Overdetermined Problems and Shape Optimization in Cones

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Mostly Maximum Principle
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Let ω be an open connected smooth domain on S^{N-1} and Σ_ω the cone defined by:

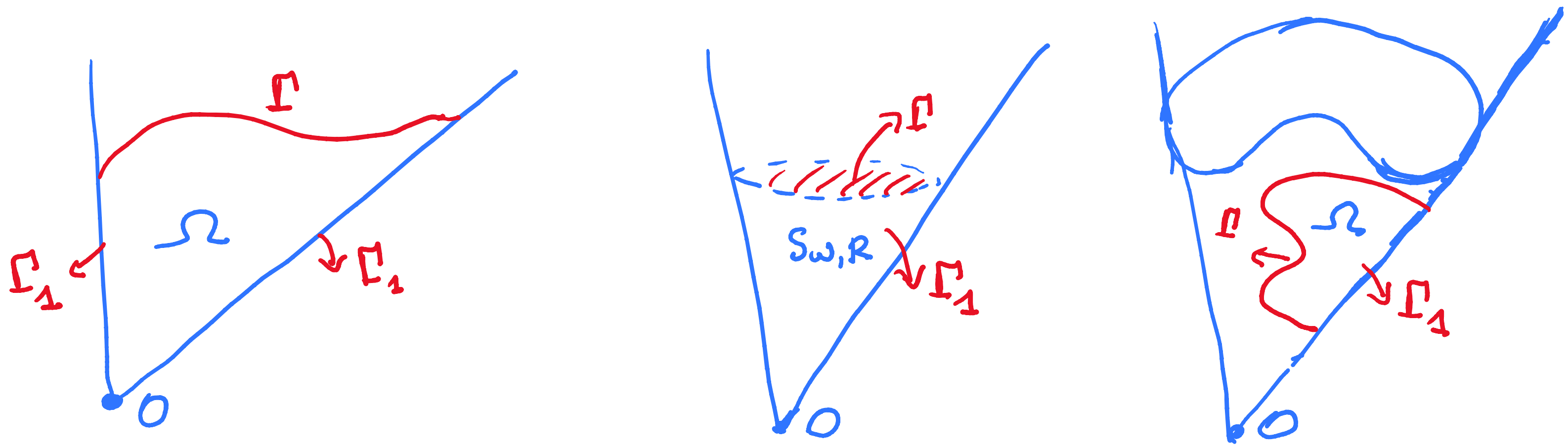
$$\Sigma_\omega = \{tx : x \in \omega, \ t \in (0, +\infty)\}$$

For a domain $\Omega \subset \Sigma_\omega$ we set:

$\Gamma = \partial\Omega \cap \Sigma_\omega$ (relative boundary)

$\Gamma_1 = \partial\Omega \setminus \bar{\Gamma}$

We assume that Γ is a smooth manifold and $H_{N-1}(\Gamma) > 0$, $H_{N-1}(\Gamma_1) > 0$



We consider the overdetermined problem :

$$(OP) \begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1 \\ \quad + & \\ \dfrac{\partial u}{\partial \nu} = -c < 0 & \text{on } \Gamma \end{cases}$$

If $\Omega = S_{\omega,R} = B_R \cap \Sigma_\omega$ then the radial function :

$$u(x) = \frac{N^2c^2 - |x|^2}{2N}$$

is a solution of (OP) .

(Solution of Serrin's overdetermined problem in the ball)

QUESTION : Is $S_{\omega,R}$ the only domain for which (OP) has a solution?

It depends on the cone and hence on $\omega \subset S^{N-1}$.

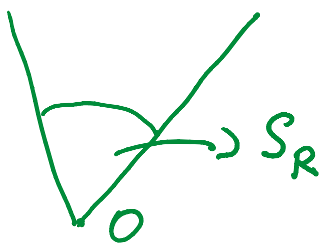
THEOREM - [G. TRALLI - F.P., 2020] -

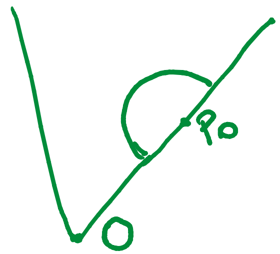
If Σ_ω is a convex cone and u is a classical solution of (OP) such that $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ then:

$$\Omega = \Sigma_\omega \cap B_R(P_0) \quad [B_R(P_0) \text{ centered at } P_0,\ R = Nc]$$

$$u(x) = \frac{N^2c^2 - |x - P_0|^2}{2N}, \quad \text{for some } P_0.$$

Either $P_0 = 0$ or $P_0 \in \partial\Sigma_\omega$ and Γ is a half-sphere lying on a flat part of $\partial\Sigma_\omega$.





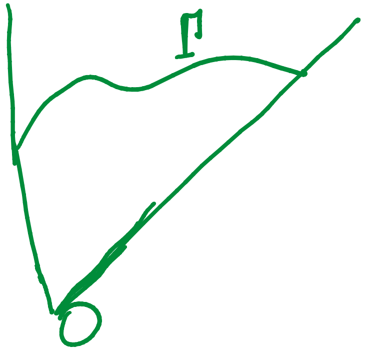
Hence spherical sectors centered at O (or half balls in the case when $\partial\Sigma_\omega$ has a flat portion) are the only smooth domains inside a convex cone for which (OD) has a solution.

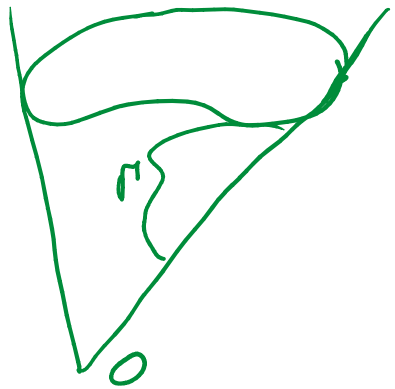
REMARK. The assumption $u \in W^{1,\infty}(\Omega) \cap W^{2,2}(\Omega)$ is a "gluing condition". It is automatically satisfied whenever Γ $(= \partial\Omega \cap \Sigma_\omega)$ and $\partial\Sigma_\omega$ intersect orthogonally.

The proof is by integral identities as the one of Weinberger for the Dirichlet problem.

A similar proof yelds a result for CMC surfaces in cones.

Let $\Gamma \subset \Sigma_\omega$ be a smooth connected $(N-1)$-dimensional manifold with smooth boundary contained in $\partial\Sigma_\omega$





THEOREM ([Choe-Park-2011], [G. Tralli - F.P. 2020])

If Σ_ω is a smooth convex cone and Γ is a CMC surface inside Σ_ω, intersecting $\partial\Sigma_\omega$ orthogonally, then

$$\Gamma = \Sigma_\omega \cap \partial B_R(P_0)$$

and either $P_0 = 0$ or $P_0 \in \partial\Sigma_\omega$ and Γ is a half sphere lying on a flat part of $\partial\Sigma_\omega$.

in [G. Tralli - F.P., 2020] a more general gluing condition between Γ and $\partial\Sigma_\omega$ is allowed.

The CMC problem in cones has a variational formulation

It is related to a relative isoperimetric inequality in the same way as Aleksandrov's result is related to the classical isoperimetric inequality.
Indeed the "quite smooth" CMC surfaces which intersect orthogonally a cone Σ_ω are the relative boundaries of the critical points of the relative (to Σ_ω) Perimeter functional [Ritoré-Rosales, 2004].

Let E be any measurable set contained in Σ_ω with $|E| < \infty$ and let $P_{\Sigma_\omega}(E)$ be the relative (to Σ_ω) perimeter of E, i.e. the measure of the part of ∂E which is contained in Σ_ω.

THEOREM – (Relative isoperimetric inequality)
If Σ_ω is a convex cone then

$$P_{\Sigma_\omega}(E) \geq N \omega_N^{1/N} |E|^{\frac{N-1}{N}} \qquad \forall E \subset \Sigma_\omega$$

and equality holds if and only if E is a spherical sector centered at 0 $\quad \omega_N = |B_1 \cap \Sigma_\omega|$

Equivalently we can say that the only measurable sets $E \subset \Sigma_\omega$ which minimize P_{Σ_ω}, under a volume constraint, are the spherical sectors $B_R \cap \Sigma_\omega = S_{\omega, R}$ -

The relative isoperimetric inequality was proved in:
[P.L. Lions - F.P., 1990], [M. Ritoré - C. Rosales, 2004]
[A. Figalli - E. Indrei, 2013], [X. Cabré - X. Ros Oton - J. Serra 2016]

All different proofs and all require the convexity of the cone, Figalli-Indrei's proof also gives a stability result.

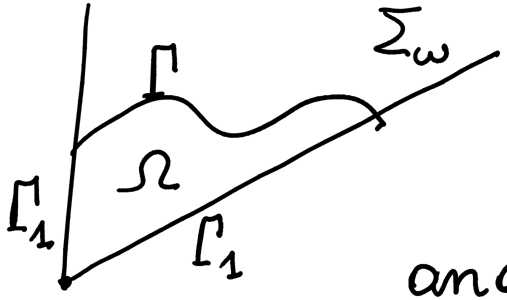
It does not hold in general nonconvex cones, counterexample in [P.L. Lions–F.P., 1990]. It holds in "almost" convex cones [Baer - Figalli, 2017], [Indrei-F.P., 2020].

The relative isoperimetric inequality inspired the result for CMC surfaces in convex cones and also the one for the (OP) problem.

In convex cones the answer for the two problems is the same.

? A clear relation between them is not yet understood!

Also the (OD) problem has a variational formulation in terms of critical points of the torsional energy.



Let Ω be a domain in Σ_ω

$\partial\Omega = \Gamma \cup \Gamma_1 \cup \partial\Gamma$

and consider the energy functional:

$$J_\omega(\Omega) = \frac{1}{2}\int_\Omega |\nabla u_\Omega|^2 dx - \int_\Omega u_\Omega \, dx$$

where u_Ω is the unique weak solution in $W_0^{1,2}(\Omega \cup \Gamma_1)$ of the problem:

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega \\ u = 0 & \text{on } \Gamma \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1 \end{cases}$$

$\Gamma = \partial\Omega \cap \Sigma_\omega$

$\Gamma_1 = \partial\Omega \cap \partial\Sigma_\omega$

The functional J_ω represents a "relative" version of the classical torsional energy functional usually defined using the solution of the analogous Dirichlet problem.

The question of minimizing the classical torsional energy functional among smooth bounded domains of fixed volume is known as Saint-Venant problem.

We are interested in the same mimization problem but in the relative setting.

First we characterize the domains which are critical points of $J_\omega(\Omega)$, with a volume constraint.

Using domain derivative technique, as for similar problems in shape-optimization theory we prove that

Ω is a critical point of J_ω with a fixed volume $\iff$ $\dfrac{\partial u_\Omega}{\partial \nu} \equiv$ constant on Γ

i.e. the stationary points of J_ω (with fixed volume) are the the domains $\Omega \subset \Sigma_\omega$ for which the (OD) problem has a solution.

Thus the result of [G. Tralli - F.P. 2020] claims that in convex cones the only smooth critical points of J_ω are either the spherical sectors centered at O or half balls lying over a flat portion of $\partial\Sigma_\omega$.

Next we would like to characterize the minimizers of J_ω.

Let us call "isoperimetric" a cone which has the property that minimizers (fixing the volume) exist and the only minimizers are the spherical sectors centered at O.
(All convex cones are isoperimetric).

THEOREM. ([G. Tralli - F.P., 2020]) If Σ_ω is an isoperimetric cone then the only minimizers of J_ω (with a fixed volume) are the spherical sectors centered at the origin.

Proof by ω-symmetrization ([P.L. Lions - F.P. - M. Tricarico, 1989])

QUESTION: Does the reverse implication hold?

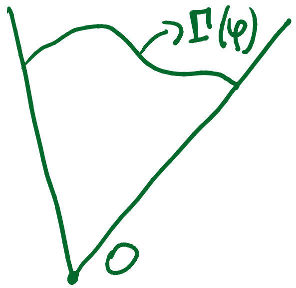
We would like to understand when in a nonconvex cone there exist nonradial minimizers.

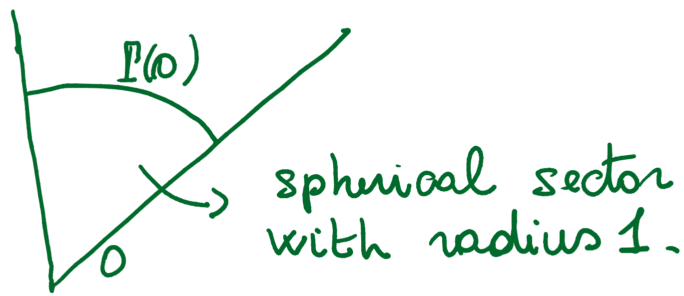
We study the torsional energy functional restricted to the class of strictly starshaped domain $\Omega \subset \Sigma_\omega$, i.e. $\Gamma = \partial\Omega \cap \Sigma_\omega$ is a radial graph of a function φ on ω -

$$\Gamma(\varphi) = \{ e^{\varphi(z)} z \,,\; z \in \omega \} \,, \qquad \varphi \in C^2(\omega, \mathbb{R}) \cap C^0(\bar{\omega})$$

$$\Omega_\varphi = \{ p \in \mathbb{R}^N \,,\; p = \lambda z \,,\; 0 < \lambda < e^{\varphi(z)},\; z \in \omega \}$$

In particular for $\varphi = 0$, $\Gamma(0)$ is the unit spherical cap.





We consider the torsional energy functional J_ω restricted to the starshaped domains Ω_φ which becomes $J_\omega(\varphi)$.
We can compute the first and second variation of J_ω as a function of φ.
The volume of Ω_φ is:

$$V(\varphi) = \frac{1}{N}\int_\omega e^{N\varphi(z)}\,d\sigma$$

and we consider the set

$$M = \{\varphi \in C^2(\omega,\mathbb{R}),\ V(\varphi) = c\}$$

and the restriction I of the torsional energy to M

$$I''_\omega(\varphi)(v,w) = J''_\omega(\varphi)(v,w) - \lambda V''(\varphi)(v,w)$$

Obviously $\varphi = 0$ is a "trivial" critical point of I_ω.

We want to study its stability.

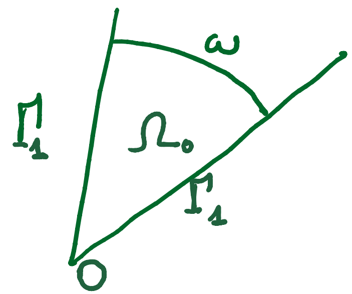
If $\varphi = 0$ is not a stable critical point of I_ω it cannot be a minimizer.

At $\varphi = 0$ $I''(0)$ has a simple expression:

$$I''(0)(v,v) = -\frac{1}{N^2}\int_\omega v^2(z)\,d\sigma + \frac{1}{N}\int_\omega v(z)\frac{\partial u'}{\partial \nu}(z)\,d\sigma$$

for v s.t. $\int_\omega v = 0$ and u' solution of:

$$\begin{cases} -\Delta u' = 0 & \text{in } \Omega_0 = \Omega(\varphi = 0) \\ u' = \frac{1}{N} v & \text{on } \omega \\ \frac{\partial u'}{\partial \nu} = 0 & \text{on } \Gamma_1 = \partial\Omega_0 \setminus \omega \end{cases}$$



THEOREM _ [A. Iacopetti - F.P. - T. Weth, 2021]
Let $\lambda_1(\omega)$ be the first (nontrivial) eigenvalue of the Laplace-Beltrami operator $-\Delta_{S^{N-1}}$ on ω with zero Neumann boundary condition.

If : $\lambda_1(\omega) < N-1$
then $\varphi = 0$ is not a stable critical point.

If : $\lambda_1(\omega) > N-1$
then $\varphi = 0$ is a stable critical point.

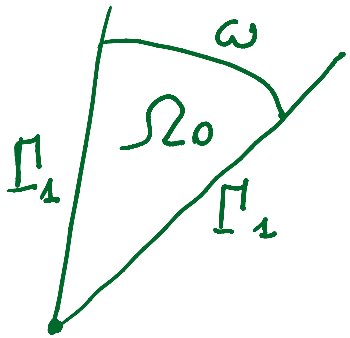
Proof. Let w_1 a $L^2(\omega)$-normalised eigenfunction corresponding to $\lambda_1(\omega)$, hence $\int_\omega w_1 = 0$. Consider its extension to Σ_ω :

$$\tilde{w}_1(r,z) = \frac{1}{N} r^{\alpha_1} w_1(z), \quad z \in \omega, \ r > 0$$

$$\alpha_1 = -\frac{N-2}{2} + \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda_1}$$

Then $\tilde{w}_1\big|_{\Omega_0}$ uniquely solves

$$\begin{cases} -\Delta \tilde{w}_1 = 0 & \text{in } \Omega_0 \\ \tilde{w}_1 = \frac{1}{N} w_1 & \text{on } \omega \\ \frac{\partial \tilde{w}_1}{\partial \nu} = 0 & \text{on } \Gamma_1 \end{cases}$$



and $\frac{\partial \tilde{w}_1}{\partial \nu} = \frac{\alpha_1}{N} w_1$. Hence

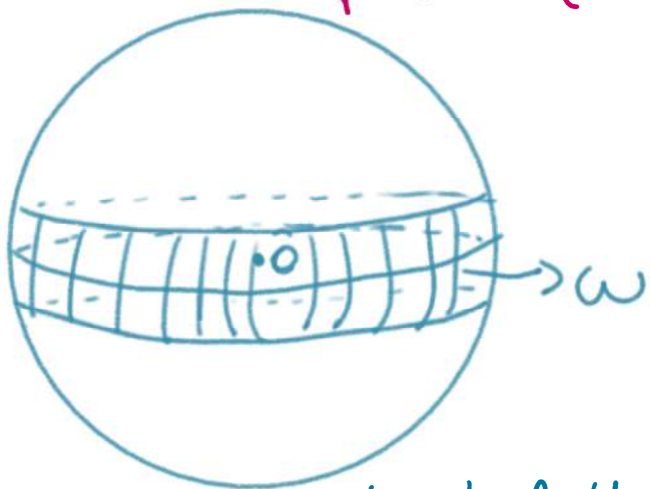
$$\underline{I''(0)(w_1,w_1) = -\frac{1}{N^2}} + \underline{\frac{\alpha_1}{N^2} < 0} \iff \underline{\alpha_1 < 1}$$

which is equivalent to $\underline{\lambda_1(\omega) < N-1}$ ▥

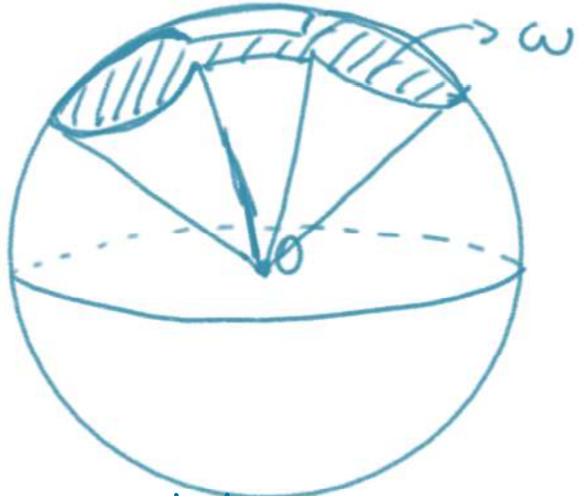
We can construct examples of nonconvex $\omega \subset S^{N-1}$ such that $\lambda_1(\omega) < N-1$.

In such cases the spherical sectors do not minimize the torsional energy functional J_ω with fixed volume.

Hence a minimizer set Q_0, if quite smooth, is a domain, different from a spherical sector, for which a solution of the (OD) problem exists.



ω is a neighborhood of the equator.



ω is made by two discs on S^{N-1} connected by a narrow tunnel.

QUESTION: Do minimizers of $J_\omega(\Omega)$ exist in all cones?

Recall that if Σ_ω is isoperimetric the only minimizers are the spherical sector and the torsion function is radial

The existence of a minimizer in a general cone is not so obvious because the cone is an unbounded set.

First we observe that the question of the existence of a minimizer should be addressed in the class of "quasi-open sets" and in the corresponding Sobolev spaces.

(The class of open sets is "too small" to look for minimizers)

So we consider:

$$O_c(\Sigma_\omega) = \inf\{J_\omega(\Omega),\ \Omega \subset \Sigma_\omega \text{ quasi open},\ |\Omega| \le c\}$$

We study the behavior of a minimizing sequence $\{\Omega_K\}$. Since the cone is not bounded $\{\Omega_K\}$ could not be compact, in particular the sets Ω_K could escape at ∞! -

We use the "Concentration - Compactness Principle" of P.L. Lions on the corresponding sequence of torsion functions $\{u_{\Omega_K}\}$ to show that under a suitable hypothesis on Σ_ω a minimizer exists and is a quasi-open set Ω^* which is the level set $\{u^*>0\}$ of the limit function u^* of u_{Ω_K} -

The CC Principle to study shape-optimization problems was first used in [D. Bucur, 2000] in the Dirichlet case -
We could not follow straightforwardly the approach of Bucur since our problem is different and new difficulties arise -

In applying the CC Principle to the sequence $\{u_{\Omega_K}\}$ three possibilities can occur:

i) vanishing

ii) dicothomy

iii) compactness

Tipically to get the convergence of $\{u_{\Omega_K}\}$ a limit problem plays a role, which, in our case, is the mixed boundary problem in a half ball of measure c (since the cone is asymptotically flat).

Thus we prove:

THEOREM [A. Iacopetti - F.P. - T. Weth, 2021]

If $\mathcal{O}_c(\Sigma_\omega) < \mathcal{O}_c(S^+_{N-1})$ then $\mathcal{O}_c(\Sigma_\omega)$ is achieved.

Considering the explicit expression of the torsion function in the half ball or in a spherical sector we obtain:

THEOREM - [A. Iacopetti - F.P. - T. Weth, 2021]
If $H_{N-1}(\omega) < H_{N-1}(S^+_{N-1})$ then $O_c(\Sigma_\omega)$ is achieved.

In particular this holds for the examples above.

Once the minimizer Ω^* exists we can show that it has the following properties:

i) Ω^* is bounded, open, connected

ii) there exists a critical dimension d^* ($=5$ or 6 or 7) such that the relative boundary $\Gamma^* = \partial\Omega^* \cap \Sigma_\omega$:

- is smooth if the dimension $N < d^*$
- can have countable isolated singularities if $N = d^*$
- can have a singular set of dimension $N - d^*$, if $N > d^*$

iii) on the regular part of Γ^* the overdetermined condition holds.

Thus, if Σ_ω is a nonconvex cone such that

$$\boxed{H^{N-1}(\omega) < H^{N-1}(S^+_{N-1}) \text{ and } \lambda_1(\omega) < N-1} \qquad (*)$$

the minimizer Ω^* exists and is a nonradial domain such that (OD) has a solution.

? What can we say about the shape of this
• domain?

The conditions (*) also allow to say that Σ_ω is not isoperimetric and there exists a minimizer $\tilde{\Omega}$ for the relative perimeter functional P_{Σ_ω} which is not the spherical sector.
This implies that the relative boundary Γ of $\tilde{\Omega}$ is a CMC surface which intersect the cone orthogonally and is not a spherical cap.

? What kind of CMC surface is? A new one?