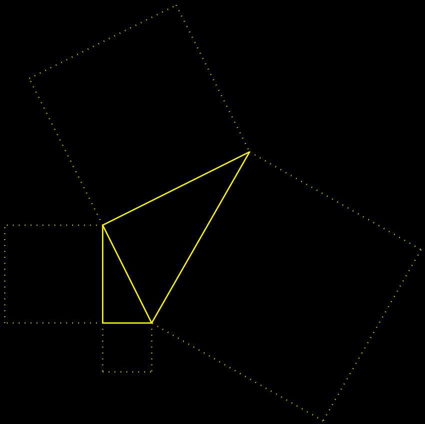


MDM TEAM
Frontal Section
840 mm - 475 mm
Length
1060 mm
Weight
45 kg
Reachable Depth
100 m
Controlled D.O.Fs
6
Battery Life
3 hr
Maximum Speed
3 kn

Dynamic simulations







Pythagorean–hodograph curves

- smooth PH spline paths
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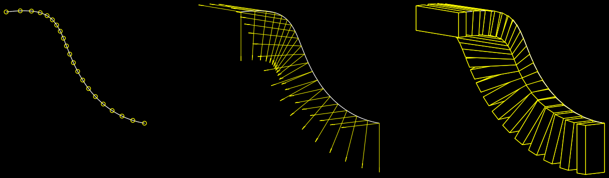
- real test case: Zeno AUV



- smooth PH spline paths
- data stream interpolation
- real test case: Zeno AUV