Non coercive unbounded deterministic Mean Field Games

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Cortona, 30 May 2022

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Plan

- Introduction to the classical MFG problem.
- Some non-coercive cases with unbounded coefficients:
  1. the player has forbidden directions;
  2. the player controls the acceleration.

A brief introduction to Mean Field Games

The Mean Field Games model (MFG) was proposed by Lasry-Lions, and independently by Huang-Malhamé-Caines, in 2006.

The MFG model describes interactions among a very large number of identical agents.

Aim of MFG theory is to relate individual actions to mass behavior.

Deterministic Mean Field Game

$$\begin{cases} (HJ) & -\partial_t u + H(x, Du) = F(x, m), & R^n \times (0, T) \\ (C) & \partial_t m - div(m\, D_p H(x, Du)) = 0, & R^n \times (0, T) \\ & m(x, 0) = m_0(x),\ u(x, T) = G(x, m(T)), & R^n \end{cases}$$

• u is the value function of the generic player:
dynamics $\quad x'(s) = \alpha,\ x(t) = x$
cost $\quad J_t^m(\alpha, x) = \int_t^T \frac{1}{2}|\alpha|^2 + F(x(s), m(x(s), s))ds + G(x(T), m(x(T), T))$

• m is the density distribution of the agents

• F and G encode the interactions between the agent and the whole population.

• H is strictly convex w.r.t p (ex. $H(x, p) = \frac{1}{2}|p|^2$).

Coercive case: PL Lions, lecture notes by P. Cardaliaguet

- Dynamics of the generic agent: the player controls his/her velocity

$$x'(s) = \alpha(s), \qquad x(t) = x, \quad s \in [t, T].$$

- The player can move in every direction and in all $\mathbb{R}^n$.

- $H(x, p) = \frac{1}{2}|p|^2$

- F and G strongly regularizing nonlocal terms: $F[m]$, $G[m]$.

MAIN RESULT (PL Lions, notes by P. Cardaliaguet)

- system MFG has a solution (u, m);
- m is the image of the initial distribution through a flow defined by the optimal control problem associated to the Hamilton Jacobi equation: $\begin{cases} x'(s) = -Du(x(s), s), \\ x(0) = x. \end{cases}$

Basic References for MFG theory:

- Lasry, Lions, C.R.A.S. (2006), Jpn. J. Math. 2 (2007),
- Huang, Malhamé, Caines, Commun. Inf. Syst. (2006).
- Cardaliaguet, Notes on Mean Field Games (from Lions lectures at College de France), `www.ceremade.dauphine.fr/ cardalia/`, `www.college-de-france.fr`
- Achdou, Lecture Notes in Math., CIME Found. (2013)
- Bensoussan, Frehse, Yam, "Mean field games and mean field type control theory", Springer (2013)
- Gomes, Pimentel, Voskanyan, Regularity Theory for MFG-systems, Springer (2016)

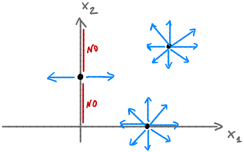
Class 1. First example: forbidden direction in some points.

$x = (x_1, x_2) \in \mathbb{R}^2$, $p = (p_1, p_2)$.

- Dynamics of the player $\begin{cases} x_1'(s) = \alpha_1(s), \\ x_2'(s) = x_1(s)\,\alpha_2(s) \end{cases}$
- Hamiltonian $H(x,p) = \frac{1}{2}(p_1^2 + x_1^2 p_2^2)$ is not coercive and unbounded;

$$B(x) = \begin{pmatrix} 1 & 0 \\ 0 & x_1 \end{pmatrix} \to \quad x' = \alpha B, \quad H(x,p) = \frac{|pB(x)|^2}{2}.$$

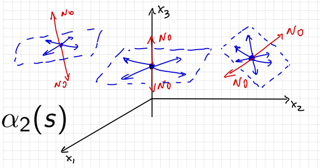
Forbidden direction in some points:



Class 1. Second example: forbidden direction at EVERY point $\to$ Heisenberg example

$x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $p = (p_1, p_2, p_3)$ $\qquad B(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -x_2 & x_1 \end{pmatrix}$

- Dynamics $\begin{cases} x_1'(s) = \alpha_1(s), \\ x_2'(s) = \alpha_2(s), \\ x_3'(s) = -x_2(s)\,\alpha_1(s) + x_1(s)\,\alpha_2(s) \end{cases}$



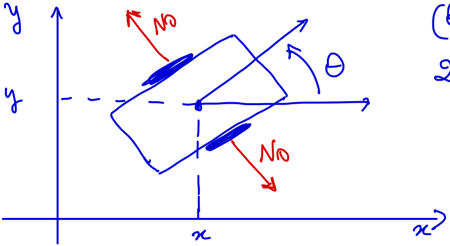
  We can move only tangent to the plane generated by $(1, 0, -x_2)$, $(0, 1, x_1)$.

  Nonholonomic integrator.

- Hamiltonian: $H(x, p) = \frac{1}{2}((p_1 - x_2 p_3)^2 + (p_2 + x_1 p_3)^2)$ is not coercive and unbounded.

- $\det BB^T(x) = 0$, for any $x \in \mathbb{R}^3$.

Car parking



(θ, x, y)

2 choices of mouvement

$$\begin{cases} \dot{\theta} = \alpha_1 \\ \dot{x} = \cos\theta\, \alpha_2 \\ \dot{y} = \sin\theta\, \alpha_2 \end{cases}$$

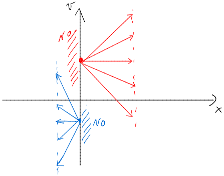
$$B = \begin{pmatrix} 1 & 0 \\ 0 & \cos\theta \\ 0 & \sin\theta \end{pmatrix}$$

Forbidden direction

State variable $y = (x, v) \in \mathbb{R}^{2n}$, x is the position, v is the velocity.

Dynamics of the generic player is controlled by the acceleration (double integrator).

$$\begin{cases} x'(s) = v(s), \\ v'(s) = \alpha(s), \\ x(t) = x, \ v(t) = v. \end{cases}$$



The control α is involved only in the second component of the state variable.
Hamiltonian $\quad H(x, v, p_x, p_v) = -v \cdot p_x + \frac{|p_v|^2}{2} \quad$ is not coercive with respect to $p = (p_x, p_v)$ and unbounded w.r.t. v.

Class 1: the MFG system in the Heisenberg example

$$B(x) := \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -x_2 & x_1 \end{pmatrix}, \quad \text{for } x = (x_1, x_2, x_3).$$

MFG system

$$\begin{cases} (HJ) & -\partial_t u + \frac{|DuB(x)|^2}{2} = F[m(t)](x) \\ (C) & \partial_t m - \text{div}(m\, DuB(x)B(x)^T) = 0 \\ & m(x,0) = m_0(x), \quad u(x,T) = G[m(T)](x), \end{cases}$$

$p = (p_1, p_2, p_3)$

$$|pB(x)|^2 = ((p_1 - x_2 p_3)^2 + (p_2 + x_1 p_3)^2),$$
$$pB(x)B(x)^T = (p_1 - x_2 p_3, p_2 + x_1 p_3, -p_1 x_2 + p_2 x_1 + p_3(x_1^2 + x_2^2)).$$

Weak solution

- $u \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^3 \times [0,T])$.
- $m \in C([0,T]; \mathcal{P}_1(\mathbb{R}^3))$; $\forall t \in [0,T]$, $m(t)$ is AC w.r.t. Lebesgue measure and m is bounded.
- (HJ)-equation is satisfied by u in the viscosity sense.
- (C)-equation is satisfied by m in the sense of distributions.

Assumptions

- $F[\cdot]$ and $G[\cdot]$ are regularizing nonlocal coupling with: $\|F[m]\|_{C^2}$, and $\|G[m]\|_{C^2} \leq C$, $\forall m \in \mathcal{P}_1$
- the initial m_0 has a compact support.

Main Theorem

System MFG has a weak solution (u, m)

Existence of weak solution: difficulties

The lack of coercivity of H prevents the application of classical results.

- Degenerate Hamiltonian due to the dynamics of the generic player $\rightarrow$ non uniqueness of the optimal trajectories $\rightarrow$ we cannot prove a contraction property of the flow associated to the dynamics;
- Unbounded Hamiltonian and unbounded drift with quadratic growth in the continuity equation.

Existence of weak solution (Heisenberg example)

- Study of the problem associated to the completion of B:

$$B^\varepsilon(x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_2 & x_1 & \varepsilon \end{pmatrix}$$

- Existence of the solution $(u^\varepsilon, m^\varepsilon)$
- Optimal synthesis for the optimal control associated to u^ε;
- Representation formula for m^ε (Ambrosio, Gigli, Savarè);
- Uniform estimates (in particular: uniform semiconcavity and Lipschitz continuity of u^ε):
- Convergence to a weak solution (u, m).

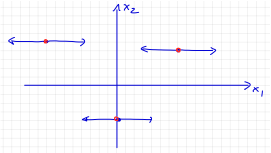
KEY ASSUMPTIONS:

1. Compactness of initial distribution of players.
2. Sublinear growth of the coefficients of B.

Generalization

- We do not need Hörmander condition, only sublinear growth of the coefficients and a particular structure of the $n \times m$ matrix B.

- Completely degenerate case: $B(x) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$



$$\begin{cases} x_1'(s) = \alpha_1(s), \\ x_2'(s) = 0 \end{cases}$$

Hamiltonian $\quad H(x,p) = \frac{1}{2} p_1^2$

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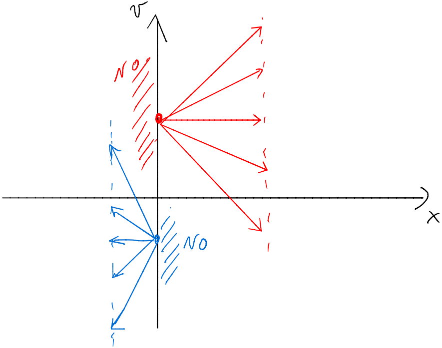
MFGs with control on acceleration, Achdou, M., Marchi, Tchou, NoDEA 2020)

State variable $y = (x, v) \in R^{2n}$, x is the position, v is the velocity.

Dynamics of the generic player is controlled by the acceleration (double integrator).

$$\begin{cases} x'(s) = v(s), \\ v'(s) = \alpha(s), \\ x(t) = x, \;\; v(t) = v. \end{cases}$$



The control α is involved only in the second component of the state variable.

Hamiltonian

$$H(x, v, p_x, p_v) = -p_x \cdot v + \frac{|p_v|^2}{2} - \frac{|v|^2}{2}$$

- is not coercive with respect to $p = (p_x, p_v)$;
- is unbounded, the term $\frac{1}{2}|v|^2$ stands for kinetic energy.

The MFG system

State variable: (x, v). $\quad u = u(x, v, t),\ m = m(x, v, t)$.

$$(MFG)\begin{cases} -\partial_t u - v \cdot D_x u + \frac{1}{2}|D_v u|^2 - \frac{1}{2}|v|^2 = F[m(t)](x, v), \\ \partial_t m - v \cdot D_x m - div_v(m\, D_v u) = 0, \\ m(x, v, 0) = m_0(x, v),\ u(x, v, T) = G[m(T)](x, v). \end{cases}$$

Drift in the continuity equation $\to (-v, D_v u)$. $\quad (|D_v u| \le C(1 + |v|))$

Theorem 1.

Under suitable assumptions,

- *system MFG has a solution* (u, m)*;*
- $m(x, v, s)$ *is the image of* m_0 *through the characteristic flow* $\Phi(x, v, 0, s)$ *associated to*

$$\begin{cases} x'(s) = v(s), & x(0) = x, \\ v'(s) = -D_v u(x(s), v(s), s), & v(0) = v. \end{cases}$$

DIFFICULTIES

- Degenerate Hamiltonian due to the dynamics of the generic player $\to$ we cannot prove that the associated flow has Lipschitz continuous inverse;
- Unbounded Hamiltonian w.r.t. the variable v: some estimates of the value function hold only locally w.r.t v ($|u(x, v, t)| \leq C(1 + |v|^2)$); results for equations with terms of quadratic growth.

Directions and perspectives

- General MFG with control on the acceleration and state constraints (SIAM J. Math. Analysis 2022).
- First and second order MFG with degenerate Hamiltonian under general Hörmander conditions (joint project with C. Marchi, C. Mendico).
- MFG on networks (joint project with Y. Achdou, C. Marchi, N. Tchou).

Thank You!

Structure of the matrix $n \times m$, $n \geq m$

$$B(x) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & x_1 & 0 \\ x_1 & x_2 & 0 \\ x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 \end{pmatrix} \qquad 5 \times 3 \text{ example, } x = (x_1, x_2, x_3, x_4, x_5)$$

The solution m is the push-forward of the initial distribution m_0 by the flow Φ.

Representation formula for m: interpretation of the evolution of the population's density as the push-forward of the initial distribution through a flow defined a.e. by the optimal control problem associated to the Hamilton Jacobi equation.

Push-forward of the measure m_0 by the flow Φ:

$$\int_{R^n} \varphi(x)m(x,t)dx = \int_{R^n} \varphi(\Phi(x,0,t))m_0(x)\,dx, \quad \forall \varphi \in C_0^0(R^n).$$

(or equivalently $m(A) = m_0(\Phi(\cdot,0,s)^{-1}(A))$, $\forall A \subset R^n$.

Control on the acceleration: Proof

1. We fix $\overline{m}$ in a suitable space and we find properties of u solution of

$$HJ : \begin{cases} -\partial_t u - v \cdot D_x u + \frac{1}{2}|D_v u|^2 - \frac{1}{2}|v|^2 = l(x, v, \overline{m}), \\ u(x, v, T) = G(x, v, \overline{m}(T)) \end{cases}$$

2. We find an unique $\widetilde{m}$ such that

$$\text{Continuity equation:} \begin{cases} \partial_t m + v \cdot D_x m - div_v(mD_v u) = 0 \\ m(x, v, 0) = m_0(x, v). \end{cases}$$

Fixed point: $\overline{m} \to \mathcal{T}(\overline{m}) = \widetilde{m}$.

Control on the acceleration: Difficulties

- Degenerate Hamiltonian due to the dynamics of the generic player $\to$ we cannot prove a contraction property of the flow associated to the dynamics;
- Unbounded Hamiltonian: some estimates of the value function hold only locally w.r.t v ($|u(x,v,t)| \leq C(1+|v|^2)$);
- Unbounded drift $(-v, D_v u)$ in the continuity equation ($\partial_t m - \mathrm{div}(m\,(-v, D_v u)) = 0$).

Control on the acceleration: properties of the optimal control problem

1. $u(x, v, t)$ is Lipschitz continuous, w.r.t. v and t locally in v; $|D_v u| \leq C(1 + |v|)$.
2. $u(x, v, t)$ is semiconcave w.r.t. (x, v).
3. $x(\cdot)$ optimal trajectory for $u(x, t)$, $\alpha(\cdot)$ optimal control law; then the restriction of α to $[r, T]$ $(r > t)$ is the unique optimal control with initial condition $(x(r), v(r))$ at time r.

Also here we cannot prove that the inverse of the flow Φ associated to the dynamics is Lipschitz continuous.

Uniqueness and the representation of m

Let m be any solution to the continuity equation. ($e_t(\gamma) = \gamma(t)$ is the evaluation map). Then, by

- superposition principle (Ambrosio, Gigli, Savarè)
- disintegration theorem
- the drift is $(v, -D_v u)$ (and it has a linear growth at infinity)
- optimal synthesis

$$\begin{aligned} \int_{\mathbb{R}^{2N}} \psi \, dm_t &= \int_\Gamma \psi(e_t(\gamma)) d\eta(\gamma) \\ &= \int_{\mathbb{R}^{2N}} \left(\int_{e_0^{-1}(x,v)} \psi(e_t(\gamma)) d\eta_{(x,v)}(\gamma) \right) dm_0(x,v) \\ &= \int_{\mathbb{R}^{2N}} \psi(\gamma(t)) dm_0(x,v) \quad \forall \psi \in C_0^0(\mathbb{R}^{2N}) \end{aligned}$$

where $\gamma(\cdot) = (x(\cdot), v(\cdot))$ solves

$$\begin{cases} x'(s) = v(s), & x(0) = x, \\ v'(s) = -D_v u(x(s), v(s), s), & v(0) = v. \end{cases}$$