On Bernstein type theorems for minimal graphs under Ricci lower bounds

joint works with G. Colombo, M. Magliaro and M. Rigoli

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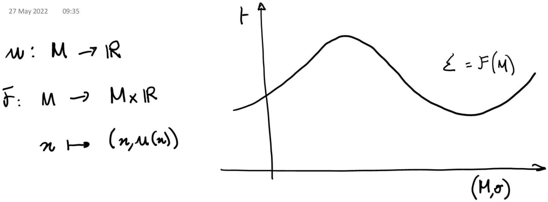
(M,σ) complete Riemannian manifold, dimension m



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$$\begin{array}{c}\Sigma \text{ is minimal} \\ \text{(stationary for area)}\end{array} \iff \mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0 \ \text{ on } M$$

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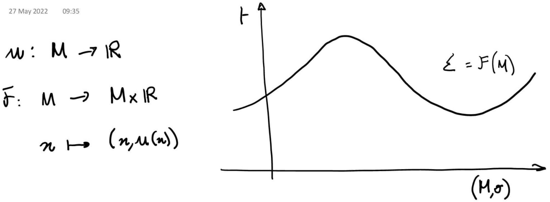

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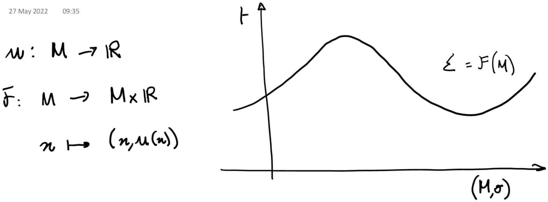

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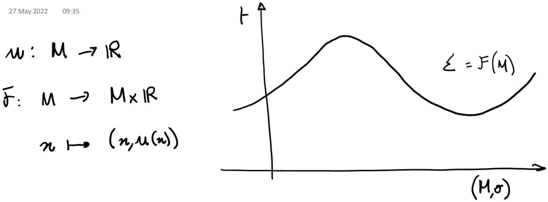

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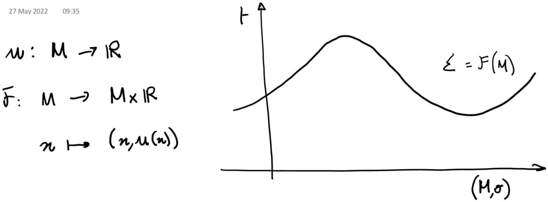

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Minimal graphs over $\mathbb{R}^m$

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GENERALIZED BERNSTEIN THEOREM: property

$$(\mathscr{B}1) \quad \text{all solutions to (MS) on } \mathbb{R}^m \text{ are affine}$$

holds if and only if $m \leq 7$.
(Bernstein '15, De Giorgi '65, Almgren '66, Simons '68, Bombieri-De Giorgi-Giusti '69)

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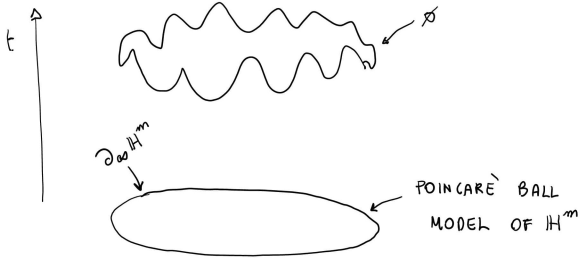

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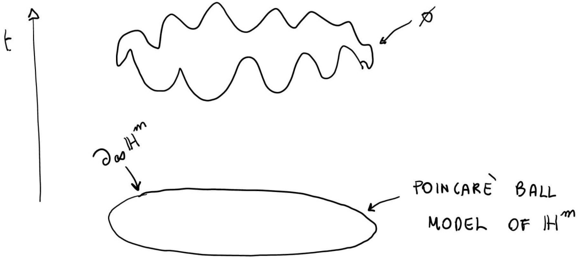

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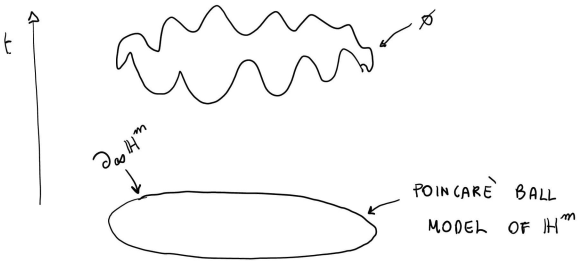

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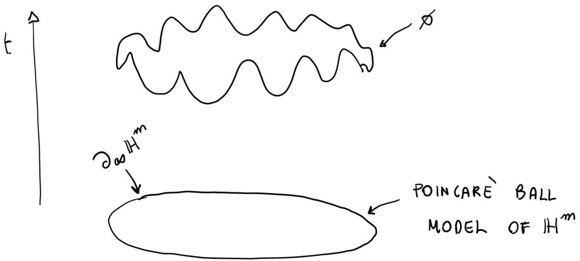

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- Ricci curvature

$$\mathrm{Ric}(X) = \sum_{j=1}^{m-1} \mathrm{Sec}(X \wedge e_j), \qquad |X| = 1, \{e_j\} \text{ o.n. basis for } X^{\perp}.$$

$(\mathscr{B}1)$, $(\mathscr{B}2)$, $(\mathscr{B}3)$ might hold if $\mathrm{Ric} \geq 0$:

Examples: $\mathbb{S}^p \times \mathbb{R}^q$, Menguy's examples, Eguchi-Hanson metric on $T^*\mathbb{S}^2$, gravitational instantons, ...

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1) Cheeger-Colding's theory is available: if $o \in M$, $\lambda_j \to +\infty$, then

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THE CASE OF SLOW VOLUME GROWTH

Theorem

Let M^m be complete, $\mathrm{Ric} \geq 0$. Fix $o \in M$ and assume that

$$\int^{\infty} \frac{r}{|B_r(o)|} \mathrm{d}r = +\infty \tag{1}$$

Let u be a non-constant solution of (MS). Then,

- *$M = N \times \mathbb{R}$ with the product metric $\sigma_N + \mathrm{d}s^2$,*
- *In the variables $(x, s) \in N \times \mathbb{R}$ it holds $u(x, s) = as + b$ for some $a, b \in \mathbb{R}$.*

Thus, ($\mathscr{B}1$) holds (hence, ($\mathscr{B}2$) and ($\mathscr{B}3$)).

In particular, it applies to surfaces with $\mathrm{Sec} \geq 0$.

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PROPERTIES $(\mathscr{B}2)$, $(\mathscr{B}3)$ IF $\mathrm{Sec} \geq 0$

Theorem (Colombo, M-, Rigoli, arXiv '22)

Let M be complete, $\mathrm{Sec} \geq 0$. Let u be a non-constant solution of (MS) *and assume that*

$$u(x) \leq C\big(1 + r(x)\big), \qquad r \text{ distance from a fixed origin } o.$$

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Theorem (Colombo, Magliaro, M-, Rigoli, 2021)

A complete manifold M with $\mathrm{Ric} \geq 0$ *satisfies* $(\mathscr{B}2)$:
positive minimal graphs over M are constant.

Independently proved by Q. Ding '21, with different techniques.

Previously shown by Rosenberg, Schulze, Spruck '13 under the further condition $\mathrm{Sec} \geq -\kappa^2$, $\kappa \in \mathbb{R}^+$.

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Let M^m be complete with $\mathrm{Ric} \geq 0$. *Let u solve* (MS) *and* $|Du| \in L^\infty(M)$. *Then, every tangent cone at infinity of M splits off a line.*

Remark: M may not split off any line!

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  (different proof: here we use heat equation techniques).

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  $$u(x) \leq C\big(1 + r(x)\big), \qquad r \text{ distance from a fixed origin } o,$$

  then we get same conclusion up to further requiring

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If M is complete and $\mathrm{Ric} \geq 0$, then positive minimal graphs are constant.

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Theorem

Let M^m complete with $\mathrm{Ric} \geq -(m-1)\kappa^2$, *for constant* $\kappa \geq 0$.

Let u be a positive solution of (MS) *on an open set* $\Omega \subset M$.

If either

(i) $\partial\Omega$ *locally Lipschitz and* $|\partial\Omega \cap B_R| \leq C_1 \exp\left\{C_2 R^2\right\}$, *or*

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$$\frac{\sqrt{1+|Du|^2}}{e^{\kappa\sqrt{m-1}u}} \leq \max\left\{1, \limsup_{x\to\partial\Omega} \frac{\sqrt{1+|Du(x)|^2}}{e^{\kappa\sqrt{m-1}u(x)}}\right\} \qquad \text{on } \Omega. \tag{3}$$

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$$\Delta_g W^{-1} + \left[\| \mathrm{II} \|^2 + \mathrm{Ric}\left(\frac{Du}{W}\right)\right] W^{-1} = 0 \qquad \text{on } \Sigma.$$

Korevaar's method: compute $\Delta_g(W\eta)$, for η a (carefully crafted) cutoff depending on u and r, the distance in (M, σ) from a fixed point.

Problem: need to evaluate $\Delta_g r = g^{ij}(D^2 r)_{ij}$, but Ric ≥ 0 only estimates $\sigma^{ij}(D^2 r)_{ij}$!

IDEA: in place of r, we use an exhaustion ϱ built via potential theory (stochastic geometry)
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The proof

Fix $C > \kappa\sqrt{m-1}, \quad z = We^{-Cu}$

- CLAIM: the following set is empty for every $\tau > 0$:

$$\Omega' := \left\{ x \in \Omega \ : \ z(x) > \max\left\{1, \limsup_{y \to \partial\Omega} \frac{W(y)}{e^{\kappa\sqrt{m-1}u(y)}}\right\} + \tau \right\}$$

Once the claim is shown, thesis follows by letting $\tau \to 0$, $C \downarrow \kappa\sqrt{m-1}$.

By contradiction: suppose that $\Omega' \neq \emptyset$.

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$$\mathscr{L}_g\phi = W^2 \mathrm{div}_g\left(W^{-2}\nabla\phi\right) \qquad \text{on } \Sigma'.$$

Since $\|\nabla u\|^2 = \frac{W^2-1}{W^2}$,

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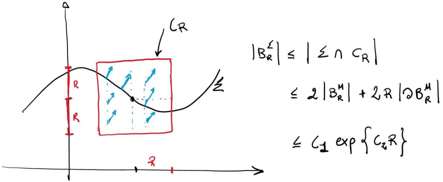
Key information: a graph has *area bounds* (calibrated):



LEMMA: in our assumptions, we can include $\overline{\Sigma'}$ isometrically a complete manifold (N^m, h) the volume of whose balls satisfies

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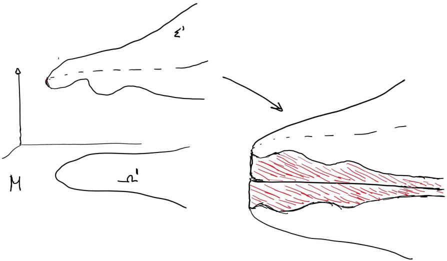

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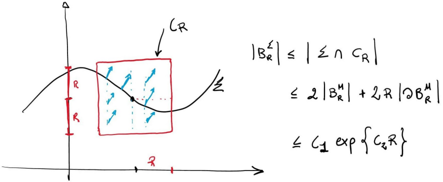

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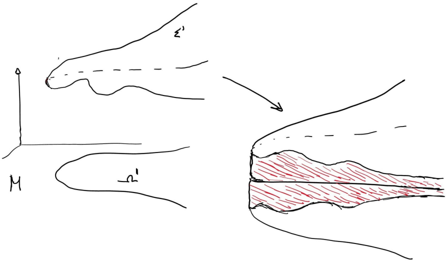

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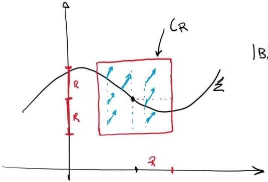

$$|B_R^\Sigma| \le |\Sigma \cap C_R|$$
$$\le 2|B_R^M| + 2R|\partial B_R^M|$$
$$\le C_1 \exp\{C_2 R\}$$

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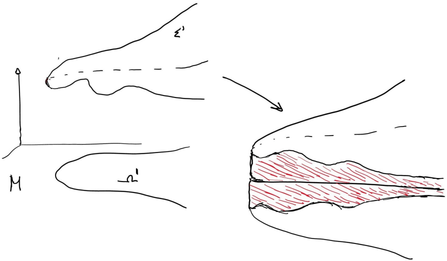

THEOREM (Grigoryan '99, Pigola-Rigoli-Setti '03):

If (N,h) is complete and

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then

$$\begin{cases} \Delta_h \omega \ge \omega & \text{on } \overline{U} \subset N, \\ \sup_U \omega < \infty \end{cases} \qquad \Longrightarrow \qquad \sup_U \omega \le \max\left\{0, \sup_{\partial U} \omega\right\}$$

The property characterizes the *stochastic completeness of* N, that is, the fact that the trajectories of the minimal Brownian motion on N have infinite lifetime almost surely.

AHLFORS-KHAS'MINSKII DUALITY
(M.-Valtorta '13, M.-Pessoa '20):
If (N,h) is stochastically complete, there exists $v \in C^\infty(N)$ solving

$$\begin{cases} \Delta_g v \le v \\ v \ge 1, \quad v \text{ exhaustion} \end{cases}$$

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The property characterizes the *stochastic completeness of* N, that is, the fact that the trajectories of the minimal Brownian motion on N have infinite lifetime almost surely.

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(M.-Valtorta '13, M.-Pessoa '20):
If (N, h) is stochastically complete, there exists $v \in C^\infty(N)$ solving

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setting $\varrho = \log v \in C^\infty(N)$,

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Let $\delta, \varepsilon', \varepsilon$ be positive, small (specified later), and set

$$z_0 = W\left(e^{-Cu-\varepsilon\varrho} - \delta\right) < z$$

For ε, δ small enough, the upper level-set

$$\Omega_0' := \left\{ x \in \Omega \ : \ z_0(x) > \max\left\{ 1, \limsup_{y \to \partial\Omega} \frac{W(y)}{e^{\kappa\sqrt{m-1}u(y)}} \right\} + \tau \right\} \subset \Omega'.$$

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We compute on the graph Σ_0'

$$\begin{aligned}\mathscr{L}_g z_0 &\geq \Big[\|C\nabla u + \varepsilon \nabla \varrho\|^2 - (m-1)\kappa^2 \|\nabla u\|^2 - \varepsilon \Delta_g \varrho\Big] z_0 \\ &\geq \Big\{\Big[C^2(1-\varepsilon') - (m-1)\kappa^2\Big]\|\nabla u\|^2 - \varepsilon\left[\Delta_g \varrho + \|\nabla \varrho\|^2\right]\Big\} z_0 \\ &> \left\{C_\tau - \varepsilon\left[\Delta_g \varrho + \|\nabla \varrho\|^2\right]\right\} z_0\end{aligned}$$

if ε' small enough and $\varepsilon << \varepsilon'$.

Using $\Delta \varrho + \|\nabla \varrho\|^2 \leq 1$ and $\varepsilon << 1$,

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contradiction at a maximum point of z_0 on Σ_0'.

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