Mostly Maximum Principle

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Regularity results for a free boundary problem governed by a nonstandard growth operator

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Abstract.
We introduce some results concerning the regularity of flat or Lipschitz free boundaries of non-homogeneous equations governed by the $p(x)-$Laplace operator.

The results are contained in
Regularity of flat free boundaries for a $p(x)$-Laplacian problem with right hand side Nonlinear Anal. 212 (2021)
and in
Regularity of Lipschitz free boundaries for a $p(x)$-Laplacian problem with right hand side, preprint (2022)
both papers have been obtained in collaboration with:

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Plan of the talk

- The problem
- Some motivations (Thermistor problem)
- The $p(x)$-Laplace operator versus the p-Laplace operator
- An overview about two-phase problems (Prandl-Batchelor problem)

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The problem

The one-phase problem that we consider is the following one:

$$\begin{cases} \Delta_{p(x)} u = f, & \text{in } \Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \\ |\nabla u| = g, & \text{on } F(u) := \partial\Omega^+(u) \cap \Omega. \end{cases} \tag{1}$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain, $p \in C^1(\Omega), f \in C(\Omega) \cap L^\infty(\Omega)$ and $g \in C^{0,\beta}(\Omega)$, $g > 0$, $\beta \in (0,1]$.

First Result

Theorem (Flatness implies $C^{1,\alpha}$-F- Claudia Lederman)

Let u be a viscosity solution to (1) *in* B_1. *Assume that* $0 \in F(u)$, $g(0) = 1$ *and* $p(0) = p_0$. *There exists a universal constant* $\bar{\varepsilon} > 0$ *such that, if the graph of u is $\bar{\varepsilon}$−flat in B_1, in the direction e_n, that is*

$$(x_n - \bar{\varepsilon})^+ \leq u(x) \leq (x_n + \bar{\varepsilon})^+, \quad x \in B_1, \tag{2}$$

and

$$\|\nabla p\|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \quad \|f\|_{L^\infty(B_1)} \leq \bar{\varepsilon}, \quad [g]_{C^{0,\beta}(B_1)} \leq \bar{\varepsilon}, \tag{3}$$

then $F(u)$ is $C^{1,\alpha}$ in $B_{1/2}$.

Second Result

Theorem (Optimal regularity-F- Claudia Lederman)

Let u be a viscosity solution to (1) in B_1. There exists a constant $C > 0$ such that

$$\|\nabla u\|_{L^\infty(B_{1/2})} \leq C.$$

Third Result

Theorem (Lipschitz implies $C^{1,\alpha}$-F- Claudia Lederman)

Let u be a viscosity solution to (1) *in* B_1, *with* $0 \in F(u)$. *If* $F(u)$ *is a Lipschitz graph in a neighborhood of* 0, *then* $F(u)$ *is* $C^{1,\alpha}$ *in a (smaller) neighborhood of* 0.

Comment: we exploited some recent results by [Si] and [SS]

Some Motivations: the thermistor problem

Let $\Omega \subset \mathbb{R}^n$ be smooth. The system

$$\begin{cases} -\mathrm{div}(|\nabla u(x)|^{\sigma(\theta(x))-2}\nabla u(x)) = \tilde{f}, \quad u_{|\partial\Omega} = 0 \\ -\Delta\theta(x) = \lambda|\nabla u(x)|^{\sigma(\theta(x))}, \quad \theta_{|\partial\Omega} = 0 \end{cases} \tag{4}$$

gives a joint description, [V.V. Zhikov, Solvability of the three-dimensional thermistor problem. (Russian. Russian summary) Tr. Mat. Inst. Steklova 261 (2008), Differ. Uravn. i Din. Sist., 101–114; translation in Proc. Steklov Inst. Math. 261 (2008), no. 1, 98–111], of the electric field (with potential u) and the temperature θ in a thermistor.

... a thermistor is a type of resistor whose resistance is strongly dependent on temperature, more so than in standard resistors...

Here $\lambda > 0$ is a parameter and $\sigma : [0, +\infty) \to \mathbb{R}$, is a bounded function so that there exist a, b positive numbers $1 < a < b$ such that $a \leq \sigma(s) \leq b$ for every $s \in [0, \infty)$.

Assuming that the temperature θ is known and denoting $p(x) = \sigma(\theta(x))$, then we obtain the free boundary problem

$$\begin{cases} \operatorname{div}(|\nabla u(x)|^{p(x)-2}\nabla u(x)) = -\tilde{f}, \quad \Omega \\ u = 0, \quad \text{on } \partial\Omega, \\ |\nabla u(x)| = g(x) := \left(-\frac{\Delta\theta(x)}{\lambda}\right)^{\frac{1}{p(x)}}, \quad \text{on } \partial\Omega, \end{cases} \tag{5}$$

that is

$$\begin{cases} \Delta_{p(x)} u = f, \quad \text{in } \Omega^+(u) := \{x \in \Omega : u(x) > 0\}, \\ |\nabla u| = g, \quad \text{on } F(u) := \partial\Omega^+(u) \cap \Omega. \end{cases} \tag{6}$$

where $\Delta_{p(x)} u := \operatorname{div}(|\nabla u(x)|^{p(x)-2}\nabla u(x))$ and $f := -\tilde{f}$.

A variational approach

Problem (1) comes out also naturally from limits of a singular perturbation problem with forcing term as in [LW1], arising in the study of flame propagation with nonlocal and electromagnetic effects.
(1) appears by minimizing the following functional

$$\mathcal{E}(v) = \int_\Omega \left(\frac{|\nabla v|^{p(x)}}{p(x)} + \chi_{\{v>0\}} + f(x)v \right) dx \tag{7}$$

studied in [LW3], as well as in the seminal paper by Alt and Caffarelli [AC] in the case $p(x) \equiv 2$ and $f \equiv 0$.
In [LW4], (1) appears in the study of an optimal design problem.

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Remarks on the p constant case

The homogeneous two phase case ($f \equiv 0$) for the p-Laplace operator (p here is constant) has been studied by John Lewis and Kay Nystrom,[LN],[LN2].

The $p-$Laplace non-homogeneous one phase case (p here is constant) has been dealt with in a paper by Leitão R, Ricarte G. **[LR]** (2018) on IFB.

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There $\Omega \subset \mathbb{R}^n$ is a bounded domain, $p > 1$ is a constant. $f \in C(\Omega) \cap L^\infty(\Omega)$ and $g \in C^{0,\beta}(\Omega)$, $g \geq 0$. The authors proved that flat free boundaries are $C^{1,\alpha}$ and that Lipschitz free boundaries are $C^{1,\alpha}$.

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Basic Background

Let $p : \Omega \to [1, \infty)$ be a measurable bounded function, called a variable exponent on Ω.

$p_{\max} = \operatorname{esssup} p(x)$ and $p_{\min} = \operatorname{essinf} p(x)$.

The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the modular $\varrho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx$ is finite.

The Luxemburg norm on this space is defined by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \|u\|_{p(\cdot)} = \inf\{\lambda > 0 : \varrho_{p(\cdot)}(u/\lambda) \leq 1\}.$$

This norm makes $L^{p(\cdot)}(\Omega)$ a Banach space.

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There holds the following relation between $\varrho_{p(\cdot)}(u)$ and $\|u\|_{L^{p(\cdot)}}$:

$$\min\left\{\left(\int_\Omega |u|^{p(x)}\,dx\right)^{1/p_{\min}}, \left(\int_\Omega |u|^{p(x)}\,dx\right)^{1/p_{\max}}\right\} \le \|u\|_{L^{p(\cdot)}(\Omega)}$$

$$\le \max\left\{\left(\int_\Omega |u|^{p(x)}\,dx\right)^{1/p_{\min}}, \left(\int_\Omega |u|^{p(x)}\,dx\right)^{1/p_{\max}}\right\}.$$

Moreover, the dual of $L^{p(\cdot)}(\Omega)$ is $L^{p'(\cdot)}(\Omega)$ with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

$W^{1,p(\cdot)}(\Omega)$ denotes the space of measurable functions u such that u and the distributional derivative ∇u are in $L^{p(\cdot)}(\Omega)$. The norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \||\nabla u|\|_{p(\cdot)}$$

makes $W^{1,p(\cdot)}(\Omega)$ a Banach space.

The space $W_0^{1,p(\cdot)}(\Omega)$ is defined as the closure of the $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.

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Basic differences with respect to the p constant case.

If $u \geq 0$ is a weak solution to

$$\Delta_{p(\cdot)} u = 0, \quad \Omega,$$

then there exists a positive constant $C = C(u)$ such that for $B_{4R}(x_0) \subset\subset \Omega$

$$\sup_{B_R(x_0)} u \leq C(\inf_{B_R(x_0)} u + R).$$

The dependence of C on u can not be removed, see [HKLMP].
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If $u \geq 0$ is a weak solution to

$$\Delta_{p(\cdot)} u = f \in L^q, \quad \Omega,$$

and $||u||_{L^\infty} < +\infty$, then there exists a positive constant $C = C(u)$ such that for $B_{4R}(x_0) \subset\subset \Omega$

$$\sup_{B_R(x_0)} u \leq C(\inf_{B_R(x_0)} u + R + \mu R).$$

where $\mu = \mu(||f||_{L^q}, p_{\min}, p_{\max}, R)$ and C depends on u and the other parameters as well in a complicate, but clear way, as well, see [Wo] .

Definition

Given $u, \varphi \in C(\Omega)$, we say that φ touches u from below (resp. above) at $x_0 \in \Omega$ if $u(x_0) = \varphi(x_0)$, and

$$u(x) \geq \varphi(x) \quad (\text{resp. } u(x) \leq \varphi(x)) \quad \text{in a neighborhood } O \text{ of } x_0.$$

If this inequality is strict in $O \setminus \{x_0\}$, we say that φ touches u strictly from below (resp. above).

Definition

Assume that $1 < p_{\min} \le p(x) \le p_{\max} < \infty$ with $p(x)$ Lipschitz continuous in Ω and $\|\nabla p\|_{L^\infty} \le L$, for some $L > 0$ and $f \in L^\infty(\Omega)$. We say that u is a weak solution to $\Delta_{p(x)} u = f$ in Ω if $u \in W^{1,p(\cdot)}(\Omega)$ and, for every $\varphi \in C_0^\infty(\Omega)$, there holds that

$$-\int_\Omega |\nabla u(x)|^{p(x)-2} \nabla u \cdot \nabla \varphi \, dx = \int_\Omega \varphi f(x) \, dx.$$

Definition

Let u be a continuous nonnegative function in Ω. We say that u is a viscosity solution to (1) in Ω, if the following conditions are satisfied:

1. $\Delta_{p(x)} u = f$ in $\Omega^+(u)$ in the weak sense.
2. For every $\varphi \in C(\Omega)$, $\varphi \in C^2(\overline{\Omega^+(\varphi)})$. If φ^+ touches u from below (resp. above) at $x_0 \in F(u)$ and $\nabla\varphi(x_0) \neq 0$, then

$$|\nabla\varphi(x_0)| \leq g(x_0) \quad (\text{resp. } \geq g(x_0)).$$

Theorem

Assume that $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$ *with* $p(x)$ *Lipschitz continuous in* Ω *and* $\|\nabla p\|_{L^\infty} \leq L$*, for some* $L > 0$ *and* $f \in L^\infty(\Omega)$*. Assume moreover that* $f \in C(\Omega)$ *and* $p \in C^1(\Omega)$*. Let* $u \in W^{1,p(\cdot)}(\Omega) \cap C(\Omega)$ *be a weak solution to* $\Delta_{p(x)} u = f$ *in* Ω*. Then* u *is a viscosity solution to* $\Delta_{p(x)} u = f$ *in* Ω*.*

Theorem

Let u be a viscosity solution to (1) in Ω. Then the following conditions are satisfied:

1. $\Delta_{p(x)} u = f$ *in* $\Omega^+(u)$ *in the viscosity sense, that is:*
   - (ia) *for every* $\varphi \in C^2(\Omega^+(u))$ *and for every* $x_0 \in \Omega^+(u)$, *if* φ *touches* u *from above at* x_0 *and* $\nabla\varphi(x_0) \neq 0$, *then* $\Delta_{p(x_0)}\varphi(x_0) \geq f(x_0)$, *that is, u is a viscosity subsolution;*
   - (ib) *for every* $\varphi \in C^2(\Omega^+(u))$ *and for every* $x_0 \in \Omega^+(u)$, *if* φ *touches* u *from below at* x_0 *and* $\nabla\varphi(x_0) \neq 0$, *then* $\Delta_{p(x_0)}\varphi(x_0) \leq f(x_0)$, *that is, u is a viscosity supersolution.*
2. *For every* $\varphi \in C(\Omega)$, $\varphi \in C^2(\overline{\Omega^+(\varphi)})$. *If* φ^+ *touches u from below (resp. above) at* $x_0 \in F(u)$ *and* $\nabla\varphi(x_0) \neq 0$, *then*

$$|\nabla\varphi(x_0)| \leq g(x_0) \quad (\text{resp. } \geq g(x_0)).$$

Lemma

Assume that $1 < p_{\min} \le p(x) \le p_{\max} < \infty$ *with* $p(x)$ *Lipschitz continuous in* Ω *and* $\|\nabla p\|_{L^\infty} \le L$*, for some* $L > 0$*. Let* $x_0 \in \Omega$ *and* $0 < R \le 1$ *such that* $B_{4R}(x_0) \subset \Omega$*. Let* $v \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ *be a nonnegative solution to*

$$\operatorname{div}(|\nabla v + e|^{p(x)-2}(\nabla v + e)) = f \quad \text{in } \Omega, \tag{9}$$

where $f \in L^\infty(\Omega)$ *with* $||f||_{L^\infty(\Omega)} \le 1$ *and* $e \in \mathbb{R}^n$ *with* $|e| = 1$*. Then, there exists* C *such that*

$$\sup_{B_R(x_0)} v \le C\left[\inf_{B_R(x_0)} v + R\left(||f||_{L^\infty(B_{4R}(x_0))}^{\frac{1}{p_{\max}-1}} + C\right)\right]. \tag{10}$$

The constant C *depends only on* n, $p_{\min}$, $p_{\max}$, $||v||_{L^\infty(B_{4R}(x_0))}$ *and* L.

Lemma

Let $x_0 \in B_1$ *and* $0 < \bar{r}_1 < \bar{r}_2 \leq 1$. *Assume that* $1 < p_{\min} \leq p(x) \leq p_{\max} < \infty$ *and* $\|\nabla p\|_{L^\infty} \leq \varepsilon^{1+\theta}$, *for some* $0 < \theta \leq 1$. *Let* c_0, c_1, c_2 *be positive constants and let and* $c_3 \in \mathbb{R}$. *There exist positive constants* $\gamma \geq 1$, $\bar{c}$, ε_0 *and* ε_1 *such that the functions*

$$w(x) = c_1|x - x_0|^{-\gamma} - c_2,$$

$$v(x) = q(x) + \frac{c_0}{2}\varepsilon(w(x) - 1), \quad q(x) = x_n + c_3$$

satisfy, for $\bar{r}_1 \leq |x - x_0| \leq \bar{r}_2$,

$$\Delta_{p(x)} w \geq \bar{c}, \quad \textit{for } 0 < \varepsilon \leq \varepsilon_0, \tag{11}$$

$$\frac{1}{2} \leq |\nabla v| \leq 2, \qquad \Delta_{p(x)} v > \varepsilon^2, \quad \textit{for } 0 < \varepsilon \leq \varepsilon_1. \tag{12}$$

Here $\gamma = \gamma(n, p_{\min}, p_{\max})$, $\bar{c} = \bar{c}(p_{\min}, p_{\max}, c_1)$,

Lemma (Improvement of flatness, F-Claudia Lederman)

Let u satisfy (1) *in B_1 and* $\|f\|_{L^\infty(B_1)} \leq \varepsilon^2, ||g-1||_{L^\infty(B_1)} \leq \varepsilon^2$,

$$||\nabla p||_{L^\infty(B_1)} \leq \varepsilon^{1+\theta}, ||p-p_0||_{L^\infty(B_1)} \leq \varepsilon, \tag{13}$$

for $0 < \varepsilon < 1$, *for some constant* $0 < \theta \leq 1$. *Suppose that*

$$(x_n - \varepsilon)^+ \leq u(x) \leq (x_n + \varepsilon)^+ \quad \text{in } B_1, \quad 0 \in F(u). \tag{14}$$

If $0 < r \leq r_0$ *for* r_0 *universal, and* $0 < \varepsilon \leq \varepsilon_0$ *for some* ε_0 *depending on* r, *then*

$$(x \cdot \nu - r\varepsilon/2)^+ \leq u(x) \leq (x \cdot \nu + r\varepsilon/2)^+ \quad \text{in } B_r, \tag{15}$$

with $|\nu| = 1$ *and* $|\nu - e_n| \leq \tilde{C}\varepsilon$ *for a universal constant* $\tilde{C}$.

The basic step in the improvement of flatness.
Let

$$U_\beta(t) = \alpha t^+ - \beta t^-, \ \ \beta \geq 0, \ \ \alpha = G(\beta) \equiv \sqrt{1+\beta^2}$$

and ν is a unit vector which plays the role of the normal vector at the origin. $U_\beta(x \cdot \nu)$ is a so-called *two plane solution*.

The classical two-phase inhomogeneous problem

$$
\begin{cases}
\Delta u = f_+ & \text{in } B_1^+(u) := \{x \in B_1 : u(x) > 0\}, \\
\Delta u = f_- & \text{in } B_1^-(u) := \{x \in B_1 : u(x) \leq 0\}^\circ, \\
u_\nu^+ = \sqrt{1 + (u_\nu^-)^2} & \text{on } F(u) := \partial B_1^+(u) \cap B_1.
\end{cases}
\tag{16}
$$

B_1 is the unit ball in $\mathbb{R}^n$, centered at the origin. Instead of $r \to \sqrt{1 + r^2}$ we may consider a continuous function $r \to G(r)$ such that G is strictly increasing and such that $G(0) > 0$ as well.

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Moreover:
$f_{\pm} \in C(B_1) \cap L^{\infty}(B_1)$,

$$B_1^+(u) := \{x \in B_1 : u(x) > 0\}, \quad B_1^-(u) := \{x \in B_1 : u(x) \leq 0\}^{\circ}.$$

u_ν^+ and u_ν^- denote the normal derivatives in the inward direction to $B_1^+(u)$ and $B_1^-(u)$ respectively.

The problem comes from several applied contexts:
the Prandtl-Bachelor model in fluid-dynamics (see e.g. [B1],[EM]), the eigenvalue problem in magnetohydrodynamics ([FL]), or in flame propagation models ([LW]).
B=Batchelor; EM= Elcrat-Miller; FL=Friedman-Liu; LW=Lederman-Wolanski

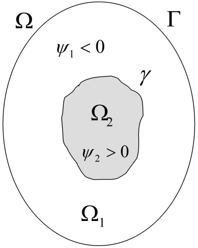
Batchelor model

A bounded 2-dimensional domain is delimited by two simple closed curves γ, Γ.
For given constants $\mu < 0, \omega > 0$, consider functions ψ_1, ψ_2 satisfying

$$\Delta\psi_1 = 0 \text{ in } \Omega_1, \psi_1 = 0 \text{ on } \gamma, \psi_1 = \mu \text{ on } \Gamma,$$

$$\Delta\psi_2 = \omega \text{ in } \Omega_2, \psi_2 = 0 \text{ on } \gamma.$$

and $\Omega_1 := \{\psi_1 < 0\}, \Omega_2 := \{\psi_2 > 0\}$.



Prandtl-Batchelor flow configuration

- ψ_1 is the stream functions of an irrotational flow in Ω_1
- ψ_2 is the stream of a constant vorticity flow in Ω_2.

The model proposed by Batchelor comes from the limit of large Reynold number in the steady Navier-Stokes equation.

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For the flow of that type is hypothesized that there is a jump in the tangential velocity along γ, namely

$$|\nabla\psi_2|^2 - |\nabla\psi_1|^2 = \sigma$$

for some positive constant σ and γ had to be determined:

γ:=Free boundary.

$$
\begin{cases}
\Delta\psi_2 = \omega & \text{in } \Omega_2 \equiv \Omega^+(\psi_2) := \{x \in \Omega : \psi_2(x) > 0\}, \\
\\
\Delta\psi_1 = 0 & \text{in } \Omega_1 \equiv \Omega^-(\psi_1) := \{x \in \Omega : \psi_1(x) \leq 0\}^\circ, \\
\\
\psi_2 = 0 = \psi_1, & \text{on } \gamma \equiv F(\psi_2) := \partial\Omega_2^+(\psi_2) \cap \Omega, \\
|\nabla\psi_2|^2 - |\nabla\psi_1|^2 = \sigma & \text{on } \gamma \equiv F(\psi_2) := \partial\Omega_2^+(\psi_2) \cap \Omega, \\
\psi_1 = \mu, & \text{on } \Gamma \equiv \partial\Omega
\end{cases}
\tag{17}
$$

Expected Results

- Existence of Lipschitz viscosity solutions and weak regularity properties of the free boundary.
- Strong regularity results.
- Higher regularity results for the free boundary.

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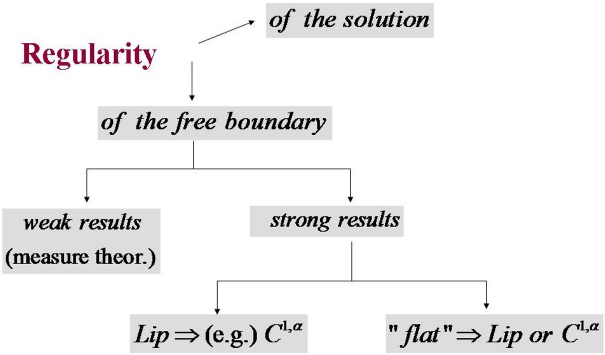
Regularity
of the solution
of the free boundary
weak results
(measure theor.)
strong results
Lip ⇒ (e.g.) $C^{1,\alpha}$
"flat" ⇒ Lip or $C^{1,\alpha}$

Known results

- Homogeneous case, i.e. $f_{\pm} \equiv 0$ for Δ : strong regularity properties of the f.b., Louis Caffarelli, [C1],[C2].
- Existence of Lipschitz viscosity solutions, [C3] based on [ACF].
- Inhomogeneous case: Lipschitz regularity was obtained by Caffarelli, Jerison and Kenig in [CJK].
- Further results for homogeneous free boundary problem (for fully nonlinear operators) see for example: [Fe1,Fe2, CFS, FS1,F1, W1, W2, MT].
- In the case of the non-homogeneous setting: [D], in one phase, and [DFS, DFS2,DFS3, DFS4, DFS5] for two-phase problems (linear operators with variable coefficients, fully nonlinear operators).
- Recently: [LaYoTe]

D=Daniela De Silva, S:= Sandro Salsa, W=Wang, Fe=Feldman,

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Theorem (Flatness $\to C^{1,\gamma}$, [De Silva-F-Salsa])

Let u be a solution of our n.h.f.b. problem (16)*. There exists a universal constant $\bar{\delta} > 0$ such that, if $0 \leq \delta \leq \bar{\delta}$ and*

$$\{x_n \leq -\delta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \delta\}, \qquad (\delta - \textit{flatness}) \quad (18)$$

then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$, with γ universal.

Theorem

Let u be a solution of our n.h.f.b. problem (16)*. If $F(u)$ is a Lipschitz graph in B_1, then $F(u)$ is $C^{1,\gamma}$ in $B_{1/2}$, with γ universal.*

Theorem ([De Silva-F-Salsa-5])

Let u be a (Lipschitz) viscosity solution to (16) *in B_1. There exists a universal constant $\bar{\eta} > 0$ such that, if*

$$\{x_n \leq -\eta\} \subset B_1 \cap \{u^+(x) = 0\} \subset \{x_n \leq \eta\}, \quad \text{for } 0 \leq \eta \leq \bar{\eta}, \tag{19}$$

then $F(u)$ is C^{2,γ^*} in $B_{1/2}$ for a small γ^* universal, with the C^{2,γ^*} norm bounded by a universal constant.

Theorem ([De Silva-F-Salsa-5])

Let k be a nonnegative integer. Assume that u is a solution of (16) and it is endowed by a flat free boundary and $f_{\pm} \in C^{k,\gamma}(B_1)$. Then $F(u) \cap B_{1/2}$ is C^{k+2,γ^*}. If $f_{\pm}$ are C^{∞} or real analytic in B_1, then $F(u) \cap B_{1/2}$ is C^{∞} or real analytic, respectively.

Lemma ([De Silva-F-Salsa])

Let u satisfy (16) and

$$U_\beta(x_n - \varepsilon) \le u(x) \le U_\beta(x_n + \varepsilon), \quad in \quad B_1, \quad 0 \in F(u),$$

with $0 < \beta \le L$ *and*

$$||f||_{L^\infty(B_1)} \le \varepsilon^2 \beta.$$

If $0 < r \le r_0$ *for* r_0 *universal, and* $0 < \varepsilon \le \varepsilon_0$ *for some* ε_0 *depending on r, then*

$$U_{\beta'}(x \cdot \nu_1 - r\frac{\varepsilon}{2}) \le u(x) \le U_{\beta'}(x \cdot \nu_1 + r\frac{\varepsilon}{2}) \quad in \ B_r, \tag{20}$$

with $|\nu_1| = 1$, $|\nu_1 - e_n| \le \tilde{C}\varepsilon$, *and* $|\beta - \beta'| \le \tilde{C}\beta\varepsilon$ *for a universal constant* $\tilde{C}$.

Consequence

Assume the lemma above holds. To prove the Theorem "Flatness $\to C^{1,\gamma}$" in hypotheses of flatness conditions.

We rescale considering a blow up sequence

$$u_k(x) = \frac{u(\rho_k x)}{\rho_k} \quad \rho_k = \bar{r}^k, \quad x \in B_1 \tag{21}$$

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We iterate to get, at the kth step,

$$U_{\beta_k}(x\cdot\nu_k-\rho_k\varepsilon_k)\leq u_k(x)\leq U_{\beta_k}(x\cdot\nu_k+\rho_k\varepsilon_k)\quad\text{in } B_{\rho_k},$$

with $\varepsilon_k=2^{-k}\tilde{\varepsilon}$, $|\nu_k|=1$, $|\nu_k-\nu_{k-1}|\leq\tilde{C}\varepsilon_{k-1}$,

$$|\beta_k-\beta_{k-1}|\leq\tilde{C}\beta_{k-1}\varepsilon_{k-1},\ \ \varepsilon_k\leq\beta_k\leq L.$$

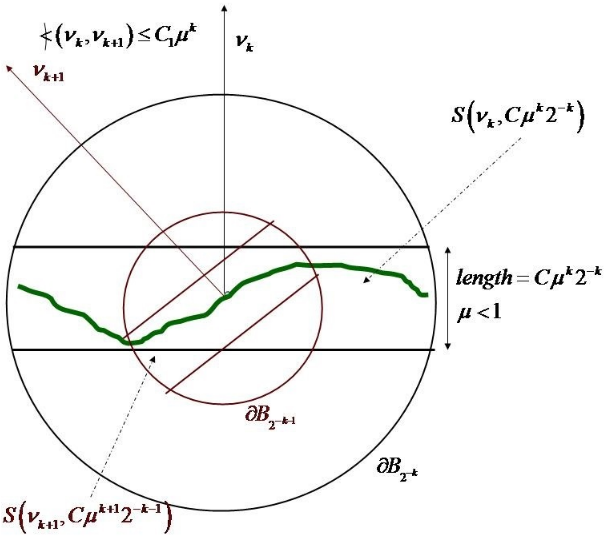
Thus, since

$$f_k(x) = \rho_k f(\rho_k x), \quad x \in B_1$$

(recall that $\bar{\eta} = \tilde{\varepsilon}^3$)

$$\|f_k\|_{L^\infty(B_1)} \le \rho_k \tilde{\varepsilon}^3 \le \tilde{\varepsilon}_k^2 \beta_k = \tilde{\varepsilon}_k^2 \min\{\alpha_k, \beta_k\}.$$

The figure below describes the step from k to $k+1$.


$\measuredangle(\nu_k, \nu_{k+1}) \le C_1 \mu^k$
ν_k
ν_{k+1}
$S(\nu_k, C\mu^k 2^{-k})$
$length = C\mu^k 2^{-k}$
$\mu < 1$
$\partial B_{2^{-k-1}}$
$\partial B_{2^{-k}}$
$S(\nu_{k+1}, C\mu^{k+1} 2^{-k-1})$

This implies that $F(u)$ is $C^{1,\gamma}$ at the origin. Repeating the procedure for points in a neighborhood of $x = 0$, (all estimates are universal), we conclude that there exists a unit vector $\nu_\infty = \lim \nu_k$ and $C > 0$, $\gamma \in (0, 1]$, both universal, such that, in the coordinate system $e_1, ..., e_{n-1}, \nu_\infty$, $\nu_\infty \perp e_j$, $e_j \cdot e_k = \delta_{jk}$, $F(u)$ is $C^{1,\gamma}$ graph, say $x_n = g(x')$, with $g(0') = 0$ and

$$\left|g(x') - \nu_\infty \cdot x'\right| \leq C \left|x'\right|^{1+\gamma}$$

in a neighborhood of $x = 0$.

Proof of Lemma [De Silva-F-Salsa]

We argue by contradiction.

Step 1. Fix $r \leq r_0$, to be chosen suitably. Assume that for a sequence $\varepsilon_k \to 0$ there is a sequence u_k of solutions of our free boundary problem in B_1, with right hand side f_k such that $\|f_k\|_{L^\infty(B_1)} \leq \varepsilon_k^2 \min\{\alpha_k, \beta_k\}$, and

$$U_{\beta_k}(x_n - \varepsilon_k) \leq u_k(x) \leq U_{\beta_k}(x_n + \varepsilon_k) \quad \text{in } B_1, \ \ 0 \in F(u_k), \qquad (22)$$

with $0 \leq \beta_k \leq L$, $\alpha_k = \sqrt{1 + \beta_k^2}$, but the conclusion of Lemma (Main) does not hold for every $k \geq 1$.

Construct the corresponding sequence of renormalized functions

$$\tilde{u}_k(x) = \begin{cases} \frac{u_k(x) - \alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(u_k) \cup F(u_k) \\ \frac{u_k(x) - \beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(u_k). \end{cases}$$

and

$$-1 \leq \tilde{u}_k(x) \leq 1, \quad \text{for } x \in B_1.$$

At this point we need compactness to show that the graphs of $\tilde{u}_k$ converge in the Hausdorff distance to a Hölder continuous $\tilde{u}$ in $B_{1/2}$.

The compactness is provided by an appropriate Harnack result and a sharp version of Ascoli Arzelà's theorem.

Moreover, up to a subsequence $\beta_k \to \tilde{\beta}$ so that $\alpha_k \to \tilde{\alpha} = \sqrt{1 + \tilde{\beta}^2}$.

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Moreover, up to a subsequence $\beta_k \to \tilde{\beta}$ so that $\alpha_k \to \tilde{\alpha} = \sqrt{1 + \tilde{\beta}^2}$.

Step 2: Transmission problem.
$\tilde{u}$ solves the "linearized problem" ($\tilde{\alpha} \neq 0$)

$$\begin{cases} \Delta \tilde{u} = 0 & \text{in } B_1 \cap \{x_n \neq 0\}, \\ \tilde{\alpha}^2 (\tilde{u}_n)^+ - \tilde{\beta}^2 (\tilde{u}_n)^- = 0 & \text{on } B_1 \cap \{x_n = 0\}. \end{cases} \tag{23}$$

Moreover according with the following result

Theorem (Regularity of the transmission problem)

Let $\tilde{u}$ be a viscosity solution to (23) in B_1 such that $\|\tilde{u}\|_\infty \leq 1$. Then $\tilde{u} \in C^\infty\left(\bar{B}_1^\pm\right)$ and in particular, there exists a universal positive constant $\bar{C}$ such that

$$|\tilde{u}(x) - \tilde{u}(0) - (\nabla_{x'}\tilde{u}(0) \cdot x' + \tilde{p}x_n^+ - \tilde{q}x_n^-)| \leq \bar{C}r^2, \quad in \ B_r \tag{24}$$

for all $r \leq 1/2$ and with $\tilde{\alpha}^2\tilde{p} - \tilde{\beta}^2\tilde{q} = 0$.

Step 3 (Contradiction). We can prove the last step.
We can show that (for k large and $r \leq r_0$)

$$\widetilde{U}_{\beta_k'}(x \cdot \nu_k - \varepsilon_k \frac{r}{2}) \leq \tilde{u}_k(x) \leq \widetilde{U}_{\beta_k'}(x \cdot \nu_k + \varepsilon_k \frac{r}{2}), \quad \text{in } B_r$$

where again we are using the notation:

$$\widetilde{U}_{\beta_k'}(x) = \begin{cases} \frac{U_{\beta_k'}(x) - \alpha_k x_n}{\alpha_k \varepsilon_k}, & x \in B_1^+(U_{\beta_k'}) \cup F(U_{\beta_k'}) \\ \frac{U_{\beta_k'}(x) - \beta_k x_n}{\beta_k \varepsilon_k}, & x \in B_1^-(U_{\beta_k'}). \end{cases}$$

This will clearly imply that

$$U_{\beta_k'}(x\cdot\nu_k-\varepsilon_k\frac{r}{2})\leq u_k(x)\leq U_{\beta_k'}(x\cdot\nu_k+\varepsilon_k\frac{r}{2}),\quad \text{in } B_r$$

leading to a contradiction with the assumption that the thesis of the Lemma is false.

Indeed, recalling the Theorem (Regularity of the transmission problem), it is sufficient to show that in B_r :

$$\widetilde{U}_{\beta_k'}(x \cdot \nu_k - \varepsilon_k \frac{r}{2}) \leq (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-) - Cr^2$$

and

$$\widetilde{U}_{\beta_k'}(x \cdot \nu_k + \varepsilon_k \frac{r}{2}) \geq (x' \cdot \nu' + \tilde{p}x_n^+ - \tilde{q}x_n^-) + Cr^2.$$

This can be shown after some elementary calculations as long as $r \leq r_0$, r_0 universal, and $\varepsilon \leq \varepsilon_0(r)$.

Improvement of flatness proof in the $p(x)-$Laplace one-phase case

Step 1: Compactness. Fix $r \leq r_0$ with r_0 universal. Assume by contradiction that there exists a sequence $\varepsilon_k \to 0$ and a sequence u_k of solutions to (1) in B_1 with right hand side f_k, exponent p_k and free boundary condition g_k satisfying (13) with $\varepsilon = \varepsilon_k$, such that u_k satisfies (14), i.e.,

$$(x_n - \varepsilon_k)^+ \leq u_k(x) \leq (x_n + \varepsilon_k)^+ \quad \text{for } x \in B_1, 0 \in F(u_k), \qquad (25)$$

but u_k does not satisfy the conclusion (15) of the lemma.
Set

$$\tilde{u}_k(x) = \frac{u_k(x) - x_n}{\varepsilon_k}, \quad x \in \Omega_1(u_k).$$

Then, (25) gives

$$-1 \leq \tilde{u}_k(x) \leq 1 \quad \text{for } x \in \Omega_1(u_k). \qquad (26)$$

With a compactness argument and Ascoli-Arzelà theorem it is possible to prove that there exists a convergent subsequence to a function $\tilde{u}$.

Step 2. Transmission problem. The function $\tilde{u}$ solves the following linearized problem

$$\begin{cases} \mathcal{L}_{p_0}\tilde{u} = 0 & \text{in } B_{1/2} \cap \{x_n > 0\}, \\ \tilde{u}_n = 0 & \text{on } B_{1/2} \cap \{x_n = 0\}. \end{cases} \tag{27}$$

Here $1 < p_{\min} \leq p_0 \leq p_{\max} < \infty$, $\tilde{u}_n$ denotes the derivative in the e_n direction of $\tilde{u}$ and

$$\mathcal{L}_{p_0}u := \Delta u + (p_0 - 2)\partial_{nn}u. \tag{28}$$

Definition

Let $\tilde{u}$ be a continuous function on $B_\rho \cap \{x_n \geq 0\}$. We say that $\tilde{u}$ is a viscosity solution to (28), if given a quadratic polynomial $P(x)$ touching $\tilde{u}$ from below (resp. above) at $\bar{x} \in B_\rho \cap \{x_n \geq 0\}$,

(i) if $\bar{x} \in B_\rho \cap \{x_n > 0\}$ then $\mathcal{L}_{p_0} P \leq 0$ (resp. $\mathcal{L}_{p_0} P \geq 0$), i.e. $\mathcal{L}_{p_0} \tilde{u} = 0$ in the viscosity sense in $B_\rho \cap \{x_n > 0\}$;

(ii) if $\bar{x} \in B_\rho \cap \{x_n = 0\}$ then $P_n(\bar{x}) \leq 0$ (resp. $P_n(\bar{x}) \geq 0$).

Step 3: Improvement of flatness. From the previous step, $\tilde{u}$ solves (28) and from (26),

$$-1 \leq \tilde{u}(x) \leq 1 \quad \text{in } B_{1/2} \cap \{x_n \geq 0\}.$$

From

Theorem

Let $\tilde{u}$ be a viscosity solution to (28) in $B_{1/2} \cap \{x_n \geq 0\}$. Then, $\tilde{u} \in C^2(B_{1/2} \cap \{x_n \geq 0\})$ and it is a classical solution to (28). Moreover, if $\|\tilde{u}\|_\infty \leq 1$, then there exists a constant $\bar{C} > 0$, depending only on $n, p_{\min}$ and $p_{\max}$, such that

$$|\tilde{u}(x) - \tilde{u}(0) - \nabla \tilde{u}(0) \cdot x| \leq \bar{C} r^2 \quad \text{in } B_r \cap \{x_n \geq 0\}, \tag{29}$$

for all $r \leq 1/4$.

by the bound above (29) in Theorem 19 we find that, for the given r,

$$|\tilde{u}(x) - \tilde{u}(0) - \nabla\tilde{u}(0)\cdot x| \leq C_0 r^2 \quad \text{in } B_r \cap \{x_n \geq 0\}$$

and now the iterative argument is the same that has been applied in the linear case.

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