The question of uniqueness of steady states for Reaction–Diffusion Equations in General Domains

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Mostly Maximum Principle
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Joint work with Cole GRAHAM (Brown Univ.)

Stationary states of Reaction-diffusion Equations

$\partial_t u - \Delta u = f(u)$, in general domains $\Omega \subset \mathbb{R}^N$
Stationary states $\to$ semi-linear elliptic equations :

$$\begin{cases} -\Delta u = f(u), \quad u > 0 \quad \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

[1]HB & C. Graham, Annales IHP Analyse non lin., 2022

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Various types of reaction terms f, domains Ω, and boundary conditions on boundary $\partial\Omega$:

- Dirichlet: $u = 0$,
- Neumann: $\partial_\nu u = 0$,
- Robin: $\rho\partial_\nu u + (1-\rho)u = 0$, $(\rho \in [0,1])$[1].

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Classical types of reaction terms

Look for $0 \leq u \leq 1$; f is $C^1[0,1]$, $f(0) = f(1) = 0$.

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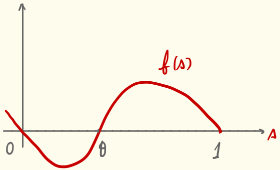
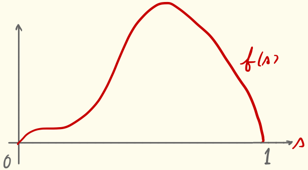
- Bistable f
- Positive f
  - General
  - Weak KPP
  - Strong KPP

Classical types of reaction terms: bistable, positive





Bistable f

$f < 0$ on $(0, \theta)$

$f > 0$ on $(\theta, 1)$

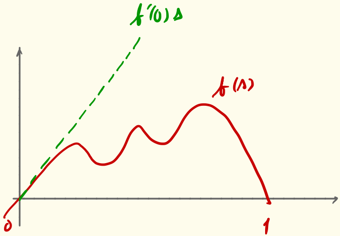
$f'(0) < 0$, $f'(1) < 0$

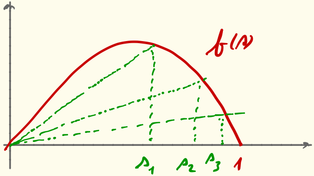
Positive f

$f > 0$ on $(0, 1)$

$f'(0) > 0$, $f'(1) < 0$.

KPP reaction terms





Weak KPP:

$f(s) \le f'(0)s \quad \forall s \in (0,1)$

Strong KPP

$\frac{f(s)}{s} \searrow$ in $s \in (0,1)$.

A classical result: strong KPP in bounded domains

$$\begin{cases} -\Delta u = f(u), \quad u > 0 \quad \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Assume that f is strongly KPP:

$$s \in (0,1) \mapsto \frac{f(s)}{s} \quad \text{is decreasing on} \quad (0,1).$$

Theorem

Under this assumption, suppose that Ω is a **bounded** *smooth domain. Then, when it exists, the positive solution u is unique.*[a]

[a] HB, *Le nombre de solutions de certains problèmes semi-linéaires elliptiques*, J. Funct. Anal. **40**, 1981.

Remarks

- Nec. and suff. cond. for existence: $\lambda_1^D(-\Delta, \Omega) < f'(0)$.
- The result is true for a general elliptic operator:

$$\begin{cases} -a_{i,j}(x)\partial_{i,j}u + b_i(x)\partial_i u \ = \ f(x,u), \quad u > 0 \quad \text{in} \quad \Omega \\ u \ = \ 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

- Extensions to unbounded domains not straightforward. Think e.g. of $-u'' - cu' = f(u)$ on $\mathbb{R}$ with f KPP. It has multiple positive solutions when $c \geq 2\sqrt{f'(0)}$.[2]
- In all of space for $-\Delta u = f(u)$:

Theorem

Assume f is positive with $f'(0) > 0$. In $\mathbb{R}^N$, there is only one bounded positive solution of the equation $-\Delta u = f(u)$, namely $u \equiv 1$.

[2] HB, F. Hamel & L. Rossi, Ann. Mat. Pura Appl. (2007).

I. Strong KPP equations, unbounded domains (Dirichlet cond.)

What about general unbounded Ω?

$$\begin{cases} -\Delta u &= f(u), \quad u > 0 \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

Consider the case f is of strong KPP type.
We require an additional non-degeneracy condition.

Generalized principal Dirichlet eigenvalue

[HB, L.Nirenberg and S. Varadhan, CPAM 1994] and unbounded domains [HB, L. Rossi, CPAM 2015]

Definition

$$\lambda_1(-L,\Omega) := \sup\left\{\lambda \mid \exists\phi > 0 \textit{ in } W^{2,N}_{loc}(\Omega)\cap C(\bar{\Omega}) \textit{ such that } (L+\lambda)\phi \le 0\right\}.$$

$\lambda_1(-L,\Omega)$ is the limit of the principal **Dirichlet eigenvalues** of $-L$ on $\Omega \cap B_R$ as $R \to \infty$ [S. Agmon, 1983].

Proposition

Suppose f is of weak KPP type. Then the equation admits a positive bounded solution if $\lambda_1(-\Delta - f'(0),\Omega) < 0$. Conversely, it has no positive bounded solution if $\lambda_1(-\Delta - f'(0),\Omega) > 0$.

- Critical case $\lambda_1 = 0$ varies; conjecture: no positive bounded solutions if f is strong KPP.
- Proposition analogous to results[3] for variable-coefficient operators in $\mathbb{R}^n$.

[3] [HB, F. Hamel and L. Rossi, Ann. Mat. Pura Appl. 2007].

Strong KPP equations, Dirichlet cond. unbounded domains

- The behavior of Ω at infinity plays a major role
- Limits of translations of Ω may be disconnected; we call their connected components *connected limits*
- Consider the "principal limiting spectrum"

$$\Sigma^*(f'(0),\Omega) := \{\lambda_1(f'(0),\Omega^*) \mid \Omega^* \text{ a connected limit of } \Omega\}.$$

Theorem

Suppose that f is of strong KPP type and that Ω is strongly noncritical in the sense that 0 is not in the closure of $\Sigma^*(f'(0),\Omega)$. Then, when it exists, the positive bounded solution of the equation is unique.

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Conjecture

In a general domain Ω, or, at least, in a uniformly smooth domain Ω, if f is strong KPP, then the equation admits at most one positive bounded solution.

Sketch of proof in noncritical domains

If $0 \notin \bar{\Sigma}^*(f'(0), \Omega)$, then limits of Ω have a "spectral gap" around 0.
We use this gap to write Ω as the union of two smooth open sets Ω_+ and Ω_- that satisfy

$$\lambda_1(-\Delta - f'(0), \Omega_+) > 0 \quad \text{and} \quad \Sigma^*(f'(0), \Omega_-) \subset (-\infty, 0).$$

This decomposition is delicate; we rely on a beautiful result of Lieb[a]:

$$\inf_{x \in \mathbb{R}^N} \lambda_1(-\Delta, A \cap (B + x)) \leq \lambda_1(-\Delta, A) + \lambda_1(-\Delta, B).$$

[a] Inventiones, 1983

Solutions cannot vanish at infinity in Ω_-, and the uniqueness proof from bounded domains goes through.
The equation satisfies the maximum principle on Ω_+, so we can "transfer uniqueness" from the overlap $\Omega_- \cap \Omega_+$ to all of Ω_+, and thus all of Ω.

Narrow and ample domains

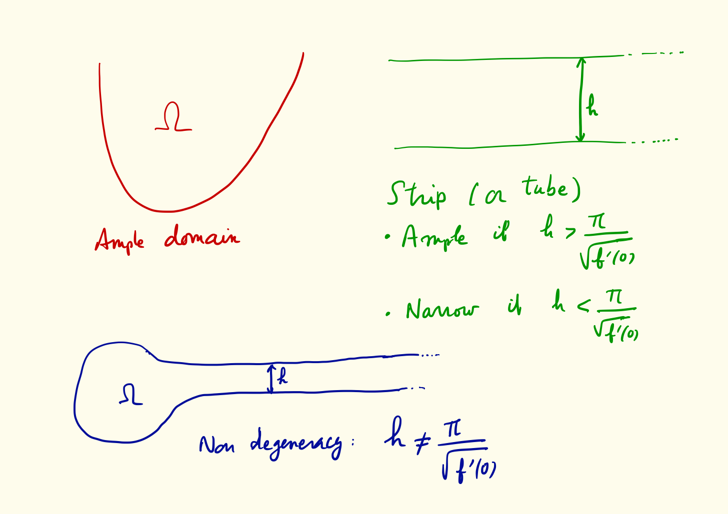
We term the components Ω_+ and Ω_- "narrow" and "ample."

Definition

Given $\mu > 0$, we say that the domain is μ-narrow if $\lambda_1(-\Delta - \mu, \Omega) > 0$. We say it is μ-ample if $\lambda_1(-\Delta - \mu, \Omega^*) < 0$, for every connected limit Ω^* of Ω.

- Balance between μ-growth in interior and absorption at the boundary
- Narrow: every point of Ω relatively close to $\partial\Omega$
- Ample: Ω sufficiently capacious even at infinity
- $\lambda_1(-\Delta - \mu, \Omega) > 0$ (narrow) iff $\lambda_1(-\Delta - \mu, \Omega^*) > 0$ for every connected limit Ω^* of Ω.

Examples of domains



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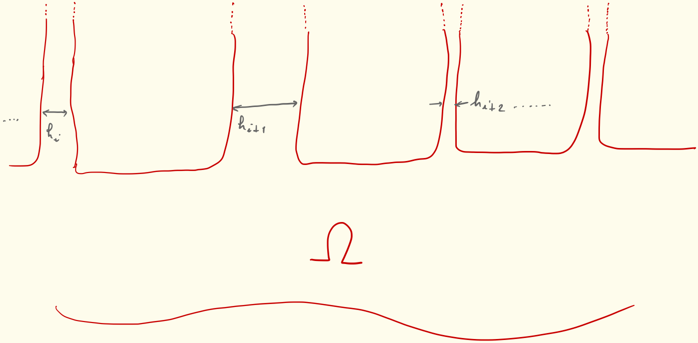


Non degeneracy: $\inf_i \left| h_i - \frac{\pi}{\sqrt{f'(0)}} \right| > 0$

Comb shape

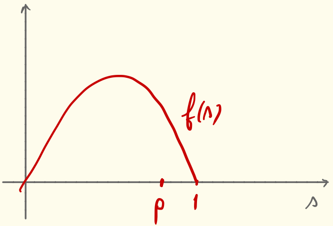
II. Positive reaction terms

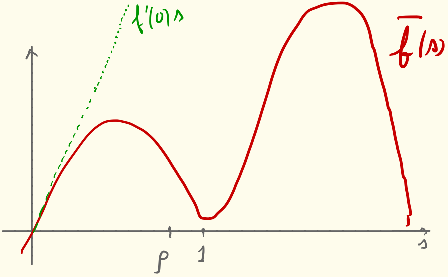
Subtle interaction between positive reactions and Dirichlet conditions.
Example of *non-uniqueness*:

Proposition

Consider Dirichlet conditions. On every bounded domain Ω, there exists a positive reaction f such that the equation with Dirichlet conditions has multiple stable positive bounded solutions. In fact, f can be chosen to satisfy the weak KPP condition : $f(u) \leq f'(0)u$.

Sketch of proof



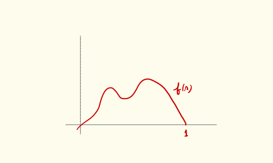


$\rho > \max_{\bar{\Omega}} u_1$ $\qquad$ $0 < u_1 < 1$ $\qquad$ $\bar{f}(s) = f(s)$, for $0 < s < \rho$

Minimize $\{E(w);\ w \in H^1_0(\Omega)\} \longrightarrow u_2 \implies$ 2 solutions

$E(w) = \int \frac{1}{2} |\nabla w|^2 - \bar{F}(w)$ $\qquad$ $0 < u_1 < u_2$.

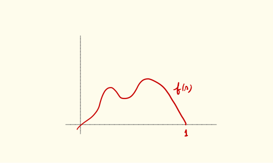
Positive reactions, Dirichlet, Lipschitz epigraphs



Theorem (HB, L. Caffarelli & L. Nirenberg, CPAM 1997)

Consider Dirichlet b.c. and f of positive *type. If* $\Omega = \{x_N > \Phi(x_1, \dots, x_{N-1})\}$ *for some globally Lipschitz function* $\Phi \colon \mathbb{R}^{d-1} \to \mathbb{R}$, *then the equation has a unique positive bounded solution. Moreover, this solution is strictly increasing in* x_N *and* $u(x) \to 1$ *as* $\mathrm{dist}(x, \partial\Omega) \to \infty$.

Positive reactions, Dirichlet, half-space



Theorem (HB, L. Caffarelli & L. Nirenberg, CPAM 1997)

Consider Dirichlet b.c. and f of positive *type. If $\Omega = \{x_N > 0\}$ is a half-space, then the solution of the equation has a unique positive bounded solution. Moreover, this solution has one-dimensional symmetry (i.e. $u = u(x_N)$) and is strictly increasing in x_N.*

III. General equations in a half space

Theorem (HB, L. Caffarelli & L. Nirenberg, Ann. Sc. Norm. Sup. Pisa, 1997)

Consider a half-plane in $\mathbb{R}^2$ or a half-space in $\mathbb{R}^3$, i.e. $\Omega = \mathbb{R}^d \times \mathbb{R}_+$ for $d = 1$ or 2, and Dirichlet b.c. Then all bounded positive solutions are one-dimensional: $u = u(x_N)$.

Corollary

For Dirichlet conditions, $\Omega = \mathbb{R}^d \times \mathbb{R}_+$ for $d = 1$ or 2, and f is bistable, ignition, or positive, then the equation has a unique bounded positive solution.

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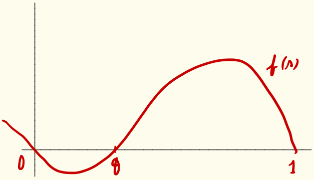
Open problem

What about higher dimensions?

It is related to the De Giorgi conjecture.

IV. Bistable equations



We assume the non-linearity is *unbalanced*, i.e.

$$\int_0^1 f(s)ds > 0,$$

(unlike the Allen-Cahn equation).

Bistable equations – non-uniqueness bounded domains

$$\begin{cases} -\Delta u = \lambda f(u), & u > 0 \quad \text{in} \quad \Omega \\ u = 0 \quad \text{on} \quad \partial\Omega. \end{cases}$$

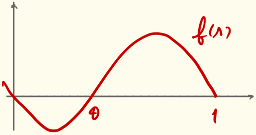
Theorem (P. Rabinowitz)

Assume that f is bistable unbalanced. Let Ω be a bounded smooth domain. There exists λ^* such that

1. *the equ. does not have positive solutions for $\lambda < \lambda^*$,*
2. *the equ. has (at least) one positive solutions for $\lambda = \lambda^*$,*
3. *the equ. has (at least) two distinct positive solutions for $\lambda > \lambda^*$,*

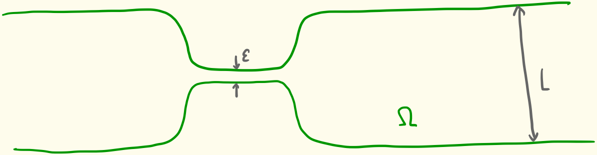
(Weakly) stable maximum solution when $\lambda \geq \lambda^*$; second solution is unstable.
Topological degree argument.

Bistable reaction terms, blocking



Bistable f

$$(*) \quad \begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$



Bistable reaction terms, blocking

Theorem

When ε is sufficiently small, there are two stable solution u_i with $u_1(x) \to 0$ as $x_1 \to +\infty$ and $u_1(x) \to w_0(y) > 0$ as $x_1 \to -\infty$, and the reverse for u_2.

Related to [HB, J. Bouhours, G. Chapuisat, Calc. Var. PDE, 2016] and [HB, F. Hamel, H. Matano, CPAM, 2009] that considered Neumann cond.

Coercive Lipschitz epigraphs

- Epigraph:

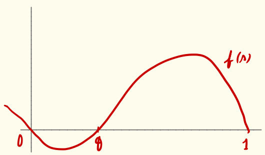
$$\Omega = \{x \in \mathbb{R}^N; x_N > \phi(x_1, \dots, x_{N-1}\}$$

- Lipschitz: ϕ globally Lipschitz
- coercive;

$$\lim_{x' \in \mathbb{R}^{N-1}, |x'| \to \infty} \phi(x') = +\infty$$

- strongly coercive: Can be written as a coercive epigraph for directions in an open cone around e_N.

Bistable reactions, Dirichlet cond., coercive Lipschitz epigraphs



Theorem

Consider Dirichlet b.c. and f of bistable unbalanced *type. In such a strongly coercive epigraph, the equation has a unique positive bounded solution. Moreover, this solution is strictly increasing in x_N and $u(x) \to 1$ as $\mathrm{dist}(x, \partial\Omega) \to \infty$.*

Bistable reaction terms, cones

Consider a cone K included in a circular cone of opening angle less than $\pi/2$ around some axis.

Theorem

For the bistable case, the equation with Dirichlet condition in K has a unique bounded positive solution.

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Theorem

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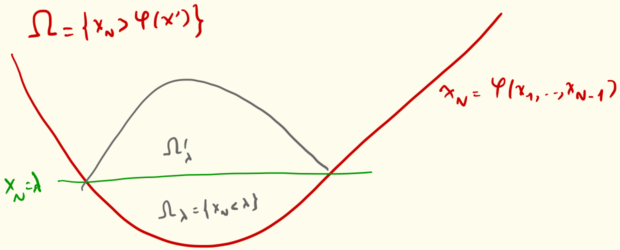
Corollary

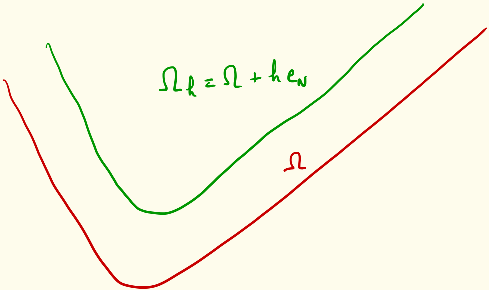
The scalar field equation $-\Delta u = u^p - u$ *does not have bounded positive solutions with Dirichlet cond. in such a cone or in a Lipschitz epigraph contained in such a graph.*

Bistable reaction terms, epigraph, Moving planes

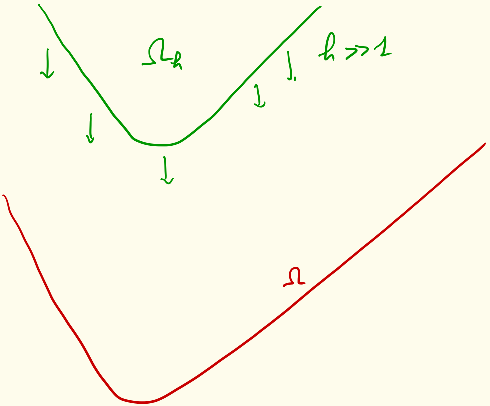


Moving plane method
(Alexandrov, Serrin ...)

Bistable reaction terms, epigraphs, sliding 1



Bistable reaction terms, sliding 2



Bistable reaction terms, sliding 3