Regularity of stable solutions to semilinear elliptic equations up to dimension 9: quantitative proofs

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Mostly Maximum Principle
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Regularity of stable solutions to semilinear elliptic equations up to dimension 9: quantitative proofs

- [Cabré, Figalli, Ros-Oton, Serra. Acta Math. 224 (2020)]
- [Cabré, A quantitative proof of the Hölder regularity... arXiv 2022]

• Semilinear elliptic PDEs : $-\Delta u = f(u)$ in $\Omega \subset \mathbb{R}^n$, bdd domain

1st variation

Energy : $E_\Omega(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 - F(u)$, $F' = f$

2nd variation is $-\Delta - f'(u)$ = linearized operator at u for the equation $-\Delta u = f(u)$

it is nonnegative iff $-\Delta - f'(u) \geq 0$

iff $\int_\Omega f'(u)\xi^2 \leq \int_\Omega |\nabla \xi|^2 \quad \forall \xi \in C_c^\infty(\Omega)$ ← Def. of stability

→ Competitors $u + \varepsilon\xi$ have all same boundary values as u

→ Our interest : nonlinearities f superlinear at $+\infty$ & $f \geq 0$

$\Downarrow$

NO absolute minimizer exists

$$E_\Omega(t\varphi) = t^2 \int_\Omega \frac{1}{2}|\nabla \varphi|^2 - \int_\Omega F(t\varphi) \xrightarrow[t \to +\infty]{} -\infty \quad (F(t\varphi) \gg t^2\varphi^2)$$

- The Barenblatt-Gelfand problem 1963 :

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \subset \mathbb{R}^n \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $f(0)>0$, nondecreasing, convex, & superlinear at $+\infty$.

- Then, $\exists \lambda^* \in (0,+\infty)$ & $0<\lambda<\lambda^* \Rightarrow \exists u_\lambda > 0$ stable classical (L^∞) sol'n

  - $u_\lambda \nearrow u^*$ as $\lambda \nearrow \lambda^*$

    ↳ $u^* \in L^1(\Omega)$ is a distributional stable solution for $\lambda = \lambda^*$

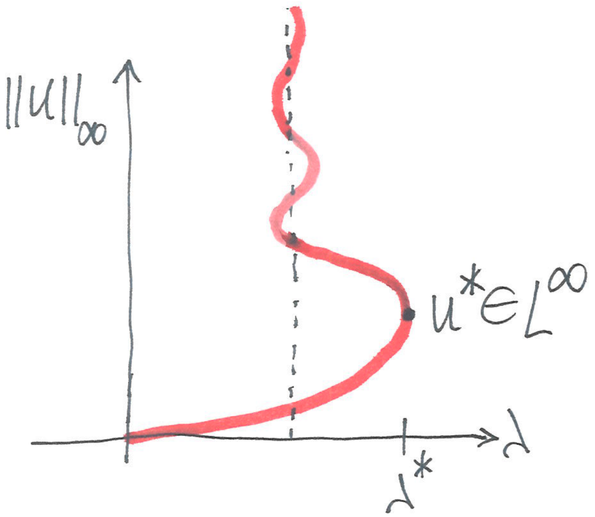
    u^* = the extremal solution of the pb.

- $\nexists$ solutions for $\lambda > \lambda^*$

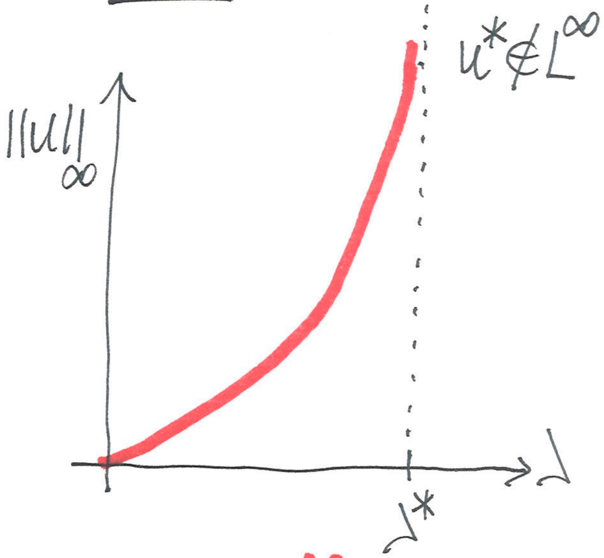
Model nonlinearities: $f(u) = e^u$ (combustion theory)

$f(u) = (1+u)^p$, $p>1$

• [Joseph-Lundgren '72] $f(u)=e^u$ & $\Omega=B_1$ (RADIAL case) :



$3\leq n\leq 9$



$n\geq 10$

▪ ODE techniques

▪ Explicit singular solution :

$u(x)=-2\log|x| \in W_0^{1,2}(B_1)$

solves $\quad -\Delta u = 2(n-2)\, e^u \quad$ in B_1, $n\geq 3$

Linearized operator $= -\Delta - 2(n-2)\dfrac{1}{|x|^2}$

(Hardy's ineq) $\to$ u stable $\Leftrightarrow$ $2(n-2)\leq \dfrac{(n-2)^2}{4} \Leftrightarrow n\geq 10$

■ [Cabré, Figalli, Ros-Oton, Serra '19]

Thm 1 $u \in C^2(B_1)$ stable sol'n of $-\Delta u = f(u)$ in B_1 & $f \geq 0 \implies$

$$\|\nabla u\|_{L^{2+\gamma}(B_{1/2})} \leq C(n) \|u\|_{L^1(B_1)} \qquad (\gamma = \gamma(n) > 0)$$

& if $n \leq 9$ then $\|u\|_{C^\alpha(\overline{B}_{1/2})} \leq C(n) \|u\|_{L^1(B_1)}$ $(\alpha = \alpha(n) > 0)$.

Corol 1 $L^\infty(\Omega)$ estimate for $n \leq 9$ (if $f \geq 0$) and any stable sol'n of $\begin{cases} -\Delta u = f(u) & \text{in } \Omega \subset \mathbb{R}^n \\ u = 0 & \text{on } \partial\Omega \end{cases}$ if Ω is bdd convex C^1 domain.

Thm 2 Ω bdd C^3 domain, $f \geq 0$, $f' \geq 0$, $f'' \geq 0$.
$u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ stable sol'n of $\begin{cases} -\Delta u = f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$ $\implies$

$$\|\nabla u\|_{L^{2+\gamma}(\Omega)} \leq C(\Omega) \|u\|_{L^1(\Omega)} \qquad (\gamma = \gamma(n) > 0)$$

& if $n \leq 9$ then $\|u\|_{C^\alpha(\overline{\Omega})} \leq C(\Omega) \|u\|_{L^1(\Omega)}$ $(\alpha = \alpha(n) > 0)$.

- <u>PROOFS</u>

$$\Delta u + f(u) = 0 \qquad \text{(EQUATION)}$$

$$\downarrow$$

$$\Delta + f'(u) \qquad \text{(LINEARIZED OPERATOR} \leq 0\text{)}$$

$$\downarrow$$

$$\int_\Omega f'(u)\,\xi^2 \leq \int_\Omega |\nabla \xi|^2 \quad \forall \xi \in C^1_c(\Omega) \qquad \text{(STABILITY)}$$

$$\downarrow \; \xi = c \cdot \eta \quad \text{with } \eta_{|\partial\Omega} \equiv 0.$$

$$\int_\Omega c\,\underline{(\Delta c + f'(u)c)}\,\eta^2 \leq \int_\Omega c^2 |\nabla \eta|^2.$$

Test function $\xi = c\eta$, $\boxed{\begin{array}{l} c = x\cdot\nabla u = r u_r \\ \eta = r^{\frac{2-n}{2}}\cdot\varphi \end{array}}$ $(r = |x|)$

Prop'n 1 [CFRS '19] u stable sol'n in B_1 & $3 \le n \le 9 \Rightarrow$

$$\int_{B_{1/4}} r^{2-n} u_r^2 \, dx \le C \int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2 dx .$$

Test function $\xi = c\eta$, $c = x\cdot\nabla u = r u_r$ $(r = |x|)$

$\eta = r^{\frac{2-n}{2}}\cdot\varphi$

Prop'n 1 [CFRS '19] u stable sol'n in B_1 & $3 \le n \le 9 \Rightarrow$

$$\int_{B_{1/4}} r^{2-n} u_r^2\, dx \le C_n \int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2 dx .$$

If we had here $\int_{B_{1/2}\setminus B_{1/4}}$ r^{2-n} $u_r^2\, dx$, then

rescale $\longrightarrow$

$$\int_{B_\rho} r^{2-n} u_r^2 \le C \int_{B_{2\rho}\setminus B_\rho} r^{2-n} u_r^2$$

$$\int_{B_\rho} r^{2-n} u_r^2 \le \frac{C}{1+C} \int_{B_{2\rho}} r^{2-n} u_r^2$$

$\rightarrow$ Algebraic decay for this adimensional quantity

$\rightarrow$ Hölder continuity

We would like

$$\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2 \le C(n) \int_{B_{1/2}\setminus B_{1/4}} u_r^2 . \qquad (*)$$

May it be true ?

If false, in the extreme case we would have

$$\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2 = 1 \quad \& \quad \int_{B_{1/2}\setminus B_{1/4}} u_r^2 = 0$$

$\rightarrow$ u is 0-homogeneous

$\Downarrow$ $-\Delta u = f(u) \geq 0$

u is a superharmonic f'cn on the sphere S^{n-1}

$u = ctt \Leftarrow$

CONTRADICTION

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CONTRADICTION

In [CFRS '19]

$\rightarrow$ We prove $(*)$ (under a doubling assumption that suffices) by COMPACTNESS using the higher integrability estimate

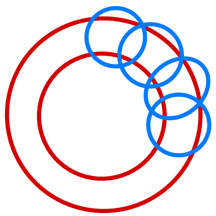
$c = |\nabla u| \Longrightarrow \|\nabla u\|_{L^{2+\gamma}} \leq C(n) \|\nabla u\|_{L^2}$

In [C'22]; Quantitative proof:

Prop'n 2 [CFRS'19] u stable sol'n in B_1 & $f \geq 0 \Rightarrow$

$$\|\nabla u\|_{L^2(B_{1/2})} \leq C_n \|u\|_{L^1(B_1)}$$

Proved using $c = |\nabla u|$.



$\hookrightarrow$ Replace $\int_{B_{1/2}\setminus B_{1/4}} |\nabla u|^2$ by $\int_{B_{1/2}\setminus B_{1/4}} |u-t|$.

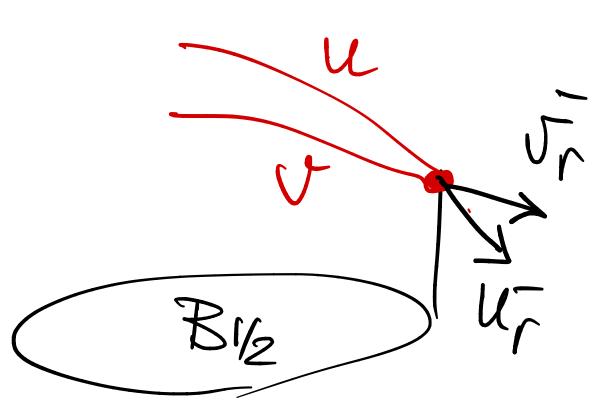
Prop'n 3 [C'22] u superharmonic in $B_1 \Rightarrow \exists\, t \in \mathbb{R}$ s.t.

$$\|u-t\|_{L^1(B_{1/2}\setminus B_{1/4})} \leq C_n \|u_r\|_{L^1(B_{1/2}\setminus B_{1/4})}.$$

Hölder continuity

Proof:

Proof: **Step 1** $\left\{\begin{array}{l} u \text{ superharm.} \\ v = \text{harmonic replacement} \\ \text{MAXIMUM PRINCIPLE} \end{array}\right.$



$$\Rightarrow \quad u_r^- \geq v_r^- \geq 0$$

But $$\int_{\partial B_{1/2}} v_r^- = \int_{\partial B_{1/2}} v_r^+ = \frac{1}{2}\int_{\partial B_{1/2}} |v_r| .$$

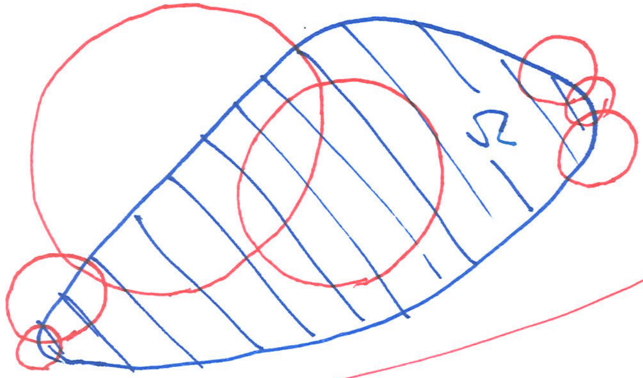
$$\int_{\partial B_{1/2}} |u_r| \geq \int_{\partial B_{1/2}} u_r^- \geq \int_{\partial B_{1/2}} v_r^-$$

$$\Delta v = 0 \Rightarrow \text{flux} = 0$$

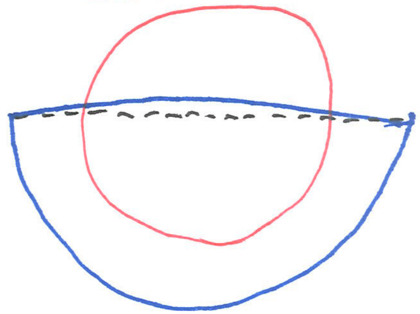
Step 2 v harmonic $\longrightarrow \left\{\begin{array}{ll} \Delta v = 0 & \text{in } B_{1/2} \\ \dfrac{\partial v}{\partial \nu} = v_r =: g & \text{on } \partial B_{1/2} \end{array}\right.$

Control of v-ctt in L^1 by g in L^1: OK ■

- Boundary regularity

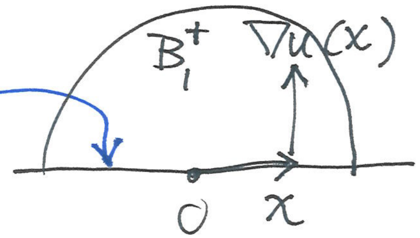


→ Almost half-balls



Simplest case:

Half-balls ; flat boundary :

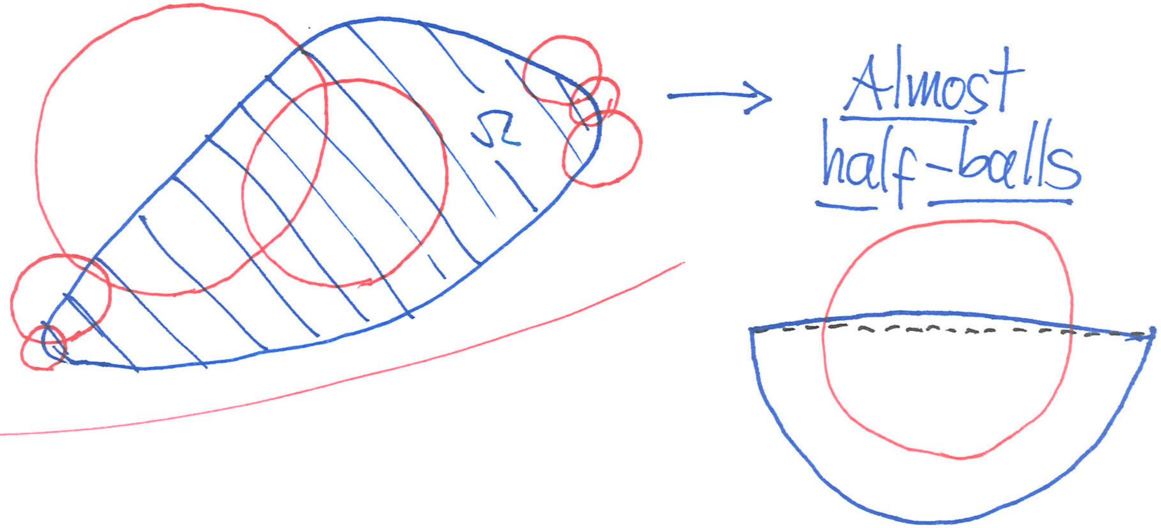


$$\xi(x) = (x\cdot\nabla u)\,|x|^{\frac{2-n}{2}}\varphi(x)$$ vanishes on the flat bdry ☺

$$\Downarrow$$

$$(n-2)(10-n)\int_{B_\rho^+} |x|^{2-n} u_r^2 \leq C \int_{B_{2\rho}^+ \setminus B_\rho^+} |x|^{2-n} |\nabla u|^2$$

• Boundary regularity



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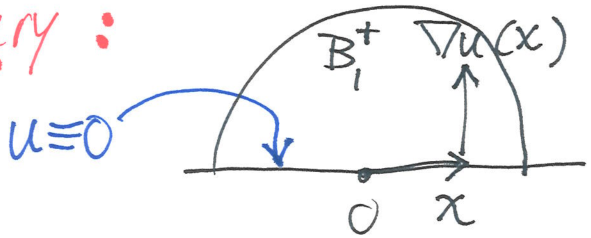


$$\downarrow$$

$\xi(x) = (x \cdot \nabla u)\,|x|^{\frac{2-n}{2}}\,\varphi(x)$ vanishes on the flat bdry ☺

$$\Downarrow$$

$$(n-2)(10-n)\int_{B_\rho^+} |x|^{2-n} u_r^2 \leq C \int_{B_{2\rho}^+ \setminus B_\rho^+} |x|^{2-n} |\nabla u|^2$$

We ask ¿ $\exists u$, $u=0$ on $\{x_n=0\}$, $\Delta u \leq 0$ in $\{x_n>0\}$,

$$\int_{B_2^+ \setminus B_1^+} |\nabla u|^2 = 1 \quad \& \quad \int_{B_2^+ \setminus B_1^+} u_r^2 = 0 \ ?$$

Yes ☹ $u(r,\theta) = \sin\theta$

Proof in [CFRS'19]

key remark: u cannot solve $-\Delta u = f(u)$ if $u = u(\theta)$

-2 homogeneous

0 homogeneous

Question: Can one pass to the limit the condition $-\Delta u = f(u)$?

Thm 4 [CFRS'19] Let u_k be stable sol'ns of $-\Delta u_k = f_k(u_k)$ in $U \subset \mathbb{R}^n$ open, with $f_k' \geq 0$, $f_k'' \geq 0$; $u_k \in W^{1,2}_{loc}(U)$

$\downarrow$ in $L^1_{loc}(U)$.

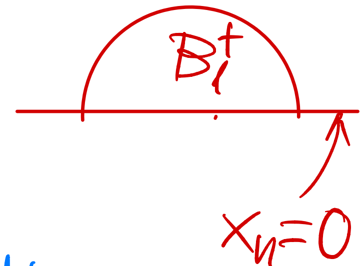
Then u

$u \in W^{1,2}_{loc}(U)$ is a stable solution of $-\Delta u = f(u)$ in U for some f nondecreasing and convex, $f: (-\infty, M) \to \mathbb{R}$.

Quantitative proof in [C'22]

Thm 5 [C'22] $u \geq 0$ stable sol'n of $-\Delta u = f(u)$ in B_1^+, $u = 0$ on $\{x_n = 0\} \cap \partial B_1^+$. Assume $f \geq 0$, $f' \geq 0$, $f'' \geq 0$.
Then,

$$\|u\|_{L^1(B_1^+ \setminus B_{1/2}^+)} \leq C_n \|u_r\|_{L^1(B_1^+ \setminus B_{1/2}^+)}.$$



- Not true for superharmonic f'cns.
- Very delicate proof, but easier to extend to other frameworks (nonflat bdry, variable coeff's, ...)
- Needs $f \geq 0, f' \geq 0, f'' \geq 0$ as [CFRS'19] proof (but uses very different arguments)

[CFRS'19] needed:

Compactness in flat bdry ⊕ Delicate blow-up + Liouville to reduce to flat bdry

• **Proof of Thm 5** uses a replacement for

$$-2\Delta u + \Delta(x\cdot\nabla u) = -f'(u)\,x\cdot\nabla u \quad :$$

$u^\lambda(x) := u(\lambda x)$

$$-2\lambda^{-3}\Delta u_\lambda + \lambda^{-2}(\lambda^{-1}x\cdot\nabla u_\lambda) = -\frac{d}{d\lambda}f(u_\lambda) = -f'(u_\lambda)\lambda^{-1}x\cdot\nabla u_\lambda$$

$$\boxed{\cdot\,|\,\Sigma \oplus \int_{B_1^+} \oplus \int_1^{1.1} \cdot\, d\lambda}$$

$\geq c\|u\|_{L^1} - C\|u_r\|_{L^1}$

$\leq C\varepsilon\|u\|_{L^1} + \varepsilon^{-\alpha}\|u_r\|_{L^1}$

delicate. Uses $f' \geq 0$, $f'' \geq 0$, & $W^{1,2+\gamma}$ estimate for u. ∎

Thanks for your attention