Comparison and regularity results for nonlocal PDE with coercive Hamiltonians

Olivier Ley

IRMAR, INSA de Rennes, France
http://ley.perso.math.cnrs.fr

Collaboration with Adina Ciomaga (Paris), Minh Tri Le (Vienna), Erwin Topp (Santiago)

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Statement of the problem

Integro-differential Hamilton-Jacobi equation

$$\lambda u - I(u,x) + H(x,Du) = f(x) \quad \text{in } \mathbb{R}^N$$

the Hamiltonian satisfies

$$H(x,Du) \geq b|Du|^m - C, \quad b, C > 0, \boxed{m > 1}$$

and $I(u,x)$ is a nonlocal diffusion

$$I(u,x) = \int_{\mathbb{R}^N} [u(x+z) - u(x) - 1_B(z) Du(x) \cdot z] \nu_x(dz)$$

$\nu_x(dz)$ is a x-**dependent** nonnegative Borel measure satisfying

$$\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \min\{1, |z|^2\} \nu_x(dz) < \infty$$

- prove **comparison principle**
- prove **Hölder/Lipschitz regularity**

Examples of nonlocal operator

General measures

$$I(u,x) = \int_{\mathbb{R}^N} [u(x+z) - u(x) - 1_B(z) Du(x) \cdot z] \nu_x(dz)$$

Levy-Ito form

$$I(u,x) = \int_{\mathbb{R}^N} [u(x+j(x,z)) - u(x) - 1_B(z) Du(x) \cdot j(x,z)] \nu(dz)$$

Measures absolutely continuous with respect to Lebesgue measure

$$I(u,x) = \int_{\mathbb{R}^N} [u(x+z) - u(x) - 1_B(z) Du(x) \cdot z] K(x,z) dz$$

$0 \leq K(x,z) \leq \frac{C}{|z|^{N+\sigma}}$ ➪ nonlocal op. of order $\boxed{\sigma \in (0,2)}$

$K(x,z) = \frac{C}{|z|^{N+\sigma}}$ ➪ $I(u,x) = (-\Delta)^{\sigma/2} u(x)$ fractional Lapl.

Comparison principle, existing results

Sublinear/subquadratic type nonlinearities

- $\nu_x(\mathbb{R}^N) < \infty$ (no singularity) [Alvarez-Tourin 96], [Alibaud 07]
- $\int_{\mathbb{R}^N} |z|\nu_x(dz) < \infty$ "$\sigma < 1$" [Soner 86], [Awatif 91]
- Levy-Ito Form
  with local 2nd order terms [Barles-Imbert 08],
  quasilinear nonlocal equations [Chasseigne-Jakobsen 17]
- General measures
  $\sigma \in (0,2)$, a priori regularity for sol's [Mou-Swiech 15]
  $\sigma < 1$, optimal transport tools [Guillen-Mou-Swiech 19]

Superlin./coercive nonlinearity $\boxed{|H(x, Du)| \geq |Du|^m - C, \; m > 1}$

- Levy-Ito form [Barles-Koike-OL-Topp 15]
- Levy-Ito form with 2nd order terms [Barles-OL-Topp 17]
- General measures, $\sigma < 1$, [Barles-Topp 16]

$\lambda u - I(u,x) + H(x, Du) = f(x)$ in $\mathbb{R}^N$
Assumptions (A-I)

$$I(x,u) = \int_{\mathbb{R}^N} [u(x+z) - u(x) - 1_B(z) Du(x) \cdot z] \nu_x(dz)$$

Levy condition : $\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \min\{1, |z|^2\} \nu_x(dz) < \infty$

To deal with infinity : $\sup_{x \in \mathbb{R}^N} \int_{B_R^C} \nu_x(dz) = m(R) \underset{R \to \infty}{\to} 0$

Regularity wrt x : $\exists$ modulus $\omega(r) \underset{r \to 0}{\to} 0$ s.t. $\forall r > 0$,

$$\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \min\{1, |z|^2\} |\nu_x - \nu_y|(dz) \leq \omega(r)|x-y|$$

Typical example : $\nu_x(dz) = K(x,z)dz$, $\sigma \in (0,2)$
$0 \leq K(x,z) \leq \frac{C}{|z|^{N+\sigma}}, \quad |K(x,z) - K(y,z)| \leq \frac{C}{|z|^{N+\sigma}} |x-y|$

$\lambda u - I(u,x) + H(x,Du) = f(x)$ in $\mathbb{R}^N$
Assumptions (A-H) for $H(x,Du)$, $f(x)$

Regularity for f : bounded and Lipschitz continuous.

Coercivity and superlinearity $\boxed{m>1}$ $\forall\, 0<\mu<1$

$$\mu H(x,\mu^{-1}p) - H(x,p) \geq (1-\mu)(|p|^m - C),$$

Regularity wrt x : $\exists$ modulus m s.t.

$$H(y,p+q)-H(x,p) \leq \omega(|x-y|)(1+|p|^m)+\omega(|q|)(1+|p|^{m-1}).$$

Typical example : $H(x,Du) = b(x)|Du|^m$ with $b(x) > \underline{b} > 0$ bounded and Lipschitz continunous

Comparison result

Theorem Let $\sigma \in (0, 2)$ and assume **(A-I)-(A-H)**.
Let u and v be respectively a bounded USC viscosity subsolution and a bounded LSC supersolution.
Then $u \leq v$ in $\mathbb{R}^N$.

Corollary There exists a unique bounded viscosity solution.

Remarks

- local case $I(u, x) \to \mathrm{Trace}(A(x)D^2u)$
  [Ishii-Lions 90], [Alvarez 96,97], [Da Lio-OL 06], [Koike-OL 11]...
- possible to add 2nd order local terms, to consider evolution equations...

Proof for $u - I(u,x) + |Du|^m = f(x)$ in $\mathbb{R}^N$
Preliminaries

- $\bar{u} = \mu u, \quad 0 < \mu < 1, \quad \mu \approx 1$

$$u - I(u,x) + |Du|^m \leq f \Rightarrow \bar{u} - I(\bar{u},x) + \frac{1}{\mu^{m-1}}|D\bar{u}|^m \leq \mu f$$

- By contradiction, assume

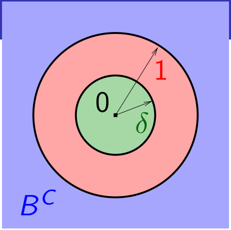
$$0 < M := \sup_{x,y \in \mathbb{R}^N} \bar{u}(x) - v(y) - \underbrace{\frac{|x-y|^2}{\epsilon^2}}_{=:\phi(x,y)} - \text{localization terms}$$

$$M = \bar{u}(\bar{x}) - v(\bar{y}) - \frac{|\bar{x}-\bar{y}|^2}{\epsilon^2}$$

$$\frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} = o_\epsilon(1) \underset{\epsilon \to 0}{\to} 0$$

$$D^+\bar{u}(\bar{x}) = D_x\phi(\bar{x},\bar{y}) = -D_y\phi(\bar{x},\bar{y}) = D^-v(\bar{y}) = 2\frac{\bar{x}-\bar{y}}{\epsilon^2} =: \bar{p}$$

Proof. Viscosity inequalities (1)



$$I(\bar u,\bar x) = I^{\delta}(\bar x) + I^{\delta 1}(\bar x) + I^{1}(\bar x)$$

$$I^{\delta}(\bar x) := \int_{B_\delta} [\phi(\bar x+z,\bar y) - \phi(\bar x,\bar y) - \bar p\cdot z]\nu_{\bar x}(dz)$$

$$I^{\delta 1}(\bar x) := \int_{B\setminus B_\delta} [\bar u(\bar x+z) - \bar u(\bar x) - \bar p\cdot z]\nu_{\bar x}(dz)$$

$$I^{1}(\bar x) := \int_{B^C} [\bar u(\bar x+z) - \bar u(\bar x) - 1_B(z) D\bar u(\bar x)\cdot z]\nu_{\bar x}(dz)$$

$\bar u$ subsolution ➪

$$\bar u(\bar x) - I^{\delta}(\bar x) - I^{\delta 1}(\bar x) - I^{1}(\bar x) + \frac{1}{\mu^{m-1}}|\bar p|^m \le \mu f(\bar x)$$

v supersolution ➪

$$v(\bar y) - I^{\delta}(\bar y) - I^{\delta 1}(\bar y) - I^{1}(\bar y) + |\bar p|^m \ge f(\bar y)$$

Proof. Viscosity inequalities (2)

At the end :

$$0 < M \leq \bar{u}(\bar{x}) - v(\bar{y}) + \left(\frac{1}{\mu^{m-1}} - 1\right)|\bar{p}|^m \leq I^\delta + I^{\delta 1} + I^1 + \mu f(\bar{x}) - f(\bar{y})$$

where

$$I^\delta = \int_{B_\delta} [\phi(\bar{x}+z,\bar{y}) - \phi(\bar{x},\bar{y}) - \bar{p}\cdot z]\nu_{\bar{x}}(dz) - \int_{B_\delta} [\phi(\bar{x},\bar{y}+z) - \phi(\bar{x},\bar{y}) - \bar{p}\cdot z]\nu_{\bar{y}}(dz)$$

$$I^{\delta 1} = \int_{B\setminus B_\delta} [\bar{u}(\bar{x}+z) - \bar{u}(\bar{x}) - \bar{p}\cdot z]\nu_{\bar{x}}(dz) - \int_{B\setminus B_\delta} [v(\bar{y}+z) - v(\bar{y}) - \bar{p}\cdot z]\nu_{\bar{y}}(dz)$$

$$I^1 = \int_{B^C} [\bar{u}(\bar{x}+z) - \bar{u}(\bar{x})]\nu_{\bar{x}}(dz) - \int_{B^C} [v(\bar{y}+z) - v(\bar{y})]\nu_{\bar{y}}(dz)$$

Goal : to reach a contradiction

Proof. Estimate of H and f

$$1 \approx \mu < 1 \;\Rightarrow\; \boxed{(\frac{1}{\mu^{m-1}} - 1)|\bar{p}|^m \geq b_m(1-\mu)|\bar{p}|^m}$$

$$\begin{aligned}\mu f(\bar{x}) - f(\bar{y}) &= (\mu - 1)f(\bar{x}) + f(\bar{x}) - f(\bar{y}) \\ &\leq (1-\mu)||f||_\infty + ||Df||_\infty |\bar{x} - \bar{y}|\end{aligned}$$

$$\Rightarrow\; \boxed{\mu f(\bar{x}) - f(\bar{y}) \leq C(1-\mu) + o_\epsilon(1)}$$

Proof. Estimate of I^δ (smoothness of ϕ)

Same estimate for $I^\delta(\bar{x})$ and $I^\delta(\bar{y})$

$$I^\delta(\bar{x}) := \int_{B_\delta} [\phi(\bar{x}+z,\bar{y}) - \phi(\bar{x},\bar{y}) - \bar{p}\cdot z]\nu_{\bar{x}}(dz)$$

$$\begin{aligned}
&\phi(\bar{x}+z,\bar{y}) - \phi(\bar{x},\bar{y}) - \bar{p}\cdot z \\
&\qquad = \phi(\bar{x}+z,\bar{y}) - \phi(\bar{x},\bar{y}) - D_x\phi(\bar{x},\bar{y})\cdot z \\
&\qquad = \tfrac{1}{2} D^2_{xx}\phi(\bar{x},\bar{y})z\cdot z + |z|^2 o_\delta(1)
\end{aligned}$$

with $||D^2_{xx}\phi|| \leq \frac{2}{\epsilon^2}$

➪ $|I^\delta(\bar{x})|, |I^\delta(\bar{y})| \leq \dfrac{C}{\epsilon^2} \sup_{x\in\mathbb{R}^N} \displaystyle\int_{B_\delta} |z|^2 \nu_x(dz) \leq \dfrac{o_\delta(1)}{\epsilon^2}$

➪ $\boxed{I^\delta = I^\delta(\bar{x}) - I^\delta(\bar{y}) \leq \frac{o_\delta(1)}{\epsilon^2}}$

Proof. Estimate of I^1 (finiteness of the measure outside B)

$$I^1 = I^1(\bar{x}) - I^1(\bar{y})$$

$$= \int_{B^c} [\bar{u}(\bar{x}+z) - \bar{u}(\bar{x})]\nu_{\bar{x}}(dz) - \int_{B^c} [v(\bar{y}+z) - v(\bar{y})]\nu_{\bar{y}}(dz)$$

$$= \int_{B^c} \underbrace{[(\bar{u}(\bar{x}+z) - v(\bar{y}+z)) - (\bar{u}(\bar{x}) - v(\bar{y}))]}_{\leq 0 \ (*)} \underbrace{\nu_{\bar{x}}(dz)}_{\geq 0}$$

$$+ \int_{B^c} [v(\bar{y}+z) - v(\bar{y})](\nu_{\bar{x}}(dz) - \nu_{\bar{y}}(dz))$$

$$\leq 2||v||_\infty \Big(\underbrace{\int_{1\leq |z| \leq R} |\nu_{\bar{x}} - \nu_{\bar{y}}|(dz)}_{\leq \omega(R)|\bar{x}-\bar{y}|} + \underbrace{\int_{|z|\geq R} \nu_{\bar{x}}(dz) + \int_{|z|\geq R} \nu_{\bar{y}}(dz)}_{\leq m(R)\to 0 \text{ as } R\to\infty} \Big)$$

$$\boxed{I^1 \leq \omega(R)o_\epsilon(1) + m(R)} \quad \forall R > 0$$

$(*)$ $(\bar{x}, \bar{y})$ maximum point of $\bar{u}(x) - v(y) - \frac{|x-y|^2}{\epsilon^2}$

➪ $\bar{u}(\bar{x}+z) - v(\bar{y}+z) - \frac{|(\bar{x}+z)-(\bar{y}+z)|^2}{\epsilon^2} \leq \bar{u}(\bar{x}) - v(\bar{y}) - \frac{|\bar{x}-\bar{y}|^2}{\epsilon^2}$

Proof. A first estimate of $I^{1\delta}$ (in annulus $A_\delta := B \setminus B_\delta$) (1)

Hahn-Jordan decomposition of the signed measure $\bar\nu := \nu_{\bar x} - \nu_{\bar y}$

$\bar\nu = \bar\nu^+ - \bar\nu^-$ with $\bar\nu^+, \bar\nu^- \geq 0$

Assumption **(A-I)** ➪ $\int_{B_\delta} |z|^2 \bar\nu^\pm(dz) \leq o_\delta(1)|\bar x - \bar y|$

$$\tilde\nu := (1 - 1_{\operatorname{supp}\bar\nu^+})\bar\nu^+ + 1_{\operatorname{supp}\bar\nu^+}\bar\nu^+ \geq 0$$

$$\begin{aligned} I^{\delta 1} &= \int_{A_\delta} [\bar u(\bar x + z) - \bar u(\bar x) - \bar p \cdot z]\nu_{\bar x}(dz) \\ &\quad - \int_{A_\delta} [v(\bar y + z) - v(\bar y) - \bar p \cdot z]\nu_{\bar y}(dz) \\ &= \int_{A_\delta} \underbrace{[\bar u(\bar x + z) - v(\bar y + z) - (\bar u(\bar x) - v(\bar y))]}_{\leq 0 \text{ (maximum point)}} \underbrace{\tilde\nu(dz)}_{\geq 0} \end{aligned}$$

$$+ \int_{A_\delta} [\bar u(\bar x + z) - \bar u(\bar x) - \bar p \cdot z]\bar\nu^+(dz) - \int_{A_\delta} [v(\bar y + z) - v(\bar y) - \bar p \cdot z]\bar\nu^+(dz)$$

Two last terms are delicate.

Proof. A first estimate of $I^{1\delta}$ (in annulus $A_\delta := B \setminus B_\delta$) (2)

$(\bar{x}, \bar{y})$ maximum point of $\bar{u}(x) - v(y) - \frac{|x-y|^2}{\epsilon^2}$

$\Rightarrow$ $\bar{u}(\bar{x}+z) - v(\bar{y}) - \frac{|\bar{x}+z-\bar{y}|^2}{\epsilon^2} \leq \bar{u}(\bar{x}) - v(\bar{y}) - \frac{|\bar{x}-\bar{y}|^2}{\epsilon^2}$

$$\int_{A_\delta} [\bar{u}(\bar{x}+z) - \bar{u}(\bar{x}) - \bar{p}\cdot z]\bar{\nu}^+(dz)$$

$$\leq \int_{A_\delta} [\frac{|\bar{x}-\bar{y}+z|^2}{\epsilon^2} - \frac{|\bar{x}-\bar{y}|^2}{\epsilon^2} - 2\frac{\bar{x}-\bar{y}}{\epsilon^2}\cdot z]\bar{\nu}^+(dz)$$

$$\leq \frac{1}{\epsilon^2}\int_{A_\delta} |z|^2 \bar{\nu}^+(dz) \leq \omega(1)\frac{|\bar{x}-\bar{y}|}{\epsilon^2} = C|\bar{p}|$$

$$\boxed{I^{\delta 1} \leq C|\bar{p}|}$$

Remark

Actually we did a "naive" proof.

In the local case, it would consists in doing

$$\mathrm{trace}(A(\bar x)X) - \mathrm{trace}(A(\bar x)Y)$$

$$= \mathrm{trace}(\underbrace{(A(\bar x) - A(\bar x))}_{||...||\leq C|\bar x - \bar y|}\underbrace{X}_{||X||\leq \frac{C}{\epsilon^2}}) + \mathrm{trace}(A(\bar y)\underbrace{(X-Y)}_{\leq 0})$$

$$\leq C\frac{|\bar x - \bar y|}{\epsilon^2} = C|\bar p|$$

Ishii-Jensen Lemma gives a better result :
if $A^{1/2}$ is Lipschitz continuous, then

$$\mathrm{trace}(A(\bar x)X) - \mathrm{trace}(A(\bar x)Y) \leq \frac{||A(\bar x)^{1/2} - A(\bar x)^{1/2}||}{\epsilon^2} \leq C\frac{|\bar x - \bar y|^2}{\epsilon^2} \underset{\epsilon\to 0}{\to} 0$$

Proof. Boundedness of $\bar{p}$

Plugging the estimates into the viscosity inequalities

$$M + (1-\mu)|\bar{p}|^m$$
$$\leq \frac{o_\delta(1)}{\epsilon^2} + \omega(R)o_\epsilon(1) + m(R) + C|\bar{p}| + C(1-\mu) + o_\epsilon(1)$$

Fix R big enough such that $m(R) \leq \frac{M}{2}$, send $\delta \to 0$

➪ $(1-\mu)|\bar{p}|^m \leq C(1+|\bar{p}|)$

➪ $|\bar{p}| \leq C$ with C independent of ϵ

Proof. A second estimate of $I^{1\delta}$ (in annulus $A_\delta := B \setminus B_\delta$)

$$\forall \delta < \rho < 1, \quad A_\delta = \{\delta \le |z| \le \rho\} \cup \{\rho < |z| < 1\}$$

$$\int_{A_\delta} [\bar{u}(\bar{x}+z) - \bar{u}(\bar{x}) - \bar{p}\cdot z]\bar{\nu}^+(dz)$$

$$\le \int_{\delta \le |z| \le \rho} \frac{|z|^2}{\epsilon^2}\bar{\nu}^+(dz) + \int_{\rho \le |z| \le 1} \underbrace{[\bar{u}(\bar{x}+z) - \bar{u}(\bar{x}) - \bar{p}\cdot z]}_{\text{bounded because } |\bar{p}| \le C} \bar{\nu}^+(dz)$$

$$\le \omega(\rho)|\bar{p}| + \frac{C}{\rho^2}\int_{\rho \le |z| \le 1} |z|^2 \bar{\nu}^+(dz)$$

➪ $\boxed{I^{\delta 1} \le C\omega(\rho) + C\omega(1)\rho^{-2}|\bar{x}-\bar{y}| = o_\rho(1) + \rho^{-2}o_\epsilon(1)}$

Proof. Conclusion

Coming back to the viscosity inequalities with this new estimate

$$\frac{M}{2} < \frac{M}{2} + (1-\mu)|\bar{p}|^m \\ \leq \frac{o_\delta(1)}{\epsilon^2} + \omega(R)o_\epsilon(1) + o_\rho(1) + \rho^{-2}o_\epsilon(1) + C(1-\mu) + o_\epsilon(1)$$

Fix ρ small enough and μ close enough to 1 such that $o_\rho(1) + C(1-\mu) < \frac{M}{4}$

➪ $0 < \frac{M}{4} \leq \frac{o_\delta(1)}{\epsilon^2} + \omega(R)o_\epsilon(1) + \rho^{-2}o_\epsilon(1)$

Send $\delta \to 0$ then $\epsilon \to 0$ to reach a contradiction. This ends the proof.

Regularity results (degenerate equations)

Sublinear nonlinearities, elliptic/parab. equations

- Hölder/Lipschitz estimates

[Barles-Chasseigne-Imbert 11], [Barles-Chasseigne-Ciomaga-Imbert12]

- More regularity $C^{1,\alpha}/C^{\sigma,\alpha}$ [Caffarelli-Silvestre 09,11], [Silvestre 11], [Ros Oton-Serra 14], [Serra 15]

Superlin./coercive nonlinearity, degenerate equations

$$\boxed{|H(x,Du)| \geq |Du|^m - C,\ m>1}$$

- Levy-Ito form

Hölder continuity [Cardaliaguet-Rainer 11]

Lipschitz continuity [Barles-OL-Topp 17] (weak Bernstein method)

- General operator of order $\sigma \in (0,2)$

subsolutions are $\frac{m-\sigma}{m-1}$-Hölder [Barles-Koike-OL-Topp 17]
($\frac{m-2}{m-1}$ in local case [Capuzzo Dolcetta-Leoni-Porretta 10])
solutions are $\frac{m-\sigma+1}{m}$-Hölder [Ciomaga-Le-OL-Topp, in progress]