Spreading speeds and spreading sets for reaction-diffusion equations

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Mostly Maximum Principle, 4th edition, Cortona, May-June 2022

Supported by Università degli Studi di Roma La Sapienza as Sapienza Visiting Professor

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I. The framework and the main questions

Reaction-diffusion equation in $\mathbb{R}^N$ with $N \geq 2$

$$u_t = \Delta u + f(u), \quad t > 0,\ x \in \mathbb{R}^N$$

Function $f : [0,1] \to \mathbb{R}$ of class C^1 and

$$f(0) = f(1) = 0$$

Examples of initial conditions

$$u_0(x) = \mathbf{1}_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{if } x \in \mathbb{R}^N \setminus U \end{cases}$$

with, typically, unbounded sets U such that $|U| > 0$ and $|\mathbb{R}^N \setminus U| > 0$

(*other initial conditions are possible*)

$$0 < u(t,x) < 1 \ \text{ for all } t > 0,\ x \in \mathbb{R}^N$$

The main questions

- Location as $t \to +\infty$ of the regions where $u(t,x) \simeq 0$ or $u(t,x) \simeq 1$?

- For any $\mathrm{e} \in \mathbb{S}^{N-1}$, is there a *spreading speed* $w(\mathrm{e}) > 0$ such that

$$\begin{cases} u(t, \gamma\, t\, \mathrm{e}) \underset{t \to +\infty}{\longrightarrow} 1 & \text{if } 0 \le \gamma < w(\mathrm{e}) \\ u(t, \gamma\, t\, \mathrm{e}) \underset{t \to +\infty}{\longrightarrow} 0 & \text{if } \gamma > w(\mathrm{e}) \end{cases}$$

- Formula for $w(\mathrm{e})$ in terms of e and U ?

- $w(\mathrm{e}) = +\infty$ can happen, in directions around which "U is unbounded"

- Spreading sets describing the asymptotic global shape of the level sets of u ?

- Memory of the initial support U ?

OUTLINE OF THE TALK

I. The framework and the main questions

II. The main hypothesis and preliminary results

III. Spreading speeds and spreading sets, Freidlin-Gärtner type formulas

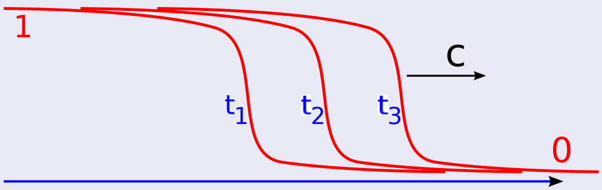
Hypothesis TF: "traveling front connecting 1 to 0 with positive speed"

The equation in $\mathbb{R}$ admits a traveling front $\varphi(x-ct)$ such that

$$\varphi(-\infty)=1>\varphi(z)>0=\varphi(+\infty)$$

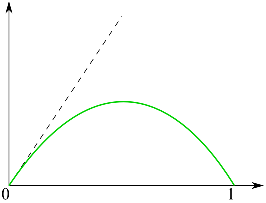
with

$$c>0$$

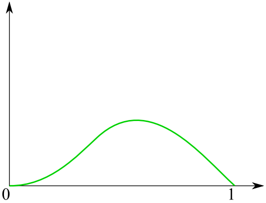


Equivalently: planar fronts $\varphi(x\cdot e-ct)$ in $\mathbb{R}^N$, with $e\in\mathbb{S}^{N-1}$

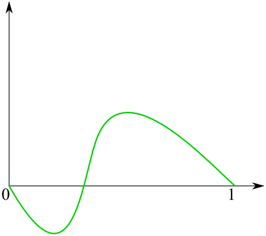
Some functions f fulfiling Hypothesis TF [Aronson, Weinberger,1978]



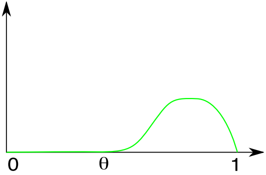
(d) KPP f



(e) positive f



(f) bistable f with $f(\theta) = 0$, $0 < \theta < 1$ and $\int_0^1 f > 0$



(g) ignition f with $f = 0$ in $[0, \theta]$, $f > 0$ in $(\theta, 1)$

More general multistable functions f are possible [Fife, McLeod, 1977]

Three preliminary results

Assume that Hypothesis TF holds. Then

1. there is a traveling front $\varphi^*(x - c^* t)$ with minimal speed $c^* > 0$
2. (*invasion property*) there exist $\alpha \in (0,1)$ and $\rho > 0$ such that, if

$$\alpha\,\mathbf{1}_{B_\rho} \le u_0 \le 1 \ \text{ in } \mathbb{R}^N$$

then

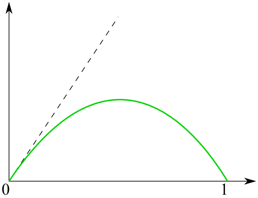
$$u(t,\cdot) \to 1 \ \text{ as } t \to +\infty \text{ locally uniformly in } \mathbb{R}^N$$

3. (*spreading*) and even

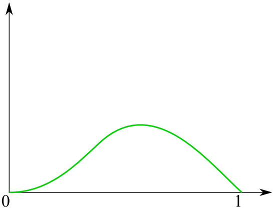
$$\begin{cases} \forall\, 0 \le \gamma < c^*, & \displaystyle\min_{|x| \le \gamma t} u(t,x) \to 1 \ \text{ as } t \to +\infty \\ \forall\, \gamma > c^*, & \displaystyle\max_{|x| \ge \gamma t} u(t,x) \to 0 \ \text{ as } t \to +\infty \text{ if } \operatorname{supp}(u_0) \text{ compact} \end{cases}$$

$\Longrightarrow$ spreading speed $w(\mathrm{e}) = c^*$ if u_0 is compactly supported

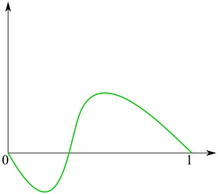
What are c^*, α and ρ in standard cases? [Aronson, Weinberger, 1978]



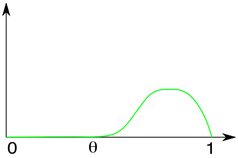
(h) KPP, $c^* = 2\sqrt{f'(0)}$, hair-trigger effect: $\alpha \in (0,1)$ and $\rho > 0$ arbitrary



(i) positive, $c^* \geq 2\sqrt{f'(0)}$, hair-trigger effect if $\liminf_{s\to 0^+} f(s)/s^{1+2/N} > 0$



(j) bistable, $f(\theta) = 0$, $0 < \theta < 1$, $\int_0^1 f > 0$: $c^* = c$ unique, $\alpha \in (\theta, 1)$ and $\rho > 0$ large



(k) ignition: $c^* = c$ unique, $\alpha \in (\theta, 1)$ and $\rho > 0$ large

Equivalent condition to the invasion property (with $\alpha\,\mathbf{1}_{B_\rho} \le u_0 \le 1$)

The invasion property is satisfied if and only if:

$$\begin{cases} \exists\,\alpha \in (0,1), & f > 0 \text{ in } [\alpha, 1) \\ \forall\, t \in [0,1), & \displaystyle\int_t^1 f(s)ds > 0 \end{cases}$$

and α can be the same as in the statement of the invasion property

$\Longrightarrow$ First property is immediate from the maximum principle

Second property: proof by contradiction (with, otherwise, steady solution $\phi : \mathbb{R} \to (0,1)$ such that $\phi(-\infty) = 1$)

$\Longleftarrow$ [Du, Poláčik, 2015]

Solutions of $\dot{\xi}(t) = f(\xi(t))$ with $\xi(0) = \alpha$: $\xi(t) \to 1$ as $t \to +\infty$

Variational solutions $\phi_R \in H_0^1(B_R)$ of

$$\Delta\phi_R + f(\phi_R) = 0 \text{ in } B_R$$

with $0 < \phi_R < 1$ in B_R and $\max\limits_{\overline{B_R}} \phi_R = \phi_R(0) > \alpha$

Solution v of the Cauchy problem emanating from ϕ_R: $v(t,\cdot) \underset{t\to+\infty}{\longrightarrow} 1$

Further comments

- The invasion property holds in particular if $f \geq 0$ in $[0,1]$ and $f > 0$ in $(1-\varepsilon, 1)$ for some $\varepsilon > 0$

- The invasion property is independent of the dimension N

- Invasion property $\not\Longrightarrow$ Hypothesis TF:

  examples of functions f with multiple oscillations, solutions developing into terraces of expanding fronts

  [Fife, McLeod,1977] [Ducrot, Giletti, Matano,2014] [Du, Matano,2017] [Giletti, Rossi, 2020] [Poláčik, 2020]

- But invasion property $\Longrightarrow$

  $\min_{|x|\leq\delta t} u(t,x) \to 1$ with $\delta > 0$ for the invading solutions

Compactly supported initial conditions u_0 (or $\lim_{|x|\to+\infty} u_0(x) = 0$)

- Extinction vs. invasion, thresholds for monotone families of u_0

  [Aronson, Weinberger, 1978] [Zlatoš, 2006] [Du, Matano, 2010] [Muratov, Zhong, 2013, 2017]

- Further results on location and shape of level sets of invading solutions

  [Gärtner, 1982] [Jones, 1983] [Uchiyama, 1985] [Roussier, 2004] [Ducrot,2015] [Rossi,2017] [Roquejoffre, Rossi, Roussier-Michon,2019]

- General result on the local convergence to steady states as $t \to +\infty$

  [Du, Matano, 2010]

- Local convergence and quasiconvergence

  [Du, Matano, 2010] [Du, Poláčik, 2015]

  [Matano, Poláčik, 2016, 2020] [Poláčik, 2017] [Pauthier, Poláčik, 2020]

I. The framework and the main questions

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III. Spreading speeds and spreading sets, Freidlin-Gärtner type formulas

We assume Hypothesis TF and $u_0 = \mathbf{1}_U$ with $U \supset B_\rho(x_0)$

- $\Longrightarrow \min_{\overline{B_{\gamma t}}} u(t,\cdot) \to 1$ as $t \to +\infty$ for any $0 \le \gamma < c^*$
- More precise description of the invasion of the state 0 by the state 1 ?
- Are there spreading speeds $\mathrm{e} \mapsto w(\mathrm{e})$ $(\ge c^*)$?
- Spreading sets ?
- Description of upper level sets of u with $0 < \lambda < 1$:

$$E_\lambda(t) := \{x \in \mathbb{R}^N : u(t,x) > \lambda\}$$

Some important definitions

- Directions "around which U is unbounded":

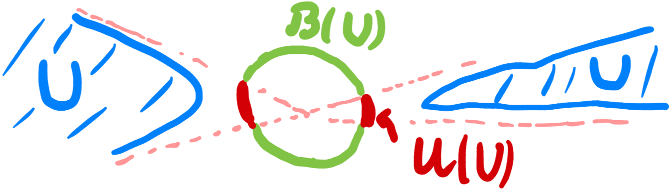
$$\mathcal{U}(U) := \Big\{\xi \in \mathbb{S}^{N-1} : \lim_{\tau\to+\infty} \frac{\mathrm{dist}(\tau\xi, U)}{\tau} = 0\Big\}$$

  (closed set in $\mathbb{S}^{N-1}$)

- Directions "around which U is bounded":

$$\mathcal{B}(U) := \Big\{\xi \in \mathbb{S}^{N-1} : \liminf_{\tau\to+\infty} \frac{\mathrm{dist}(\tau\xi, U)}{\tau} > 0\Big\}$$

  (open in $\mathbb{S}^{N-1}$, disjoint of $\mathcal{U}(U)$) ($\mathcal{B}(U) = \mathbb{S}^{N-1}$ iff U is bounded)



- Positive-distance-interior of U:

$$U_\rho := \big\{x \in U : \mathrm{dist}(x, \partial U) \geq \rho\big\}$$

  ($U_\rho \subset U$, $\mathcal{U}(U_\rho) \subset \mathcal{U}(U)$)

Theorem 1 (spreading speeds)

Assume that

$$\mathcal{B}(U) \cup \mathcal{U}(U_\rho) = \mathbb{S}^{N-1} \tag{1}$$

Then, for every $\mathrm{e} \in \mathbb{S}^{N-1}$, there is $w(\mathrm{e}) \in [c^*, +\infty]$ such that

$$\begin{cases} u(t, \gamma\, t\, \mathrm{e}) \underset{t \to +\infty}{\longrightarrow} 1 & \text{if } 0 \le \gamma < w(\mathrm{e}) \\ u(t, \gamma\, t\, \mathrm{e}) \underset{t \to +\infty}{\longrightarrow} 0 & \text{if } \gamma > w(\mathrm{e}) \end{cases}$$

Furthermore,

$$w(\mathrm{e}) = \sup_{\xi \in \mathcal{U}(U),\, \xi \cdot \mathrm{e} \ge 0} \frac{c^*}{\sqrt{1 - (\xi \cdot \mathrm{e})^2}} = \frac{c^*}{\mathrm{dist}(\mathrm{e}, \mathbb{R}^+ \mathcal{U}(U))}$$

($w(\mathrm{e}) = c^*$ if no $\xi \in \mathcal{U}(U)$ s.t. $\xi \cdot \mathrm{e} \ge 0$, $w(\mathrm{e}) = +\infty$ iff $\mathrm{e} \in \mathcal{U}(U)$)

- Freidlin-Gärtner type formula

  (for solutions of periodic KPP equations with *compactly supported* u_0)

$$w(e) = \inf_{\xi \in \mathbb{S}^{N-1},\, \xi \cdot e > 0} \frac{c^*(\xi)}{\xi \cdot e}$$

  Minimal speed $c^*(\xi)$ of pulsating fronts in the direction ξ

  [Freidlin, Gärtner, 1979] [Weinberger, 2002]
  [Berestycki, Hamel, Nadirashvili,2005] [Berestycki, Hamel, Nadin,2008]

  More general reactions [Rossi, 2017]

- Result does not hold without Hypothesis TF:

  examples of functions f with multiple oscillations, terrasses of expanding fronts

- $d_{\mathcal{H}}(U, U_\rho) < +\infty$ and U is star-shaped $\Longrightarrow \mathcal{B}(U) \cup \mathcal{U}(U_\rho) = \mathbb{S}^{N-1}$

Theorem 2 (spreading sets) (stronger property: uniformity in e)

Assume (1)

Notation: envelop set

$$\mathcal{W} := \{r\mathrm{e} : \mathrm{e} \in \mathbb{S}^{N-1},\ 0 \le r < w(\mathrm{e})\} \ = \ \mathbb{R}^+\mathcal{U}(U) + B_{c^*}$$

Then, for any compact set $C \subset \mathbb{R}^N$,

$$\begin{cases} \displaystyle\lim_{t\to+\infty} \Big(\min_{x\in C} u(t,tx)\Big) = 1 & \text{if } C \subset \mathcal{W}, \\ \displaystyle\lim_{t\to+\infty} \Big(\max_{x\in C} u(t,tx)\Big) = 0 & \text{if } C \subset \mathbb{R}^N \setminus \overline{\mathcal{W}}, \end{cases}$$

Comments

- Particular case U bounded ($\mathcal{B}(U) = \mathbb{S}^{N-1}$, $\mathcal{W} = B_{c^*}$): see II
- $\mathcal{W}$ is open, and either unbounded (iff $\mathcal{U}(U) \neq \emptyset$) or equal to B_{c^*}
- Freidlin-Gärtner formula $w(e) = \displaystyle\inf_{\xi \in \mathbb{S}^{N-1},\, \xi\cdot e>0} \frac{c^*(\xi)}{\xi \cdot e}$
  (spreading speeds in periodic equations with compactly supported u_0)
  $\overline{\mathcal{W}}$ is the Wulff shape of the envelop set of the minimal speeds $c^*(\xi)$
  $\Longrightarrow$ $\overline{\mathcal{W}}$ is a convex compact set
- Here, $\overline{\mathcal{W}}$ is neither convex nor bounded in general
- Spreading sets for (heterogeneous) reaction-diffusion equations with compactly supported u_0
  [Evans, Souganidis, 1989] [Barles, Evans, Souganidis, 1990] [Xin, 2000] [Weinberger, 2002] [Liang, Matano, 2014] [Berestycki, Nadin, 2022]
  Alternate approach: scaling $(t, x) = (t'/\varepsilon, x'/\varepsilon)$
  For non-compactly supported u_0 which are invariant by scaling
  [Gärtner, 1983] [Barles,Bronsard,Souganidis,1992]

Theorem 3 (upper level sets: local convergence)

Assume (1)

Notation, for $t > 0$, $\lambda \in (0,1)$:

$$E_\lambda(t) := \big\{x \in \mathbb{R}^N : u(t,x) > \lambda\big\}$$

Then, for every $R > 0$ and for every $0 < \lambda < 1$,

$$d_{\mathcal{H}}\Big(\overline{B_R} \cap \frac{E_\lambda(t)}{t}, \overline{B_R} \cap \mathcal{W}\Big) \underset{t\to+\infty}{\longrightarrow} 0$$

Remark: in general,

$$d_{\mathcal{H}}\Big(\frac{E_\lambda(t)}{t}, \mathcal{W}\Big) \not\to 0 \ \text{ as } t \to +\infty$$

Theorem 4 (upper level sets: global approximation)

Assume that

$$d_{\mathcal{H}}(U, U_\rho) < +\infty$$

Then, for every $0 < \lambda < 1$,

$$d_{\mathcal{H}}(E_\lambda(t), U + B_{c^* t}) = o(t) \ \text{ as } t \to +\infty$$

- Uniform interior ball condition of radius $\rho \Longrightarrow d_{\mathcal{H}}(U, U_\rho) < +\infty$
- Conditions $\mathcal{B}(U) \cup \mathcal{U}(U_\rho) = \mathbb{S}^{N-1}$ and $d_{\mathcal{H}}(U, U_\rho) < +\infty$ can not be compared in general
- Without them, the results fail in general, and the limits

$$\lim_{t \to +\infty} \frac{E_\lambda(t)}{t} = \mathcal{W} = \lim_{t \to +\infty} \frac{U}{t} + B_{c^*}$$

do not exist or do not coincide in general

Two steps of the proofs, using the maximum principle !

- Expanding sub-solutions emanating from $\mathbf{1}_{B_\rho(y)}$ with speeds $\geq c^* - \varepsilon$

- Retracting super-solutions in $B_{(c^*+\varepsilon)t}(y) \ \subset \mathbb{R}^N \setminus U$ with speeds $\leq c^* + \varepsilon$