# Stable approximation of Helmholtz solutions by evanescent plane waves

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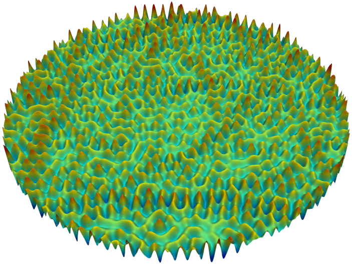
# Helmholtz equation

Homogeneous Helmholtz equation:

$$-\Delta u - \kappa^2 u = 0$$

Wavenumber $\kappa = \omega/c > 0$,
$\lambda = \frac{2\pi}{\kappa}$ = wavelength.



$u(\mathbf{x})$ represents the space dependence of time-harmonic solutions $U(\mathbf{x}, t) = \Re\{e^{-i\omega t} u(\mathbf{x})\}$ of the wave equation $\frac{1}{c^2}\frac{\partial^2 U}{\partial t^2} - \Delta U = 0$.

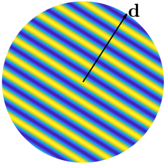
Fundamental PDE in acoustics, electromagnetism, elasticity...

- "Easy" PDE for small $\kappa$: perturbation of Laplace eq.
- "Difficult" PDE for large $\kappa$: high-frequency problems

# Propagative plane waves

A difficulty for $\kappa \gg 1$ is the approximation of Helmholtz solutions.

One can beat (piecewise) polynomial approximations using propagative plane waves (PPWs):




$$e^{i\kappa \mathbf{d}\cdot\mathbf{x}} \qquad \mathbf{d} \in \mathbb{R}^n \quad \mathbf{d}\cdot\mathbf{d} = 1$$

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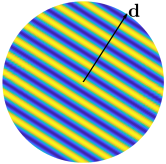


Some uses of PPWs:

- Trefftz methods:
  Galerkin schemes whose basis functions are local PDE solutions.
  E.g.: UWVF, TDG, PWDG, DEM, VTCR, WBM, LS, PUM . . .
- reconstruction of sound fields from point measurements (microphones) in experimental acoustics.

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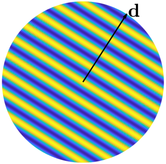


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PPWs are complex exponentials:
easy & cheap to manipulate, evaluate, differentiate, integrate...
$\rightarrow$ preferred against other Trefftz functions (e.g. circular waves)

# Approximation and instability

Rich PPW approximation theory for Helmholtz solutions:

- Cessenat, Després 1998, Taylor-based, $h$
- Melenk 1995; Moiola, Hiptmair, Perugia 2011, Vekua theory, $hp$

$\kappa$-explicit, better rates vs DOFs than polynomials.

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So why isn't everybody using plane waves?

The issue is "instability".
Increasing # of PPWs, at some point convergence stagnates.

Numerical phenomenon: due to computer arithmetic+cancellation.

PPW instability already observed in all PPW-based Trefftz methods.
Usually described and treated as ill-conditioning issue.

# Adcock–Huybrechs theory

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**Goal:** Approximate some $v \in V$ with linear combination of $\{\phi_m\} \subset V$.

**Result:** If there exists $\sum_m a_m \phi_m$ with

- good approximation of $v$,
- small coefficients $a_m$,

then the approximation of $v$ in computer arithmetic is stable,
if one uses oversampling and SVD regularization.

Stability does not depend on (LS, Galerkin,...) matrix conditioning.

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**Spoiler:**

- – PPWs can not approximate general $u$ with small coefficients.
- + Include evanescent PWs $\rightarrow$ small-coefficient approx. $\rightarrow$ stability.

Here we consider only the approximation in the unit disk $B_1 \subset \mathbb{R}^2$.

# Part I

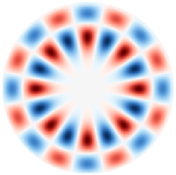
# Circular and propagative plane waves

# Circular waves — Fourier–Bessel functions

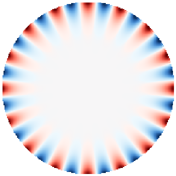
Separable solutions in polar coordinates:

$$b_p(r,\theta) := \beta_p J_p(kr)\mathrm{e}^{\mathrm{i}p\theta} \qquad \forall p \in \mathbb{Z}, \quad (r,\theta) \in B_1$$

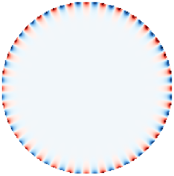
$\beta_p$ = normalization, e.g. in $H^1(B_1)$ norm. $\qquad \beta_p \sim \kappa\left(\frac{2|p|}{\mathrm{e}\kappa}\right)^{|p|}$ as $p \to \infty$.



$p = 8 = \kappa/2$
Propagative mode



$p = 16 = \kappa$



$p = 32 = 2\kappa$
Evanescent mode

$\{b_p\}_{p\in\mathbb{Z}}$ is orthonormal basis of $\quad \mathcal{B} := \left\{u \in H^1(B_1) : -\Delta u - \kappa^2 u = 0\right\}$

# PPW instability

The Jacobi–Anger expansion relates PPWs and circular waves $b_p$:

$$\mathsf{PW}_{\varphi}(\mathbf{x}) := e^{i\kappa \mathbf{d}\cdot\mathbf{x}} = \sum_{p\in\mathbb{Z}} i^p J_p(kr) e^{ip(\theta-\varphi)} = \sum_{p\in\mathbb{Z}} \left( i^p e^{-ip\varphi} {\beta_p}^{-1} \right) b_p(r,\theta)$$

$$\begin{cases} \mathbf{d} = (\cos\varphi,\ \sin\varphi) \\ \mathbf{x} = (r\cos\theta,\ r\sin\theta) \end{cases}$$

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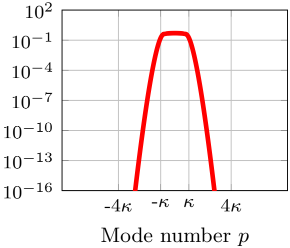


Modulus of Fourier coefficient
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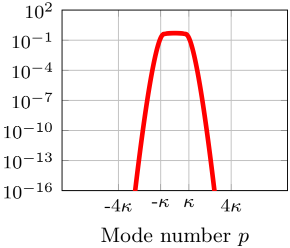


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Approximation of $u = \sum_p \widehat{u}_p b_p \in \mathcal{B}$
requires exponentially large coefficients.

$u \in H^s(B_1), s \geq 1 \iff |\widehat{u}_p| \sim o(|p|^{-s+\frac{1}{2}})$
but $|\beta_p^{-1}| \sim |p|^{-|p|}$ is much smaller!

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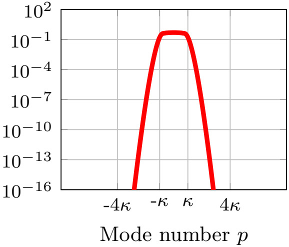


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$$\begin{matrix} \forall p \in \mathbb{Z} \\ \forall M \in \mathbb{N} \\ \forall \boldsymbol{\mu} \in \mathbb{C}^M \\ \forall \eta \in (0,1) \end{matrix} \quad \left\| b_p - \sum_{m=1}^{M} \mu_m \mathrm{PW}_{\frac{2\pi m}{M}} \right\|_{\mathcal{B}} \leq \eta \quad \Longrightarrow \quad \|\boldsymbol{\mu}\|_{\ell^1(\mathbb{C}^M)} \geq (1-\eta) \underbrace{|\beta_p|}_{\sim |p|^{|p|}}$$

Part II

# Evanescent plane waves

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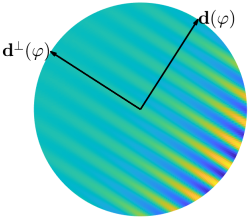
Idea from WBM (wave-based method) by Wim Desmet etc (Leuven).

Stability improves using PPWs & evanescent plane waves (EPW):

$$e^{i\kappa \mathbf{d}\cdot\mathbf{x}} \qquad \mathbf{d} \in \mathbb{C}^2 \quad \mathbf{d}\cdot\mathbf{d} = 1$$

Complex $\mathbf{d}$! Again: exponential Helmholtz solutions.



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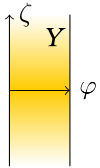


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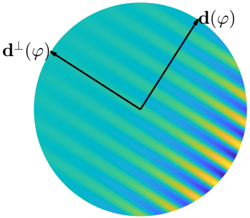
Parametrised by $\varphi$ = direction, $\zeta$ = "evanescence".
Parametric cylinder: $\mathbf{y} := (\varphi,\zeta) \in Y := [0,2\pi)\times\mathbb{R}$.

$$\mathbf{d}(\mathbf{y}) := \big(\cos(\varphi+i\zeta),\, \sin(\varphi+i\zeta)\big)\in\mathbb{C}^2$$



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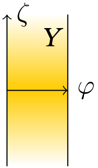


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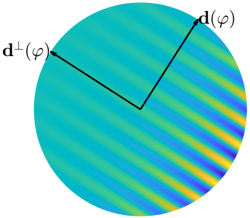
$$\mathrm{EW}_{\mathbf{y}}(\mathbf{x}) := e^{i\kappa \mathbf{d}(\mathbf{y})\cdot\mathbf{x}}$$

$$= e^{i\kappa(\cosh\zeta)\mathbf{x}\cdot\mathbf{d}(\varphi)}\; e^{-\kappa(\sinh\zeta)\mathbf{x}\cdot\mathbf{d}^{\perp}(\varphi)},$$




oscillations along $\mathbf{d}(\varphi) := (\cos\varphi, \sin\varphi)$

decay along $\mathbf{d}^{\perp}(\varphi) := (-\sin\varphi, \cos\varphi)$

# EPW modal analysis

Jacobi–Anger expansion holds also for EPWs:

$$\mathrm{EW}_{\mathbf{y}}(\mathbf{x}) = \mathrm{e}^{\mathrm{i}\kappa\mathbf{d}(\mathbf{y})\cdot\mathbf{x}} = \sum_{p\in\mathbb{Z}} \mathrm{i}^p J_p(\kappa r)\mathrm{e}^{\mathrm{i}p(\theta-[\varphi+\mathrm{i}\zeta])} = \sum_{p\in\mathbb{Z}} \left(\mathrm{i}^p \mathrm{e}^{-\mathrm{i}p\varphi} \mathrm{e}^{p\zeta} \beta_p^{-1}\right) b_p(\mathbf{x}).$$

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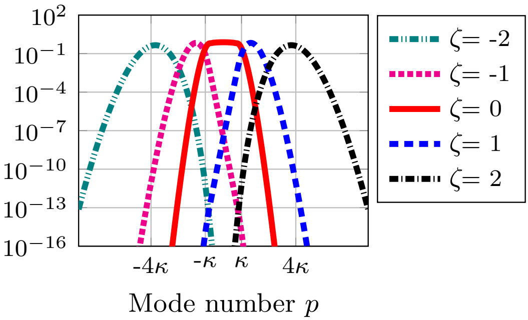
Absolute values of Fourier coefficients $|\mathrm{i}^p \mathrm{e}^{-\mathrm{i}p\varphi} \mathrm{e}^{p\zeta} \beta_p^{-1}|$, $\kappa = 16$:




Looks promising!

We can hope to approximate large-$p$ Fourier modes with EPWs & small coefficients.

# Herglotz representation with EPWs

We want to represent $u \in \mathcal{B}$ as continuous superposition of EPWs:

$$u(\mathbf{x}) = (Tv)(\mathbf{x}) = \int_Y \mathsf{EW}_{\mathbf{y}}(\mathbf{x})\ v(\mathbf{y})\ w^2(\mathbf{y})\,\mathrm{d}\mathbf{y} \qquad \mathbf{x} \in B_1$$

with density $v \in L^2(Y; w^2)$ and weight $w^2 = \mathrm{e}^{-2\kappa \sinh|\zeta| + \frac{1}{2}|\zeta|}$

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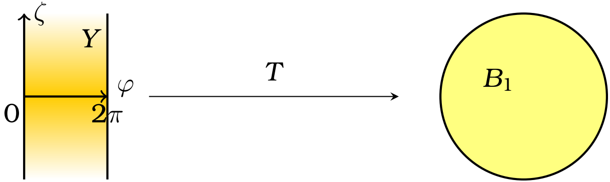
Parametric space

Physical space




Herglotz density

$v \in \mathcal{A} = \overline{\mathrm{span}\{a_p\}} \subset L^2(Y; w^2)$

Helmholtz solution

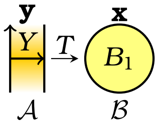
$u \in \mathcal{B} = \overline{\mathrm{span}\{b_p\}} \subset H^1(B_1)$

$a_p(\mathbf{y}) := \alpha_p\, \mathrm{e}^{p(\zeta + \mathrm{i}\varphi)}$ $\qquad$ $\alpha_p > 0$ normalization in $\|\cdot\|_{\mathcal{A}} = \|\cdot\|_{L^2(Y;w^2)}$, $p \in \mathbb{Z}$

# Helmholtz solutions are superposition of EPWs

Define Herglotz transform: (synthesis operator)

$$(Tv)(\mathbf{x}) := \int_Y \mathsf{EW}_{\mathbf{y}}(\mathbf{x})\ v(\mathbf{y})\ w^2(\mathbf{y})\,\mathrm{d}\mathbf{y} \qquad \begin{aligned} T : \mathcal{A} &\to \mathcal{B} \\ v &\mapsto u \end{aligned}$$



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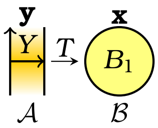


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Jacobi–Anger $\Rightarrow$ $T$ is diagonal in ONB's $\{a_p\}, \{b_p\}$:

$$\mathrm{EW}_{\mathbf{y}}(\mathbf{x}) = \sum_{p \in \mathbb{Z}} \tau_p \overline{a_p(\mathbf{y})} b_p(\mathbf{x}), \qquad \tau_p := \frac{\mathrm{i}^p}{\alpha_p \beta_p}, \qquad 0 < \tau_- \leq |\tau_p| \leq \tau_+ < \infty.$$

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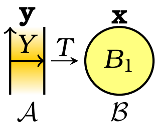


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The operator $T : \mathcal{A} \to \mathcal{B}$ is bounded and invertible:

$$T a_p = \tau_p b_p, \qquad \tau_- \|v\|_{\mathcal{A}} \le \|Tv\|_{\mathcal{B}} \le \tau_+ \|v\|_{\mathcal{A}} \qquad \forall v \in \mathcal{A}$$

Every Helmholtz solution is (continuous) linear combination of EPW with small coefficients: $\|v\|_{\mathcal{A}} \le \tau_-^{-1} \|u\|_{\mathcal{B}}$

# Part III

# Discrete EPW spaces

# Frames, RKHS, sampling

All good at continuous level, but what about finite sums of EPWs?

Call $K_{\mathbf{y}}$ the pre-images of the evanescent plane waves:

$$T : K_{\mathbf{y}} \mapsto \mathrm{EW}_{\mathbf{y}} \qquad \mathbf{y} \in Y.$$

These are Riesz representation of the evaluation functional at $\mathbf{y}$:

$$v(\mathbf{y}) = (v, K_{\mathbf{y}})_{\mathcal{A}} \qquad \forall v \in \mathcal{A}, \qquad \mathbf{y} \in Y.$$

$\mathcal{A}$ is reproducing-kernel Hilbert space, kernel: $K_{\mathbf{y}}(\mathbf{z}) = \sum_{p\in\mathbb{Z}} \overline{a_p(\mathbf{y})} a_p(\mathbf{z})$

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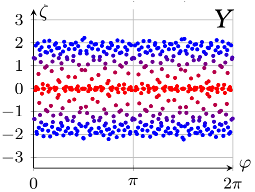
Approximation of $u$ by EPWs "maps" to
reconstruction of $v = T^{-1}u$ by point sampling:

$$\mathcal{A} \ni \qquad v \approx \sum_{m=1}^{M} \mu_m K_{\mathbf{y}_m} \qquad \underset{T^{-1}}{\overset{T}{\rightleftarrows}} \qquad u \approx \sum_{m=1}^{M} \mu_m \mathrm{EW}_{\mathbf{y}_m} \qquad \in \mathcal{B}$$

# Parameter sampling in $Y$

How to sample $\mathcal{A} = \mathrm{span}\{\alpha_p \mathrm{e}^{p(\zeta + \mathrm{i}\varphi)}\} \subset L^2(Y; w^2)$?
How to choose points $\{\mathbf{y}_m\}_m \in Y$?



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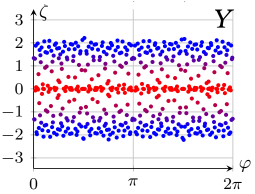

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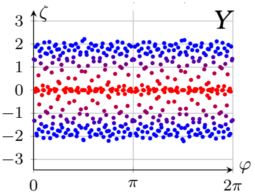


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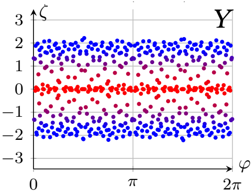


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$$\rho(\mathbf{y}) := \frac{w^2}{2P+1} \sum_{|p| \le P} |a_p(\mathbf{y})|^2 \qquad \text{on } Y$$

$\rho^{-1} =$ "Christoffel function"

and generate $M \in \mathbb{N}$ nodes $\{\mathbf{y}_m\}_{m=1,\dots,M}$ distributed according to $\rho$.

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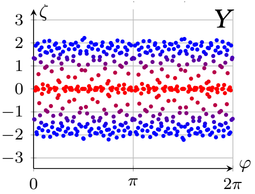


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We expect the span of the normalised sampling functionals !

$$\left\{ \mathbf{y} \mapsto \frac{1}{\sqrt{\sum_{|p| \le P} |a_p(\mathbf{y}_m)|^2}} K_{\mathbf{y}_m}(\mathbf{y}) \right\}_{m=1,\dots,M} \subset \mathcal{A}$$

to approximate any $v_P \in \mathcal{A}_P$ with small coefficients.

# Helmholtz solution approximation by EPWs

Then any $u \in \mathrm{span}\{b_p\}_{|p|\le P}$ can be approximated by EPWs

$$\left\{\mathbf{x} \mapsto \frac{1}{\sqrt{M \sum_{|p|\le P} |a_p(\mathbf{y}_m)|^2}} \mathrm{EW}_{\mathbf{y}_m}(\mathbf{x})\right\}_{m=1,\dots,M} \subset \mathcal{B}$$

with small coefficients.

Then $u$ can be stably approximated in computer arithmetic using SVD and oversampling.

The $M$-dimensional EPW space depends on truncation parameter $P$: the space is tuned to approximate the Fourier modes $b_p$ with $|p| \le P$.

# Part IV

# Numerical results

# Boundary sampling method

Given (PPW, EPW,. . . ) approximation set $\text{span}\{\phi_m\}_{m=1,\dots,M}$,
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$$A\boldsymbol{\xi} = \mathbf{c} \qquad \text{with} \qquad \begin{matrix} A_{s,m} := \phi_m(\mathbf{x}_s), \\ c_s := u(\mathbf{x}_s) \end{matrix} \quad {\scriptstyle \begin{matrix} s=1,\ldots,S \\ m=1,\ldots,M \end{matrix}} \quad \rightarrow \quad u_M = \sum_m \xi_m \phi_m \approx u.$$

Choose $\kappa^2 \neq$ Laplace–Dirichlet eigenvalue on $B_1$.

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- Oversampling: $S > M$
- SVD regularization, threshold $\epsilon$:

} required by Adcock–Huybrechs

$$A = U\,\mathrm{diag}(\sigma_1, \dots, \sigma_M)\, V^*, \qquad \Sigma_\epsilon := \mathrm{diag}(\{\sigma_m > \epsilon \max_{m'} \sigma_{m'}\}),$$

$$\boldsymbol{\xi}_\epsilon = V\Sigma_\epsilon^\dagger U^* \mathbf{c}$$

# Approximation by PPWs

Approximation of circular waves $\{b_p\}_p$ by equispaced PPWs

$\kappa = 16, \qquad \epsilon = 10^{-14}, \qquad S = \max\{2M, 2|p|\}, \qquad$ residual $\mathcal{E} = \frac{\|A\boldsymbol{\xi}_\epsilon - \mathbf{c}\|}{\|\mathbf{c}\|}$

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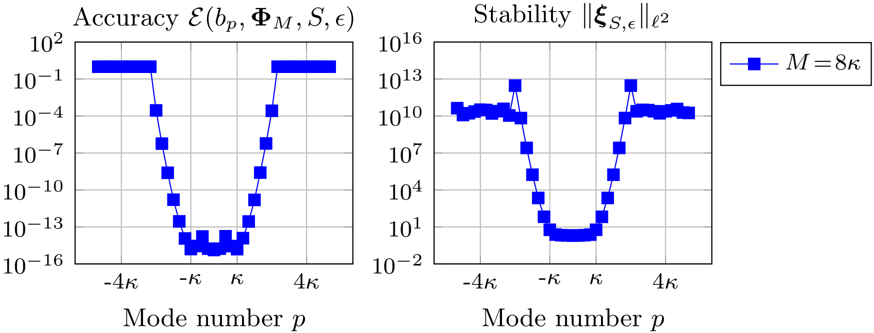


- Propagative modes $|p| \lesssim \kappa$: $\mathcal{O}(\epsilon)$ error $\forall M$, $\mathcal{O}(1)$ coeff.'s
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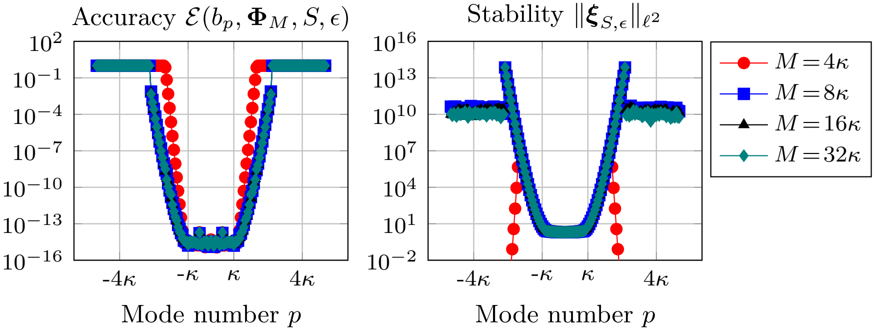


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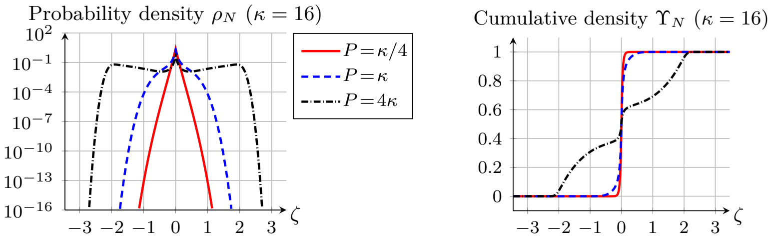
Probability density $\rho$ & cumulative d.f. as functions of evanescence $\zeta$:

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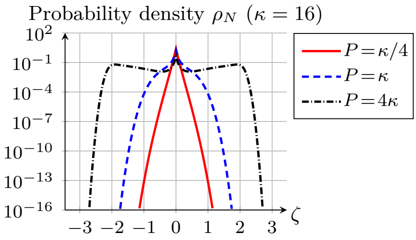


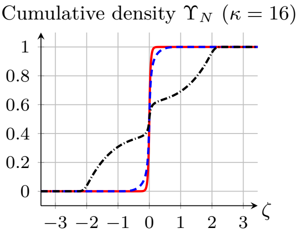


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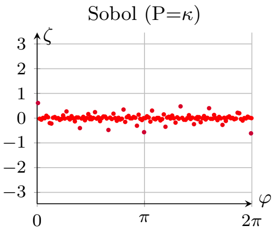


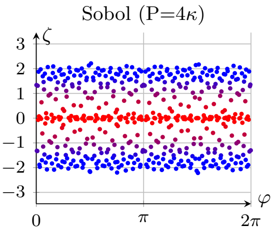


Samples computed on $(0, 1)^2$ & uniform prob., mapped to $Y$ by $\Upsilon^{-1}$.

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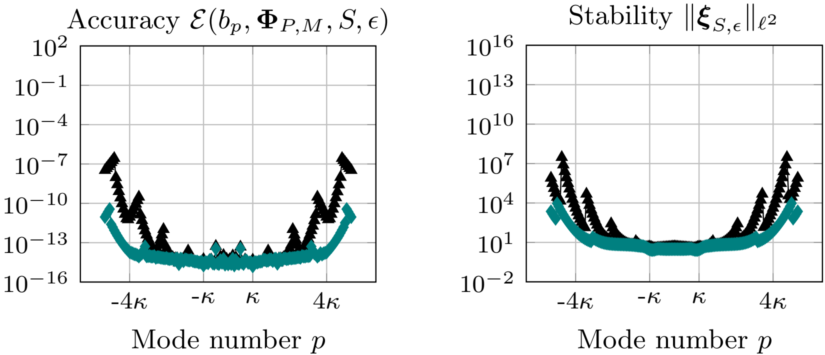
Approximation of $\{b_p\}$, $\boxed{P = 4\kappa}$, $\kappa = 16$, ▲ $M = 4P$, ◆ $M = 8P$




Discrete EPW space approximates all $b_p$s for $|p| \leq P$!

# Solution and error plots

$u = \sum_{|p|\leq P} \hat{u}_p b_p, \quad \hat{u}_p \sim (\max\{1, |p| - \kappa\})^{-1/2}, \quad \kappa = 100, \quad P = 2\kappa,$

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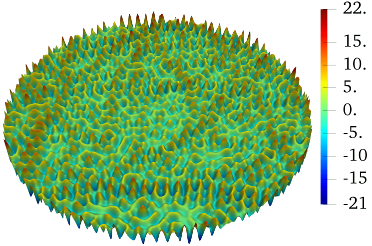
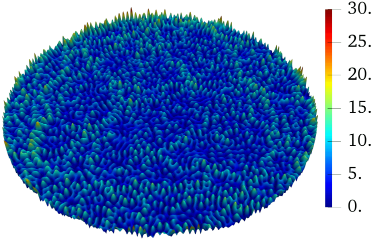



$|u|$



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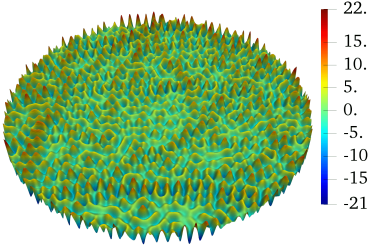


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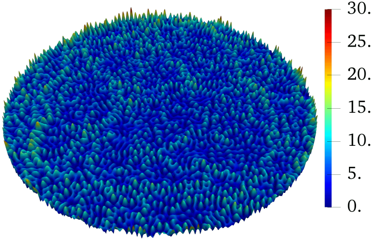


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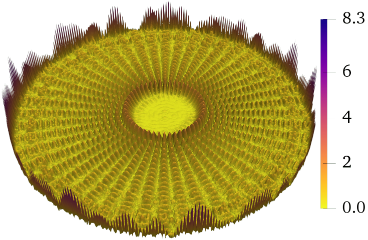
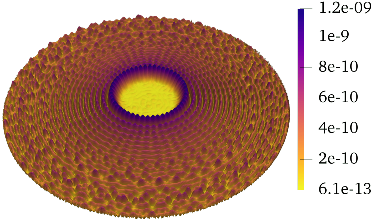



$|u - EPW|$




$\|u - PPW\|_{L^\infty} \gtrsim 7 \cdot 10^9 \, \|u - EPW\|_{L^\infty}$

DOFs/wavelength $= \lambda\sqrt{M/|B_1|} \approx 1$

# Summary

- Approximation of Helmholtz solutions by PPWs is unstable: accuracy only with large coefficients.
- Approximation by evanescent PWs seems to be stable.
- EPWs parameters chosen with sampling in $Y$.
- Key new result is stable Herglotz transform $u = Tv$.

Next steps:

- General geometries
- 3D
- Maxwell & elasticity
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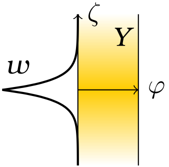
# Weighted $L^2(Y)$ space $\mathcal{A}$

Weighted $L^2$ space on parametric cylinder & orthonormal basis:

$$w(\mathbf{y}) := \mathrm{e}^{-\kappa \sinh|\zeta| + \frac{1}{4}|\zeta|} \qquad \mathbf{y} = (\varphi, \zeta) \in Y$$

$$\|v\|_{\mathcal{A}}^2 := \|v\|_{L^2(Y;w^2)}^2 = \int_Y |v(\mathbf{y})|^2 w^2(\mathbf{y})\,\mathrm{d}\mathbf{y}$$




$$a_p(\mathbf{y}) := \alpha_p\, \mathrm{e}^{p(\zeta + \mathrm{i}\varphi)}$$

$\alpha_p > 0$ normalization in $\|\cdot\|_{\mathcal{A}}$, $p \in \mathbb{Z}$

$$\mathcal{A} := \overline{\operatorname{span}\{a_p\}_{p\in\mathbb{Z}}}^{\|\cdot\|_{\mathcal{A}}} \subsetneq L^2(Y; w^2)$$

---

Jacobi–Anger: $\mathbf{x} \in B_1$ $\quad$ $\mathbf{y} \in$ 


$$\mathrm{EW}_{\mathbf{y}}(\mathbf{x}) = \sum_{p\in\mathbb{Z}} \mathrm{i}^p J_p(\kappa r) \mathrm{e}^{\mathrm{i}p(\theta - [\varphi + \mathrm{i}\zeta])} = \sum_{p\in\mathbb{Z}} \tau_p \overline{a_p(\mathbf{y})} b_p(\mathbf{x}), \qquad \tau_p := \frac{\mathrm{i}^p}{\alpha_p \beta_p}.$$

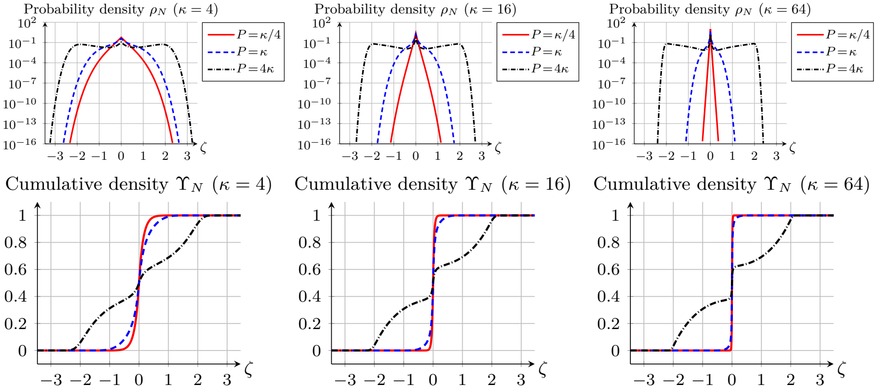
From asymptotics & choice of $w$: $\quad 0 < \tau_- \le |\tau_p| \le \tau_+ < \infty \quad \forall p \in \mathbb{Z}$.

---

$$\forall \mathbf{x} \in B_1, \qquad \mathbf{y} \mapsto \mathrm{EW}_{\mathbf{y}}(\mathbf{x}) \in \mathcal{A} \qquad \text{(not true for } \mathbf{x} \in \partial B_1\text{)}$$

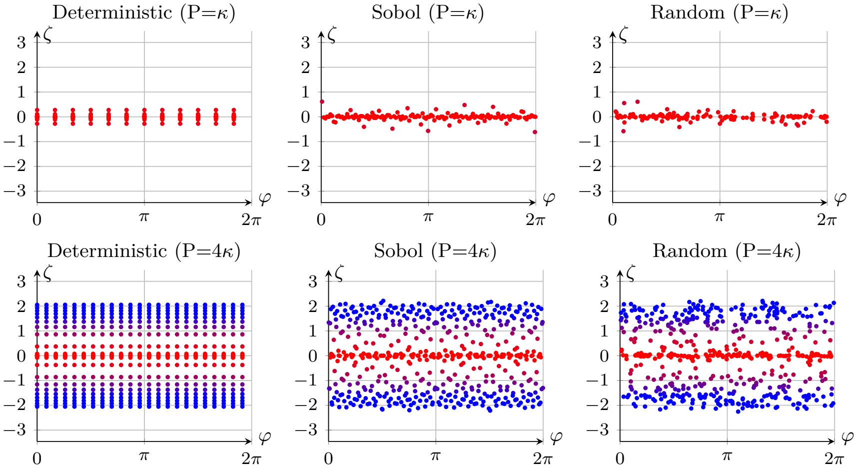
# EPW approximation: probability measure on $Y$

Probability density $\rho$ & cumulative d.f. as functions of evanescence $\zeta$:




They depend on $P$: target functions in $\text{span}\{b_p\}_{|p|\leq P}$.
Modes at $\zeta \approx \pm \log(2P/\kappa)$.
Computation of $\rho$ requires $\kappa$-dependent normalisation factors $\alpha_p$.

# Parameter samples in the cylinder $Y$

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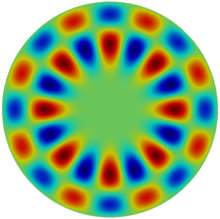
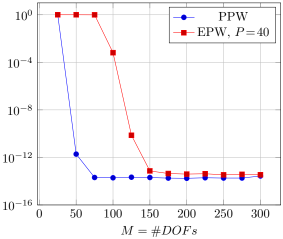
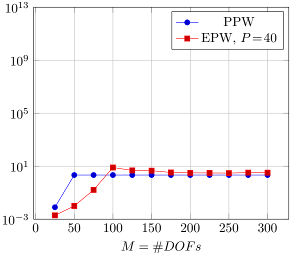
$p = 8$



$b_p$, $p = 8$, residual $\|A\boldsymbol{\xi}_\epsilon - \mathbf{c}\|/\|\mathbf{c}\|$




$b_p$, $p = 8$, coefficient norm $\|\boldsymbol{\xi}_\epsilon\|$



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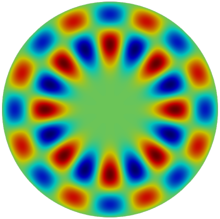
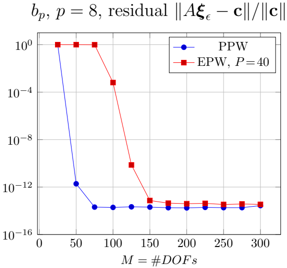
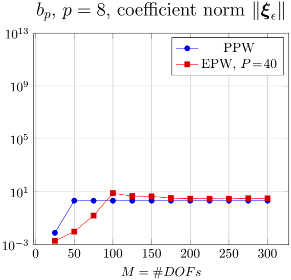
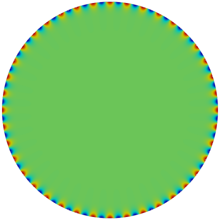
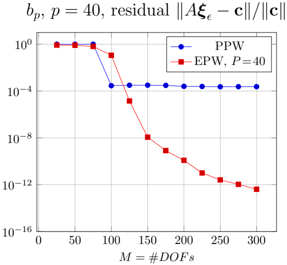


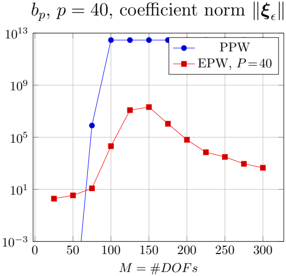



$p = 40$



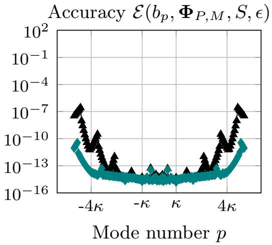


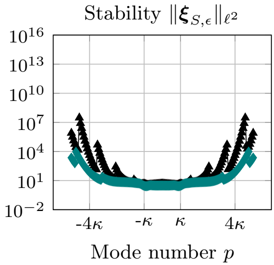



# Approximation by EPWs

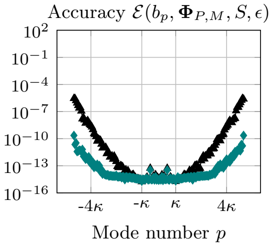
Approximation of $\{b_p\}$, ▲ $M = 4P$, ◆ $M = 8P$   $P = 4\kappa,\ \kappa = 16$

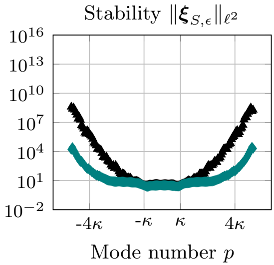
## Uniform



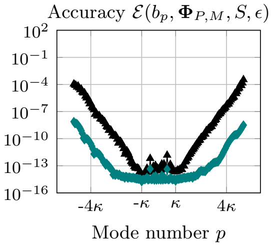



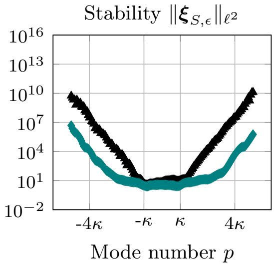

## Sobol




## Random



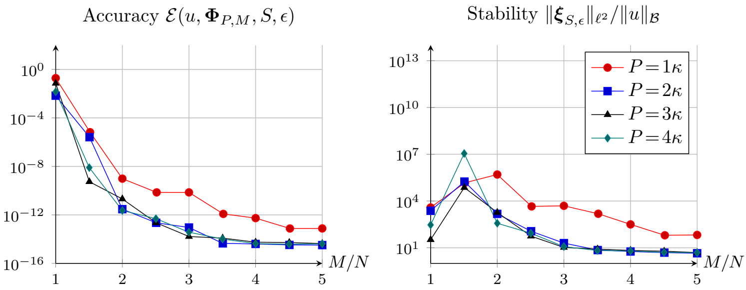



Evanescent PW approximation of rough $u$: $(S = 2M, \kappa = 16)$

$$u = \sum_{|p| \le P} \hat{u}_p b_p, \qquad \hat{u}_p \sim (\max\{1, |p| - \kappa\})^{-1/2}$$

EPWs constructed assuming that $P$ is known. Deterministic sampling.

Convergence for $M \nearrow$ plotted against $\frac{M}{2P+1} = \frac{\dim(\text{approx. space})}{\dim(\text{solution space})}$:



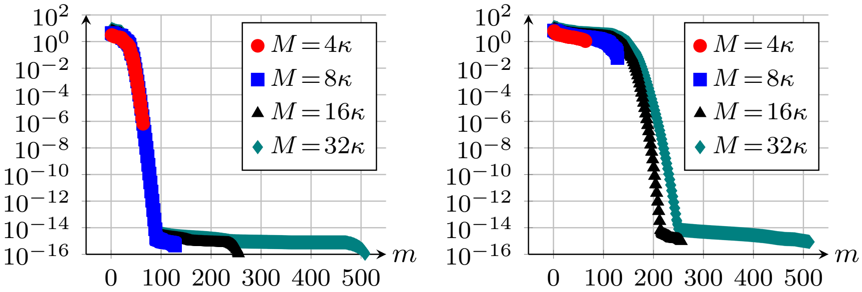

Error is $P$-independent.

# Singular values of the matrix $A$

$\kappa = 16$

PPWs

EPWs (Sobol, $P = 4\kappa$)




Comparable condition numbers, larger $\epsilon$-rank for EPWs.
Can further increase $\epsilon$-rank by raising $P$.