An Introduction to the divergence-free Virtual Element Method with focus on the Oseen Equation

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UNIVERSITÀ
DEGLI STUDI DI BARI
ALDO MORO

CALCOLO SCIENTIFICO E MODELLI MATEMATICI
alla ricerca delle cose nascoste attraverso le cose manifeste

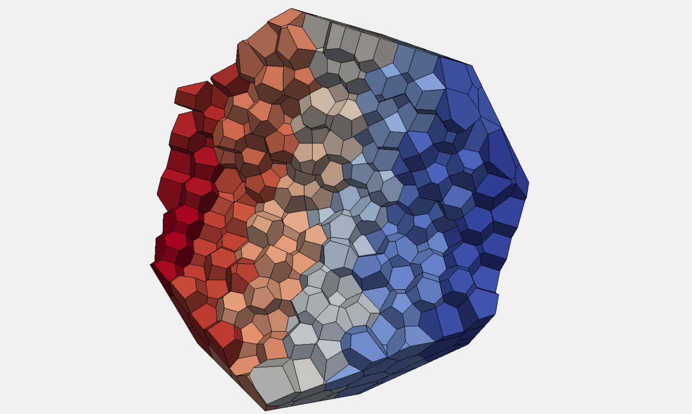
Outline of the presentation

- **Virtual Element Methods (VEMs)**
- **Divergence-free VEMs**
  - definition, DoFs, divergence-free solution,
  - kernel inclusion & advantages

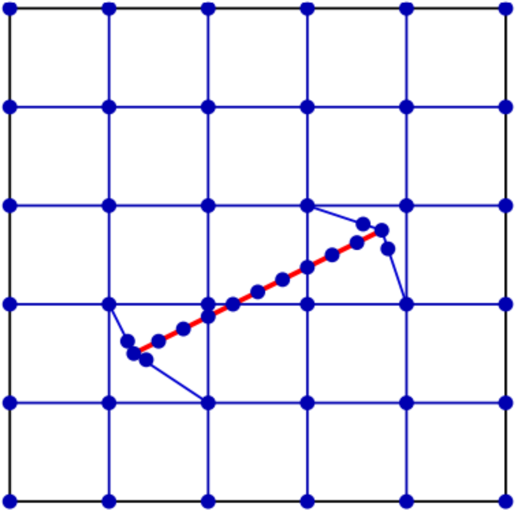
    [Beirão da Veiga, Lovadina, V., 2017], [Beirão da Veiga, Lovadina, V., 2019]
- **Vorticity stabilization for the Oseen Equation**
  - vorticity stabilization [Ahmed, Barrenechea, Burman, Guzman, Linke, Merdon, 2020]
  - VEM setting [Beirão da Veiga, Dassi, V., 2021],
- **Conclusions & Remarks**

The Virtual Element Method

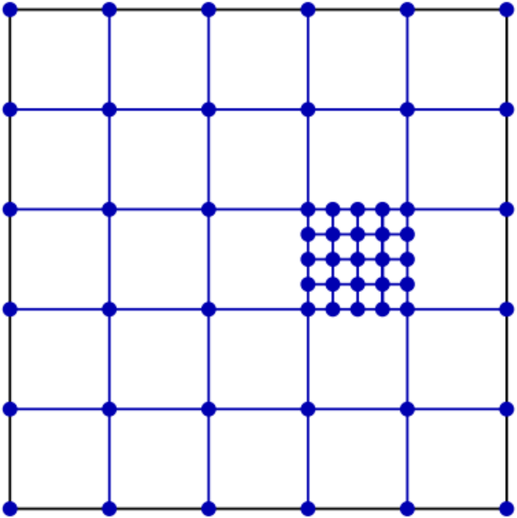
The **Virtual Element Method** (**VEM**) is a generalization of the Finite Element Method on polyhedral or polygonal meshes
[Beirão da Veiga, Brezzi, Cangiani, Manzini, Marini, Russo, M3AS, 2013]



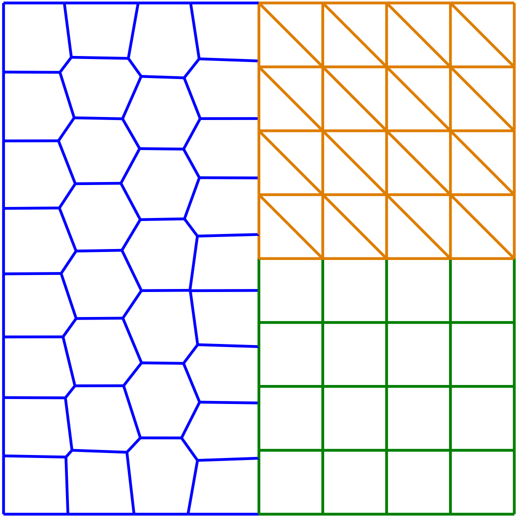
Why polygons? Fractures



Why polygons? Local Refinement



Why polygons? Mesh gluing



POlytopal Element Methods (**POEMs**)

- **Hybrid High Order methods**
  A. Di Pietro, A. Ern, et als;
- **Hybridizable Discontinuous Galerkin methods**
  B. Cockburn, J. Gopalakrishnan, et als;
- **Mimetic Finite Difference Methods**
  L. Beirão da Veiga, F. Brezzi, K. Lipnikov, G. Manzini, M. Shashkov, et als;
- **Polygonal/Polyhedral Finite Element methods**
  J. Bishop, G. Paulino, N. Sukumar, et als;
- **Polygonal/Polyhedral Discontinuous Galerkin methods**
  P. Antonietti, A. Cangiani, E. Georgoulis, P. Houston, et als;
- **Virtual Element Methods**
  L. Beirão da Veiga, F. Brezzi, L.D. Marini, A. Russo, et als;
- **Weak Galerkin Methods**
  J. Wang, X. Ye, et als.

Then mean features of VEMs

- VEMs allow to use very general polygonal and polyhedral meshes, also for high polynomial degrees,
- the VEMs spaces are similar to the usual polynomial spaces with the addition of suitable (and unknown!) **non-polynomial functions**, these functions inside each element are solutions of suitable PDEs,
- VEMs do not require the evaluation of test and trial functions at the integration points,
- the key of the VEMs is to define suitable projections onto the space of polynomials that are computable from the degrees of freedom,
- they satisfy the patch test exactly,
- **the flexibility of VEM is not limited to the mesh: C^k element, div-free method!**

The Stokes equation – primal formulation

We consider the **Stokes Problem** on a polygon $\Omega \subseteq \mathbb{R}^2$:

$$\begin{cases} -\nu\, \boldsymbol{\Delta} \boldsymbol{u} - \nabla p = \boldsymbol{f} & \text{in } \Omega, & \textit{momentum equation} \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } \Omega, & \textit{mass equation (incompressibility constraint)} \\ \boldsymbol{u} = 0 & \text{on } \partial\Omega. & \textit{boundary condition} \end{cases}$$

- $\boldsymbol{u} = (u_1, u_2)^T$ is the **fluid velocity**,
- p is the **fluid pressure**,
- $\nu > 0$ is the **fluid viscosity**,
- $\boldsymbol{f} = (f_1, f_2)^T \in [L^2(\Omega)]^2$ is the **external load**.

The Stokes equation – variational formulation

We consider

- velocities space $[H_0^1(\Omega)]^2 := \left\{ \boldsymbol{v} \in [L^2(\Omega)]^2 \quad \text{s.t.} \quad \boldsymbol{\nabla} \boldsymbol{v} \in [L^2(\Omega)]^{2\times 2}, \quad \boldsymbol{v}_{|\partial\Omega} = \boldsymbol{0} \right\}$,
- pressures space $L_0^2(\Omega) := \left\{ q \in L^2(\Omega) \quad \text{s.t.} \quad \int_\Omega q \, \mathrm{d}\Omega = 0 \right\}$.

Then the **variational formulation** of the Stokes equation is:

$$
\begin{cases}
\textit{find } (\boldsymbol{u}, p) \in [H_0^1(\Omega)]^2 \times L_0^2(\Omega) \textit{ such that} \\
\nu \int_\Omega \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} \, \mathrm{d}\Omega + \int_\Omega \mathrm{div}\, \boldsymbol{v} \, p \, \mathrm{d}\Omega = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\Omega & \textit{for all } \boldsymbol{v} \in [H_0^1(\Omega)]^2, \\
\int_\Omega \mathrm{div}\, \boldsymbol{u} \, q \, \mathrm{d}\Omega = 0 & \textit{for all } q \in L_0^2(\Omega),
\end{cases}
$$

where

$$
\boldsymbol{\nabla} u := \begin{pmatrix} u_{1,x} & u_{1,y} \\ u_{2,x} & u_{2,y} \end{pmatrix} \qquad \text{and} \qquad \boldsymbol{A} : \boldsymbol{B} := \sum_{i,j=1}^{n} a_{ij} \, b_{ij} \quad \text{for all } \boldsymbol{A}, \boldsymbol{B} \in \mathbb{R}^{n\times n}.
$$

Inf-sup stable FEMs & VEM

element name	velocity	pressure	div-free	balance approx.
P2-P0	$[\mathbb{P}_2]^2$	$\mathbb{P}_0$	NO	NO
Taylor-Hood	$[\mathbb{P}_2]^2$	$\mathbb{P}_1^{\rm cont}$	NO	✓
Mini	$[\mathbb{P}_1 + \mathbb{B}_3]^2$	$\mathbb{P}_1^{\rm cont}$	NO	NO
Crouzeix-Raviart	$[\mathbb{P}_2 + \mathbb{B}_3]^2$	$\mathbb{P}_1$	NO	✓
Scott-Vogelius$^{(*)}$	$[\mathbb{P}_k]^2$	$\mathbb{P}_{k-1}$	✓	✓
VEM	V_h	Q_h	✓	✓

$(*)$ for $k \geq 4$ and meshes without singular-vertex.

Virtual Elements for the Stokes Problem

We build a **Virtual Elements Method** for the Stokes Problem in following form

$$\begin{cases} \textit{find } (\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h \textit{ such that} \\ \nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + \int_\Omega \operatorname{div} \boldsymbol{v}_h \, p_h \, \mathrm{d}\Omega = \int_\Omega \boldsymbol{f}_h \cdot \boldsymbol{v}_h \, \mathrm{d}\Omega & \textit{for all } \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ \int_\Omega \operatorname{div} \boldsymbol{u}_h \, q_h \, \mathrm{d}\Omega = 0 & \textit{for all } q_h \in Q_h, \end{cases}$$

- $\boldsymbol{V}_h \subseteq [H^1_0(\Omega)]^2$ is a finite dimensional space,
- $Q_h \subseteq L^2_0(\Omega)$ is a finite dimensional space,
- $a_h(\cdot, \cdot) \colon \boldsymbol{V}_h \times \boldsymbol{V}_h \to \mathbb{R}$ is a bilinear form s.t.

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \approx \int_\Omega \nabla \boldsymbol{u}_h : \nabla \boldsymbol{v}_h \, \mathrm{d}\Omega \qquad \text{for all } \boldsymbol{u}_h, \, \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

- $\boldsymbol{f}_h$ is a right hand side term approximating the load term.

The pressure space and the velocities space

Let Ω_h be a **polygonal decomposition** of Ω.

For sake of simplicity we consider the consider the VEM scheme of order 2.

The **pressure space** is given by the **piecewise polynomial functions**

$$Q_h := \{q \in L^2_0(\Omega) \quad \text{s.t.} \quad q_{|_E} \in \mathbb{P}_1(E) \quad \text{for all } E \in \Omega_h\} \,.$$

The **velocity virtual space**

$$\boldsymbol{V}_h := \{\boldsymbol{v} \in [H^1_0(\Omega)]^2 \quad \text{s.t.} \quad \boldsymbol{v}_{|_E} \in \boldsymbol{V}_h(E) \quad \text{for all } E \in \Omega_h\} \,.$$

is defined, as for standard FEM, **element-wise**, by introducing

- **local spaces** $\boldsymbol{V}_h(E)$;
- **the associated local degrees of freedom**.

For $k = 1$: [Antonietti, Beirão da Veiga, Mora, Verani, SINUM, 2014].

Virtual Elements for the velocities: definition & properties

On each element $E \in \Omega_h$ we define the **local virtual velocities space**

$$\boldsymbol{V}_h(E) := \Big\{ \boldsymbol{v} \in [C^0(\overline{E})]^2 \;\; s.t. \;\; (i) \; \boldsymbol{\Delta v} + \nabla s = 0, \quad (ii) \; \mathrm{div}\, \boldsymbol{v} \in \mathbb{P}_1(E), \quad \text{for some } s \in L^2_0(E) \quad (iii) \; \boldsymbol{v}_{|e} \in [\mathbb{P}_2(e)]^2 \quad \forall e \in \partial E, \Big\}$$

- the definition of $\boldsymbol{V}_h(E)$ is associated with a Stokes-like problem on E,
- the divergence of functions in $\boldsymbol{V}_h(E)$ are polynomials of degree 1,
- polynomial inclusion: $[\mathbb{P}_2(E)]^2 \subseteq \boldsymbol{V}_h(E)$,
- the the dimension of $\boldsymbol{V}_h(E)$ is

$$\dim\big(\boldsymbol{V}_h(E)\big) = 4\, N_{\mathrm{edge}} + \dim(\mathbb{P}_1(E)) - 1\,.$$

Degrees of freedom for the velocities

$$\dim\big(\boldsymbol{V}_h(E)\big) = 4\, N_{\mathrm{edge}} + (\dim(\mathbb{P}_1(E)) - 1)$$

- $\mathbf{D_V1}$: the values at the vertices of the polygon E
- $\mathbf{D_V2}$: the values at the midpoint of every edge $e \in \partial E$

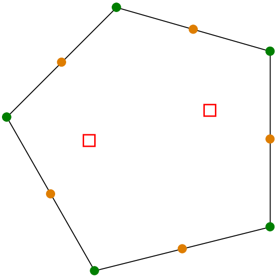


Figure: DoFs: $\mathbf{D_V1}$ green dots, $\mathbf{D_V2}$ orange dots.

Degrees of freedom for the velocities

$$\dim\big(\boldsymbol{V}_h(E)\big) = 4\,N_{\mathrm{edge}} + (\dim(\mathbb{P}_1(E)) - 1)$$

- $\mathbf{D_V3}$: the moments of $\mathrm{div}\,\boldsymbol{v}$ in E

$$\int_E (\mathrm{div}\,\boldsymbol{v})\, x \,\mathrm{d}E \qquad \int_E (\mathrm{div}\,\boldsymbol{v})\, y \,\mathrm{d}E$$

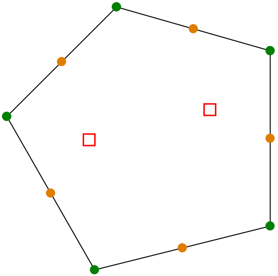


Figure: DoFs: $\mathbf{D_V3}$ red squares.

Bilinear form $a_h(\cdot,\, \cdot)$

The **approximated local form** $a_h^E(\cdot,\, \cdot)$ *mimics*

$$a_h^E(\boldsymbol{u}_h,\, \boldsymbol{v}_h) \approx \int_E \boldsymbol{\nabla} \boldsymbol{u}_h : \boldsymbol{\nabla} \boldsymbol{v}_h \,\mathrm{d}E$$

Drawback: the virtual functions are unknown inside the element!

For an arbitrary pair $\boldsymbol{u}_h,\, \boldsymbol{v}_h \in \boldsymbol{V}_h(E)$, the integral $\int_E \boldsymbol{\nabla} \boldsymbol{u}_h : \boldsymbol{\nabla} \boldsymbol{v}_h \,\mathrm{d}E$ is not computable.

- We do not attempt to approximate the virtual functions and we do require the evaluation of test and trial functions at the integration points.
- The key is to define on $\boldsymbol{V}_h(E)$ suitable projections onto the space of polynomials that are **computable from the DoFs**.

The DoFs $\boldsymbol{D_V}$ *allow us to* **compute** *the following operators*

$$\Pi_2^{0,E} \colon \boldsymbol{V}_h(E) \to [\mathbb{P}_2(E)]^2\,, \qquad \Pi_1^{0,E} \colon \boldsymbol{\nabla} \boldsymbol{V}_h(E) \to [\mathbb{P}_1(E)]^{2\times 2}$$

Divergence free velocity solution

Let us briefly recall that by definition

$$\boldsymbol{V}_h := \left\{ \boldsymbol{v} \in [H_0^1(\Omega)]^2 \quad \text{s.t.} \quad \dots \quad (\operatorname{div} \boldsymbol{v})_{|_E} \in \mathbb{P}_1(E) \quad \text{for all } E \in \Omega_h \right\}.$$

The **pressure space** is given by the **piecewise polynomial functions**

$$Q_h := \{ q \in L_0^2(\Omega) \quad \text{s.t.} \quad q_{|_E} \in \mathbb{P}_1(E) \quad \text{for all } E \in \Omega_h \}.$$

Therefore by construction

$$\operatorname{div} \boldsymbol{V}_h \subseteq Q_h \,.$$

The incompressibility constraint for the velocity solution $\boldsymbol{u}_h \in \boldsymbol{V}_h$ reads as

$$\int_\Omega \operatorname{div} \boldsymbol{u}_h \, q_h \, \mathrm{d}\Omega = 0 \qquad \text{for all } q_h \in Q_h,$$

therefore the discrete velocity $\boldsymbol{u}_h \in \boldsymbol{V}_h$ is **exactly divergence-free**.

The divergence-free property is not shared by the most popular mixed FEMs!

Kernel inclusion & Advantages

More generally, the kernels:

$$\boldsymbol{Z} = \left\{ \boldsymbol{v} \in [H^1_0(\Omega)]^2 \quad \text{s.t.} \quad \mathrm{div}\,\boldsymbol{v} = 0 \right\}$$

$$\boldsymbol{Z}_h = \left\{ \boldsymbol{v}_h \in \boldsymbol{V}_h \text{ s.t. } \int_\Omega \mathrm{div}\,\boldsymbol{v}_h\, q_h \,\mathrm{d}\Omega = 0 \quad \text{for all } q_h \in Q_h \right\} = \{ \boldsymbol{v}_h \in \boldsymbol{V}_h \text{ s.t. } \mathrm{div}\,\boldsymbol{v} = 0 \}$$

satisfy the inclusion

$$\boldsymbol{Z}_h \subseteq \boldsymbol{Z}\,.$$

Consequence of the kernel inclusion:

- **decoupling of the error**
  [Beirão da Veiga, Lovadina, V., SINUM, 2019]
- **reduced virtual element space**
  [Beirão da Veiga, Lovadina, V., M2AN, 2017]
- **coupling Stokes and Darcy flow**
  [V., M3AS, 2018]
- **underlying Stokes complex & stream formulation**
  [Beirão da Veiga, Mora, V., JSC, 2019], [Beirão da Veiga, Dassi, V., M3AS, 2020]

Convergence results

Theorem (Beirão da Veiga, Lovadina, V.)

Let Ω_h be a **polygonal decomposition** *of Ω s.t.*

- *each element $E \in \Omega_h$ is* **star-shaped with respect to a ball of uniform radius**,
- *for each element $E \in \Omega_h$, the* **length of all edges is comparable with its diameter**.

Then following error estimates hold

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{[H^1(\Omega)]^2} \lesssim h^k \, |\boldsymbol{u}|_{H^{k+1}(\Omega_h)} + \frac{h^{k+2}}{\nu} \, |\boldsymbol{f}|_{H^{k+1}(\Omega_h)}$$

$$\|p - p_h\|_{L^2(\Omega)} \lesssim h^k \, |p|_{H^k(\Omega_h)} + h^k \, |\boldsymbol{u}|_{H^{k+1}(\Omega_h)} + \frac{h^{k+2}}{\nu} \, |\boldsymbol{f}|_{H^{k+1}(\Omega_h)} \, .$$

Remark: The velocity component of the error depends on the pressure with a higher order term, via the load $\boldsymbol{f}$. Notice that for popular mixed FEMs it holds:

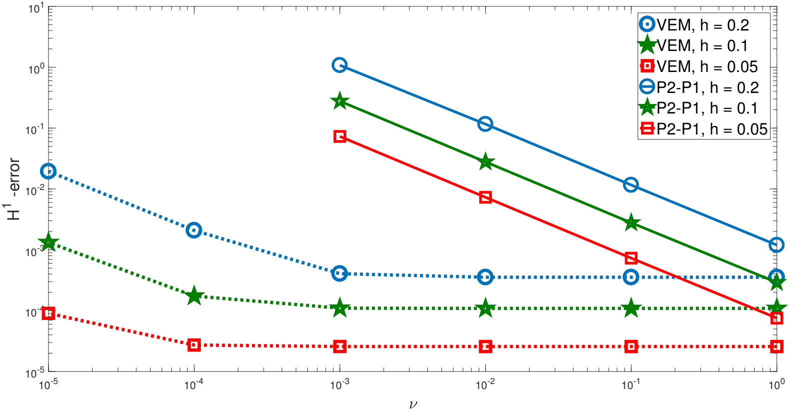
$$\|\boldsymbol{u} - \boldsymbol{u}_h^{\mathrm{FEM}}\|_{\boldsymbol{V}} \lesssim \frac{h^k}{\nu} \, |p|_{H^k(\Omega_h)} + h^k \, |\boldsymbol{u}|_{H^{k+1}(\Omega_h)} \, .$$

Navier–Stokes equation: small viscosity

Velocity error

$$\textbf{VEM:} \quad \|\boldsymbol{u} - \boldsymbol{u}_h^{\mathrm{VEM}}\|_{\boldsymbol{V}} \lesssim \frac{h^{k+2}}{\nu} \, |\boldsymbol{f}|_{k+1} + h^k \, |\boldsymbol{u}|_{H^{k+1}}$$

$$\textbf{FEM:} \quad \|\boldsymbol{u} - \boldsymbol{u}_h^{\mathrm{FEM}}\|_{\boldsymbol{V}} \lesssim \frac{h^{k}}{\nu} \, |p|_{k} + h^k \, |\boldsymbol{u}|_{H^{k+1}}$$



Reduced spaces

The div. free property of $\boldsymbol{u}_h$ implies that all the divergence moments $\boldsymbol{D_V 3}$ of $\boldsymbol{u}_h$ vanish

$$\int_E (\mathrm{div}\,\boldsymbol{u}_h)\, x \,\mathrm{d}E = 0 \qquad \int_E (\mathrm{div}\,\boldsymbol{u}_h)\, y \,\mathrm{d}E = 0$$

therefore many velocity and pressure DoFs can be eliminated from the system.

On each element $E \in \Omega_h$ we define the **reduced local virtual velocities space**

$$\widehat{\boldsymbol{V}}_h(E) := \Big\{ \boldsymbol{v} \in [C^0(\overline{E})]^2 \;\; s.t. \;\; (i) \;\; \boldsymbol{\Delta v} + \nabla s = \boldsymbol{0}, \quad (ii) \;\; \mathrm{div}\,\boldsymbol{v} \in \mathbb{P}_0(E)\,, \qquad \text{for some } s \in L^2_0(E) \quad (iii) \;\; \boldsymbol{v}_{|e} \in [\mathbb{P}_2(e)]^2 \quad \forall e \in \partial E, \Big\}$$

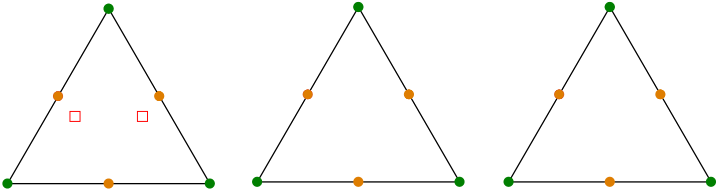
Remark: the reduced formulation allows us to solve the Stokes Problem saving $4n_P$ DoFs where n_P is the number of polygons in the mesh.

VEM & FEM (triangular elements): P2–P0

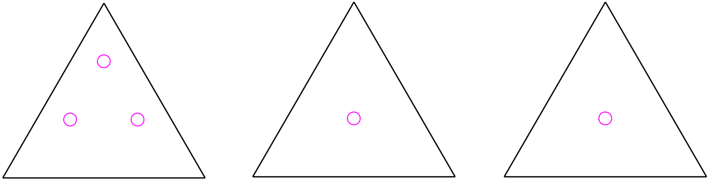
Remark: The proposed VE is different from already-known FE

VEM–2	$\widehat{\mathrm{VEM}}$–2	$\left([\mathbb{P}_2(E)]^2, \mathbb{P}_0(E)\right)$

velocities



pressures

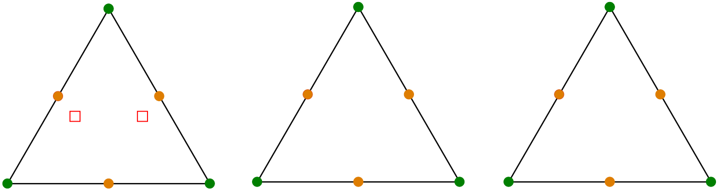


VEM & FEM (triangular elements): Taylor-Hood

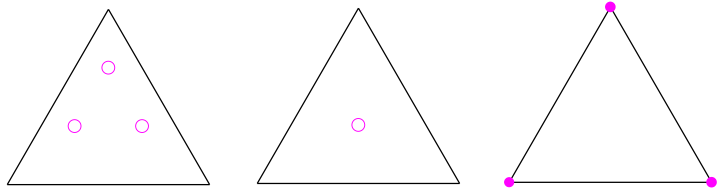
Remark: The proposed VE is different from already-known FE

VEM–2	$\widehat{\text{VEM}}$–2	$([\mathbb{P}_2(E)]^2, \mathbb{P}_1^{\text{cont}}(E))$

velocities



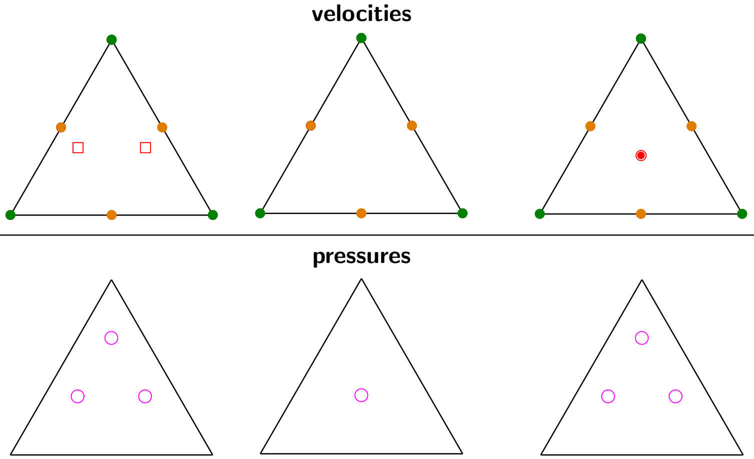
pressures



VEM & FEM (triangular elements): Crouzeix-Raviart

Remark: The proposed VE is different from already-known FE

VEM–2	$\widehat{\text{VEM}}$–2	$\left([\mathbb{P}_2(E) \oplus \mathbb{B}_3(E)]^2, \, \mathbb{P}_1(E)\right)$



Divergence free property: coupling Stokes and Darcy fluids

Consider the **Darcy Equation** (Poisson Equation in mixed form) on a polygon $\Omega \subseteq \mathbb{R}^2$:

$$\begin{cases} \text{find } (\boldsymbol{u}, p) \in H_0(\mathrm{div}, \Omega) \times L^2_0(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \mathbb{K}\boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\Omega + \int_\Omega \mathrm{div}\, \boldsymbol{v} \, p \, \mathrm{d}\Omega = 0 \qquad \text{for all } \boldsymbol{v} \in H_0(\mathrm{div}, \Omega), \\ \displaystyle\int_\Omega \mathrm{div}\, \boldsymbol{u} \, q \, \mathrm{d}\Omega = \int_\Omega g q \, \mathrm{d}\Omega \qquad \text{for all } q \in L^2_0(\Omega). \end{cases}$$

The proposed family of VEM is stable for both Stokes and Darcy problem!

Remark: Most popular FEMs are not robust for both Stokes and Darcy.

Brinkman equation

$$\begin{cases} \text{find } (\boldsymbol{u}, p) \in [H^1_0(\Omega)]^2 \times L^2_0(\Omega) \text{ such that} \\ \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, d\Omega + \int_\Omega \mathbb{K} \boldsymbol{u} \cdot \boldsymbol{v} \, d\Omega + \int_\Omega \operatorname{div} \boldsymbol{v} \, p \, d\Omega = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, d\Omega \qquad \forall \boldsymbol{v} \in [H^1_0(\Omega)]^2, \\ \int_\Omega \operatorname{div} \boldsymbol{u} \, q \, d\Omega = \int_\Omega g q \, d\Omega \qquad \forall q \in L^2_0(\Omega). \end{cases}$$

In the limit Darcy case i.e. "small" ν we recover the optimal order of accuracy.

	h	error$(\boldsymbol{u}, H^1)$	error$(\boldsymbol{u}, L^2)$	error(p, L^2)
ν = 1e-1	1/4	2.049871e-01	9.414645e-03	1.531569e-02
	1/8	4.616835e-02	8.379142e-04	2.796060e-03
	1/16	1.102679e-02	9.416547e-05	5.322283e-04
	1/32	2.654465e-03	1.104272e-05	1.261317e-04
ν = 1e-14	1/4	2.572957e-01	1.301886e-02	6.431351e-03
	1/8	5.539413e-02	1.111681e-03	1.887150e-03
	1/16	1.299961e-02	1.253090e-04	4.203480e-04
	1/32	3.003059e-03	1.394861e-05	1.026912e-04

Stokes complex

Let $\Omega \subseteq \mathbb{R}^2$ a **simply connected domain**, consider the **Stokes complex** [Mardal, Tai, Winther, SINUM, 2002] and [Falk, Neilan, SINUM, 2013]

$$0 \xrightarrow{\;\; i \;\;} H^2_0(\Omega) \xrightarrow{\;\; \mathbf{curl} \;\;} [H^1_0(\Omega)]^2 \xrightarrow{\;\; \mathrm{div} \;\;} L^2_0(\Omega) \xrightarrow{\;\; 0 \;\;} 0$$

The proposed element enjoys an **underlying discrete Stokes complex structure**

$$0 \xrightarrow{\;\; i \;\;} \Phi_h \xrightarrow{\;\; \mathbf{curl} \;\;} \boldsymbol{V}_h \xrightarrow{\;\; \mathrm{div} \;\;} Q_h \xrightarrow{\;\; 0 \;\;} 0\,,$$

that is

$$\mathbf{curl}\, \Phi_h = \boldsymbol{Z}_h$$

where Φ_h is a suitable **H^2-conforming VEM**.

Curl formulation

VEM mixed formulation

$$\begin{cases} \text{find } (\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h \text{ such that} \\ \nu a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + \int_\Omega \operatorname{div} \boldsymbol{v}_h \, p_h \, \mathrm{d}\Omega = \int_\Omega \boldsymbol{f}_h \cdot \boldsymbol{v}_h \, \mathrm{d}\Omega & \text{for all } \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ \int_\Omega \operatorname{div} \boldsymbol{u}_h \, q_h \, \mathrm{d}\Omega = 0 & \text{for all } q_h \in Q_h, \end{cases}$$

VEM curl formulation: $\mathbf{curl}\, \Phi_h = \boldsymbol{Z}_h$

$$\begin{cases} \text{find } \psi_h \in \Phi_h, \text{ such that} \\ \nu\, a_h(\mathbf{curl}\, \psi_h, \mathbf{curl}\, \varphi_h) = (\boldsymbol{f}_h, \mathbf{curl}\, \varphi_h) \quad \text{for all } \varphi_h \in \Phi_h. \end{cases}$$

- $2(n_P - 1)$ less DoFs with respect to the reduced problem;
- the **pressure** can be computed by least square;
- **definite positive linear system**;
- **higher condition number** (fourth order system).

The Oseen equation

We consider the **Oseen equation** on a polygon $\Omega \subseteq \mathbb{R}^2$:

$$\begin{cases} -\nu\, \boldsymbol{\Delta} \boldsymbol{u} + (\boldsymbol{\nabla} \boldsymbol{u})\boldsymbol{\beta} + \sigma \boldsymbol{u} - \nabla p = \boldsymbol{f} & \text{in } \Omega, \\ \mathrm{div}\,\boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = \boldsymbol{0} & \text{on } \partial\Omega \end{cases}$$

- $\nu > 0$ is the **fluid viscosity**, $\sigma > 0$ is the **reaction coefficient**,
- $\boldsymbol{\beta} \in [W_1^{\infty}(\Omega)]^2$ with $\mathrm{div}\boldsymbol{\beta} = 0$ is the **transport advective field**,
- $\boldsymbol{f} \in [L^2(\Omega)]^2$ is the **external load**.

Model problem: discretization of a time-dependent Navier–Stokes equation.

- Discretizing the Oseen equation leads to **instabilities when the convective term is dominant** with respect to the diffusive term, i.e.

$$\nu \ll \|\boldsymbol{\beta}\|_{[L^\infty(\Omega)]^2} .$$

- The majority of the **stabilizations may disrupt the divergence-free property** and related advantages.

Stabilization of the vorticity equation

We follow the approach in [Ahmed, Barrenechea, Burman, Guzman, Linke, Merdon, 2020].

Assume that $\mathrm{curl}\boldsymbol{f} \in L^2(E)$ for all $E \in \Omega_h$. We consider the **vorticity equation**

$$\mathrm{curl}\,(-\nu\,\boldsymbol{\Delta u} + (\boldsymbol{\nabla u})\boldsymbol{\beta} + \sigma\boldsymbol{u}) = \mathrm{curl}\boldsymbol{f} \quad \text{for all } E \in \Omega_h$$

Remark: in the vorticity equation the gradient of the pressure disappears!

We define the **stabilizing forms** and the **stabilizing right hand side**

$$\mathcal{L}^E(\boldsymbol{u},\boldsymbol{v}) := \tau_E \int_E \mathrm{curl}\,(-\nu\,\boldsymbol{\Delta u} + (\boldsymbol{\nabla u})\boldsymbol{\beta} + \sigma\boldsymbol{u})\,\mathrm{curl}((\boldsymbol{\nabla v})\boldsymbol{\beta})\,\mathrm{d}E$$

$$\mathcal{F}^E(\boldsymbol{v}) := \tau_E \int_E \mathrm{curl}\boldsymbol{f}\,\mathrm{curl}((\boldsymbol{\nabla v})\boldsymbol{\beta})\,\mathrm{d}E$$

where τ_E is the **stabilization parameter**.
The global forms are

$$\mathcal{L}(\boldsymbol{u},\boldsymbol{v}) := \sum_{E\in\Omega_h} \mathcal{L}^E(\boldsymbol{u},\boldsymbol{v})\,, \qquad \mathcal{F}^E(\boldsymbol{v}) := \mathcal{F}^E(\boldsymbol{v})\,.$$

The Oseen equation: kernel formulation

We consider the **stabilized Oseen equation**:

$$\begin{cases} \textit{find } (\boldsymbol{u}, p) \in [H^1_0(\Omega)]^2 \times L^2_0(\Omega) \textit{ such that} \\ \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, \mathrm{d}\Omega + \int_\Omega [(\nabla \boldsymbol{u})\boldsymbol{\beta}] \cdot \boldsymbol{v} \, \mathrm{d}\Omega + \sigma \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\Omega + \int_\Omega \mathrm{div}\, \boldsymbol{v} \, p \, \mathrm{d}\Omega + \mathcal{L}(\boldsymbol{u}, \boldsymbol{v}) \\ \qquad = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\Omega + \mathcal{F}(\boldsymbol{v}) \qquad \textit{for all } \boldsymbol{v} \in [H^1_0(\Omega)]^2, \\ \int_\Omega \mathrm{div}\, \boldsymbol{u} \, q \, \mathrm{d}\Omega = 0 \qquad \textit{for all } q \in L^2_0(\Omega). \end{cases}$$

The equation can be also written in the **kernel formulation**, i.e.

$$\begin{cases} \boldsymbol{u} \in \boldsymbol{Z} \\ \nu \int_\Omega \nabla \boldsymbol{u} : \nabla \boldsymbol{v} \, \mathrm{d}\Omega + \int_\Omega [(\nabla \boldsymbol{u})\boldsymbol{\beta}] \cdot \boldsymbol{v} \, \mathrm{d}\Omega + \sigma \int_\Omega \boldsymbol{u} \cdot \boldsymbol{v} \, \mathrm{d}\Omega + \mathcal{L}(\boldsymbol{u}, \boldsymbol{v}) \\ \qquad = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}\Omega + \mathcal{F}(\boldsymbol{v}) \qquad \textit{for all } \boldsymbol{v} \in \boldsymbol{Z}. \end{cases}$$

Stabilized Virtual Elements for the Oseen equation

Let $(\boldsymbol{V}_h, Q_h)$ be the **divergence-free** VEM couple.
We build a **stabilized Virtual Elements Method** for the Oseen in the following form

$$\begin{cases} \textit{find } (\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h \textit{ such that} \\ a_h(\nu; \boldsymbol{u}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{\beta}; \boldsymbol{u}_h, \boldsymbol{v}_h) + m_h(\sigma; \boldsymbol{u}_h, \boldsymbol{v}_h) + \int_\Omega \operatorname{div} \boldsymbol{v}_h \, p_h \, \mathrm{d}\Omega + \mathcal{L}_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \\ \qquad = \int_\Omega \boldsymbol{f}_h \cdot \boldsymbol{v}_h \, \mathrm{d}\Omega + \mathcal{F}(\boldsymbol{v}_h) \qquad \textit{for all } \boldsymbol{v}_h \in \boldsymbol{V}_h, \\ \int_\Omega \operatorname{div} \boldsymbol{u}_h \, q_h \, \mathrm{d}\Omega = 0 \qquad \textit{for all } q_h \in Q_h. \end{cases}$$

Recalling that $\boldsymbol{Z}_h \subset \boldsymbol{Z}$, we have the corresponding **kernel formulation**

$$\begin{cases} \textit{find } \boldsymbol{u}_h \in \boldsymbol{Z}_h \times Q_h \textit{ such that} \\ a_h(\nu; \boldsymbol{u}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{\beta}; \boldsymbol{u}_h, \boldsymbol{v}_h) + m_h(\sigma; \boldsymbol{u}_h, \boldsymbol{v}_h) + \mathcal{L}_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \\ \qquad = \int_\Omega \boldsymbol{f}_h \cdot \boldsymbol{v}_h \, \mathrm{d}\Omega + \mathcal{F}(\boldsymbol{v}_h) \qquad \textit{for all } \boldsymbol{v}_h \in \boldsymbol{Z}_h. \end{cases}$$

Stabilizing VEM forms

$$\mathcal{L}_h^E(\boldsymbol{u}_h, \boldsymbol{v}_h) \simeq \tau_E \int_E \operatorname{curl}\left(-\nu\, \boldsymbol{\Delta u} + (\boldsymbol{\nabla u})\boldsymbol{\beta} + \sigma \boldsymbol{u}\right) \operatorname{curl}((\boldsymbol{\nabla v})\boldsymbol{\beta})\, \mathrm{d}E$$

$$\mathcal{L}_h^E(\boldsymbol{u}_h, \boldsymbol{v}_h) := \mathcal{L}_{h,\mathrm{res}}^E(\boldsymbol{u}_h, \boldsymbol{v}_h) + \mathcal{L}_{h,\mathrm{jump}}^E(\boldsymbol{u}_h, \boldsymbol{v}_h) + \mathcal{L}_{h,\mathrm{stab}}^E(\boldsymbol{u}_h, \boldsymbol{v}_h)$$

- **stabilizing residual bilinear form**

$$\mathcal{L}_{h,\mathrm{res}}^E(\boldsymbol{u}_h, \boldsymbol{v}_h) :=$$
$$\tau_E \int_E \operatorname{curl}\left(-\nu \operatorname{div}(\Pi_{k-1}^{0,E} \boldsymbol{\nabla u}_h) + [\Pi_{k-1}^{0,E} \boldsymbol{\nabla u}_h]\boldsymbol{\beta} + \sigma \Pi_k^{0,E} \boldsymbol{u}_h\right) \operatorname{curl}([\Pi_{k-1}^{0,E} \nabla \boldsymbol{v}_h]\boldsymbol{\beta})\, \mathrm{d}E\,,$$

- **gradient jumps penalizing term**

$$\mathcal{L}_{h,\mathrm{jump}}^E(\boldsymbol{u}_h, \boldsymbol{v}_h) := \frac{1}{2} h_E^2 \int_{\partial E} [\![(\Pi_{k-1}^{0,E} \boldsymbol{\nabla u}_h)\boldsymbol{\beta}]\!] \cdot [\![(\Pi_{k-1}^{0,E} \boldsymbol{\nabla v}_h)\boldsymbol{\beta}]\!]\, \mathrm{d}e\,,$$

- **VEM stabilizing term**

$$\mathcal{L}_{h,\mathrm{stab}}^E(\boldsymbol{u}_h, \boldsymbol{v}_h) := \frac{\tau_E\, \beta_E^2}{h_E^2} S^E\left((I - \Pi_k^{0,E})\boldsymbol{u}_h, (I - \Pi_k^{0,E})\boldsymbol{v}_h\right), \qquad \beta_E := \|\boldsymbol{\beta}\|_{[L^\infty(E)]^2}\,.$$

Stability analysis

We define the norm

$$\|\boldsymbol{v}\|^2_{\mathrm{stab},E} := \nu \|\boldsymbol{\nabla} \boldsymbol{v}\|^2_{0,E} + \sigma \|\boldsymbol{v}\|^2_{0,E} +$$

$$\tau_E \|\mathrm{curl}([\Pi^{0,E}_{k-1} \boldsymbol{\nabla} \boldsymbol{v}]\boldsymbol{\beta})\|^2_{0,E} + \frac{h_E^2}{2} \|[\![(\Pi^{0,E}_{k-1} \boldsymbol{\nabla} \boldsymbol{u}_h)\boldsymbol{\beta}]\!]\|^2_{0,\partial E} + \frac{\tau_E \beta_E^2}{h_E^2} \|\boldsymbol{\nabla}(I - \Pi^{0,E}_k)\boldsymbol{v}\|^2_{0,E}$$

with global counterpart

$$\|\boldsymbol{v}\|^2_{\mathrm{stab}} := \sum_{E \in \Omega_h} \|\boldsymbol{v}\|^2_{\mathrm{stab},E} \,.$$

Proposition (Coercivity)

Let Ω_h be a **shape regular polygonal decomposition** *of Ω.*
If the parameter τ_E satisfies for any $E \in \Omega_h$

$$\tau_E \lesssim \min \left\{ \frac{h_E^4}{\nu}, \frac{h_E^2}{\sigma} \right\}$$

the following coercivity inequality holds

$$\|\boldsymbol{v}_h\|^2_{\mathrm{stab}} \lesssim a_h(\nu; \boldsymbol{v}_h, \boldsymbol{v}_h) + c_h(\boldsymbol{\beta}; \boldsymbol{v}_h, \boldsymbol{v}_h) + m_h(\sigma; \boldsymbol{v}_h, \boldsymbol{v}_h) + \mathcal{L}_h(\boldsymbol{v}_h, \boldsymbol{v}_h) \,.$$

Convergence analysis

Let Ω_h be a shape regular polygonal decomposition of Ω
Assume that for some $\varepsilon > 0$

$$\boldsymbol{u} \in [H^{3/2+\varepsilon}(\Omega)]^2 \cap [H^{k+1}(\Omega_h)]^2 , \quad \boldsymbol{f} \in [H^{k+1}(\Omega_h)]^2 , \quad \boldsymbol{\beta} \in [W^{k+1}_\infty(\Omega_h)]^2 .$$

- `Convection dominated regime` $\nu \ll h_E \beta_E$, $\sigma \ll \frac{\beta_E}{h_E}$: $\tau_E \simeq \dfrac{h_E^3}{\beta_E}$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{\mathrm{stab}} \lesssim \sum_{E \in \Omega_h} \left(\beta_E h_E^{2k+1} \left(1 + \beta_E + \frac{\beta_E h_E^3}{\max\{\nu, \sigma h_E^2\}} \right) \right) \|\boldsymbol{u}\|^2_{k+1,E} + \sum_{E \in \Omega_h} \frac{h_E^{2k+3}}{\beta_E} |\boldsymbol{f}|^2_{k+1,E} .$$

- `Diffusion dominated regime` $\beta_E h_E \lesssim \nu$, $\sigma h_E^2 \ll \nu$: $\tau_E \simeq \dfrac{h_E^4}{\nu}$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{\mathrm{stab}} \lesssim \sum_{E \in \Omega_h} \left(\nu h_E^{2k} (1 + \beta_E) \right) \|\boldsymbol{u}\|^2_{k+1,E} + \sum_{E \in \Omega_h} \frac{h_E^{2k+4}}{\nu} |\boldsymbol{f}|^2_{k+1,E} .$$

- `Reaction dominated regime` $\frac{\beta_E}{h_E} \lesssim \sigma$, $\frac{\nu}{h_E^2} \ll \sigma$: $\tau_E \simeq \dfrac{h_E^2}{\sigma}$

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|^2_{\mathrm{stab}} \lesssim \sum_{E \in \Omega_h} h_E^{2k+2} \sigma \|\boldsymbol{u}\|^2_{k+1,E} + \sum_{E \in \Omega_h} \frac{h_E^{2k+2}}{\sigma} |\boldsymbol{f}|^2_{k+1,E} .$$

Comments & Remarks

- `Diffusion dominated case` & `Reaction dominated case`:
  - the scheme recovers the **optimal orders of approximation**.

- `Convection dominated case`:
  - **"optimal" order** $h^{k+1/2}$ is recovered;
  - the degenerative term $\beta_E / \max\{\nu, \sigma h_E^2\}$ is weighted by a factor h_E^3 and can be improved using a slightly different discrete convective form;
  - the following L^2 error estimate holds:

$$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{L^2(\Omega)} \lesssim h^{k+1/2}$$

    this convergence result is not recovered by the most popular FEM.

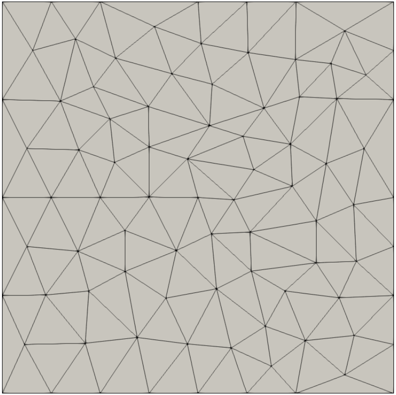
- Analogous results can be obtained for the **pressure** component of the error.

- The proposed **stabilized VEM** is **"quasi pressure-robust"**: the velocity error depends on the pressure with a higher order term via the load $\boldsymbol{f}$.

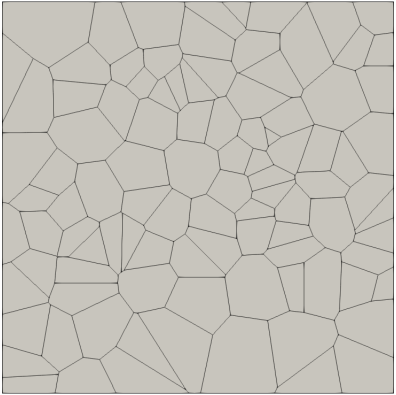
- The **VEM Stokes complex structure** is a fundamental tool for the proofs.

Numerical Test: adopted meshes

Consider the sequences of meshes



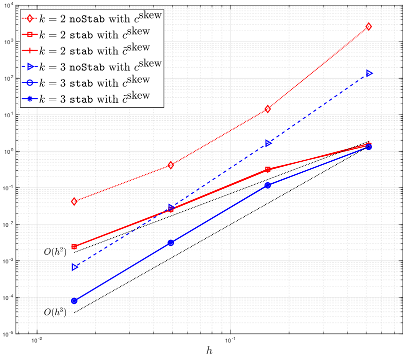
tria



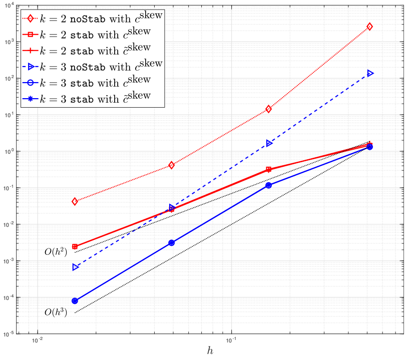
random

Numerical Test: H^1 velocity error

- **convection dominated regime**: $\nu =$ 1e-06, $\quad \boldsymbol{\beta} = (1,1)^{\mathrm{T}}, \quad \sigma = 0.$
- Loading and BCs in accordance with known exact solution.



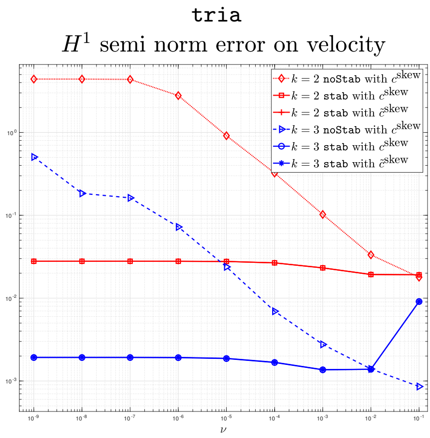
tria

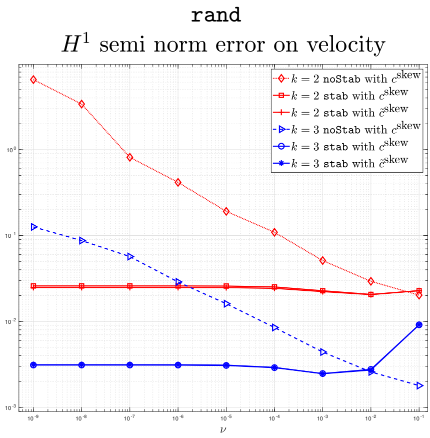


random

Numerical Test: H^1 velocity error

- **convection dominated regime**: ν varying from 1e-01 to 1e-09, $\boldsymbol{\beta} = (1,1)^{\mathrm{T}}$, $\sigma = 0$.
- Loading and BCs in accordance with known exact solution.



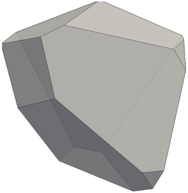


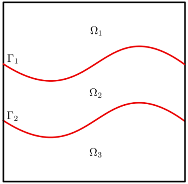
3D & Curved polygons

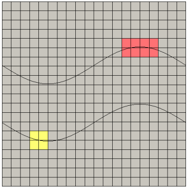
3D case [Beirão da Veiga, Dassi, V., M3AS, 2019]

Curved polygons [Beirão da Veiga, Russo, V., M2AN, 2019]

- **discrete kernel inclusion**
- **divergence free** velocity solution
- **error decoupling**
- **reduced problem**
- **Darcy limit stability**
- **underlying Stokes complex**







Conclusions

The proposed family of **Virtual Elements** has four advantages:

- it can be applied to **general polyhedral meshes**,
- it yields an **exactly divergence-free kernel**,
- the velocity error depends on the pressure with a **higher order term**,
- easier **coupling Stokes and Darcy equation**,
- enjoys a discrete **Stokes complex structure**,
- **stabilized Oseen equation**,
  - we recover the **optimal order of convergence** in all the regimes,
  - differently from other choices the stabilization proposed does not spoil such property.

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