Comparison principles by monotonicity, duality and fiberegularity

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Problem

With $\Omega \Subset X$, X open in $\mathbb{R}^n$, examine the validity of the comparison principle (comparison)

$$u \leq w \text{ on } \partial\Omega \quad \Longrightarrow \quad u \leq w \text{ on } \Omega \tag{CP}$$

for each pair $u \in \mathrm{USC}(\overline{\Omega}), w \in \mathrm{LSC}(\overline{\Omega})$ in the two (often equivalent) settings:

Nonlinear potential theory: *(u, w) is an $\mathcal{F}$-subharmonic/ $\mathcal{F}$-superharmonic pair on Ω for some subequation (contraint set)*

$$\mathcal{F} \subset \mathcal{J}^2(X) := X \times \mathcal{J}^2 := X \times \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}(n). \tag{SE}$$

Fully nonlinear PDEs: *(u, w) is a subsolution/ supersolution pair for the equation*

$$F(x, J^2_x u) := F(x, u(x), Du(x), D^2 u(x)) = 0, \quad x \in \Omega, \tag{E}$$

determined by an operator $F \in C(\mathcal{J}^2(X))$.

- All notions above are to be interpreted pointwise in the *viscosity sense* in terms of *upper/lower test jets $J^{2,\pm}_x u \subset \mathcal{J}^2, x \in \Omega$.*
- Part of a program concerning the interplay between potential theory and operator theory.

Plan of the talk

1) Nonlinear potential theory: key concepts.

2) Potential theoretic comparison: by monotonicity, duality and fiberegularity:

- comparison holds if there is "sufficient monotonicity"; there exists a (constant coefficient) monotonicity cone $\mathcal{M}$ for the constraint $\mathcal{F}$ and $\mathcal{M}$ admits a classical strict subharmonic;
- the class of monotonicity cones are well understood [Cirant-Harvey-Lawson-P; Annals of Math Studies, to appear];
- comparison does not depend on domain shape, but domain size can play a role.

3) Operator theoretic setting: constrained and unconstrained cases

4) Correspondence Principle:

- for a given operator-subequation pair $(F, \mathcal{F})$, determine conditions under which (u, w) is an $\mathcal{F}$-subharmonic/ $\mathcal{F}$-superharmonic pair **if and only if** (u, w) is a subsolution/ supersolution pair for $F(J^2 u) = 0$;
- gives equivalent formulations of the Dirichlet problem in the two settings.

N.B. F will often needed to be restricted to a suitable background constraint $\mathcal{G} \subset \mathcal{J}^2(X)$, and leads to the notion of $\mathcal{G}$-admissible viscosity sub/supersolutions. The canonical relations between $\mathcal{F}$ and F are

$$\mathcal{F} = \{(x, J) \in \mathcal{G} : \quad F(x, J) \geq 0\} \quad \text{and} \quad \partial\mathcal{F} = \{(x, J) \in \mathcal{G} : \quad F(x, J) = 0\}$$

Philosophical motivation for the larger program

Opportunities for cross-fertilization and synergy between potential theory and operator theory:

- Geometry, topology of the constraint $\mathcal{F} \leftrightsquigarrow$ structural conditions on the operator F.
- $\mathcal{F}$ "frees" a given PDE from any particular form of F (many different operators F correspond to the same constraint $\mathcal{F}$); related to a key point in [Krylov; TAMS'95].
- "Forgetting" about the operator leads to interesting questions that at first glance might not seem important for operator theory; (see the survey paper [Harvey-P., '22])

pluri-potential theory results $\Longrightarrow$ conjectures in general potential theory & PDEs

- Many new PDEs to discover; e.g. Harvey-Lawson show that every *callibrated geometry* has an underlying potential theory, but known "natural" operators are "rare gems".
- However, any given subequation $\mathcal{F}$ suggests families of "non natural" *canonical operators* such as

$$F(x, J) := \begin{cases} \operatorname{dist}(J, \partial \mathcal{F}_x) & J \in \mathcal{F}_x \\ -\operatorname{dist}(J, \partial \mathcal{F}_x) & J \in \mathcal{J}^2 \setminus \mathcal{F}_x \end{cases}$$

- On the other hand, when a natural F is known for the potential theory (F polynomial or smooth), F will have much to say about the potential theory using operator theory (e.g., taking derivatives of the equation).

Illustrative examples

Ex1: (Perturbed Monge-Ampère) With $M \in C(\Omega, \mathcal{S}(n))$ and $f \in C(\Omega, \mathbb{R})$ non-negative:

$$\det(D^2 u + M(x)) = f(x), \quad x \in \Omega \Subset \mathbb{R}^n$$

- Fails to satisfy the standard viscosity structual conditions for comparison [Crandall-Ishii-Lions, BAMS'92], which would require $M = L^2$ with $L \in \mathrm{Lip}(\Omega, \mathcal{S}(n))$
- Comparison holds on all $\Omega \Subset \mathbb{R}^n$ [Cirant-P., PM'17] where

$$\mathcal{F}_x := \{A \in \mathcal{S}(n) : \ A + M(x) \geq 0 \text{ and } F(x, A) := \det(A + M(x)) - f(x) \geq 0\}$$

defines a fiberegular subequation even if M is merely continuous.

Ex2: (Special Lagrangian potential equation from callibrated geometry) With *phase* function $h \in C(\Omega, I)$ where $I = (-n\pi/2, n\pi/2)$ consider [Harvey-Lawson, ActaM'82]:

$$G(D^2 u) := \sum_{k=1}^{n} \arctan\left(\lambda_k(D^2 u)\right) = h(x), \quad x \in \Omega \Subset \mathbb{R}^n$$

- Comparison known for constant phases [Harvey-Lawson, CPAM'09] (also Perron).
- For non-constant phases, comparison is difficult: strong degeneration of the operator G if h assumes a special phase value $\theta_k = (n - 2k)\pi/2, k = 1, \ldots n - 1$.
- Best comparison to date: phases taking values in any phase interval [Cirant-P., ME'21]

$$I_k = (\theta_{k-1}, \theta_k), \quad k = 1, \ldots n \text{ where } \mathcal{F}_x := \{A \in \mathcal{S}(n) : \ F(x, A) := G(A) - h(x) \geq 0\}$$

Ex3: (Optimal transport equations) With $f \in C(\Omega, \mathbb{R})$ non-negative and $g \in C(\mathbb{R}^n)$ having a *directional monotonicity cone* $\mathcal{D} \subset \mathbb{R}^n$ consider

$$g(Du)\det(D^2u) = f(x), \quad x \in \Omega \Subset \mathbb{R}^n$$

- Examples for g include $g(p) = p_n$ and $g(p) = p_1 \cdots p_k$ with $1 \leq k \leq n$, etc.
- Comparison for f constant in [Cirant-Harvey-Lawson-P., AoMS, to appear] and g constant in [Cirant-P., ME'21]; general case in [Cirant-P.-Redaelli, prepint '22].
- A product structure helps with the correspondence principle.

Ex4: (Equations where comparison fails on all small balls) [CHLP, AoMS, to appear] Fix $\alpha \in (1, +\infty)$ and consider $F, G \in C(\mathbb{R}^n \times \mathcal{S}(n))$ (constant coefficients):

$$F(p, A) := \lambda_{\min}(M(p, A)) \quad \text{and} \quad G(p, A) := \lambda_{\max}(M(p, A)) \quad \text{where}$$

$$M(p, A) := A + |p|^{\frac{\alpha-1}{n}}(P_{p^\perp} + \alpha P_p)) \quad \text{if } p \neq 0 \quad \text{and} \quad M(0, A) := A.$$

where for $p \neq 0$, $P_p, P_{p^\perp}$ are the projections onto the subspaces $[p], [p]^\perp$.

- The comparison principle, maximum principle and uniqueness of solutions fail on all balls.
- The *maximal monotonicity cone* for the associated (compatible) subequations $\mathcal{F}, \mathcal{G}$ is $\mathcal{M} := \{0\} \times \mathcal{P} \subset \mathbb{R}^n \times \mathcal{S}(n)$, which has empty interior.

1. Subequation: Introduced in [Harvey-Lawson, CPAM'09, JDG'11] as a class of "good" constraint sets $\mathcal{F} \subset \mathcal{J}^2(X)$ on which to base a potential theory; the axioms are:

(P) $\mathcal{F}$ satisfies the positivity condition fiberwise; that is, for each $x \in X$

$$(r, p, A) \in \mathcal{F}_x \quad \Rightarrow \quad (r, p, A + P) \in \mathcal{F}_x, \quad \forall\, P \geq 0 \text{ in } \mathcal{S}(n).$$

(N) $\mathcal{F}$ satisfies the negativity condition fiberwise; that is, for each $x \in X$

$$(r, p, A) \in \mathcal{F}_x \quad \Rightarrow \quad (r + s, p, A) \in \mathcal{F}_x, \quad \forall\, s \leq 0 \text{ in } \mathbb{R}.$$

(T) $\mathcal{F}$ satisfies three conditions of topological stability

$$\mathcal{F} = \overline{\mathcal{F}^\circ}, \quad (\mathcal{F}^\circ)_x = (\mathcal{F}_x)^\circ, \quad \mathcal{F}_x = \overline{(\mathcal{F}_x)^\circ}.$$

- $\mathcal{F}$ is closed (by (T)) and usually assumed non-empty and proper.
- (P), (N) and (T) have implications for the $\mathcal{F}$-potential theory, together with duality.
- **classical subharmonics:** $\mathcal{F} = \{(r, p, A) : \ \operatorname{tr} A \geq 0\}$
- **convex functions:** $\mathcal{F} = \{(r, p, A) : \ A \geq 0\} = \{(r, p, A) : \ \lambda_{\min}(A) \geq 0\}$
- **subaffine functions:** $\mathcal{F} = \{(r, p, A) : \ \lambda_{\max}(A) \geq 0\}$

2. Subharmonics: A function $u \in \mathrm{USC}(X)$ is *$\mathcal{F}$-subharmononic on X* if

$$J_x^{2,+} u \subset \mathcal{F}_x, \quad \forall\, x \in X \quad \text{where}$$

$$J_x^{2,+} u := \{ J_x^2 \varphi : \varphi \text{ is } C^2 \text{ near } x, \ u \leq \varphi \text{ near } x \text{ with equality in } x \},$$

is the space of *upper test jets*. Denote by $\mathcal{F}(X)$ the space of $\mathcal{F}$-subharmononics on X.

3. Duality: [Harvey-Lawson CPAM'09, JDG'11] For a given subequation $\mathcal{F} \subset \mathcal{J}^2(X)$ the *Dirichlet dual* is

$$\widetilde{\mathcal{F}} := (-\mathcal{F}^{\circ})^c = -(\mathcal{F}^{\circ})^c \quad (\text{relative to } \mathcal{J}^2(X))$$

and, by property (T), can be calculated fiberwise

$$\widetilde{\mathcal{F}}_x := (-(\mathcal{F}_x)^{\circ})^c = -((\mathcal{F}_x)^{\circ})^c \quad (\text{relative to } \mathcal{J}^2), \quad \forall\, x \in X.$$

- If $\mathcal{F}$ is a subequation, then so is $\widetilde{\mathcal{F}}$ and one has *reflexivity*: $\widetilde{\widetilde{\mathcal{F}}} = \mathcal{F}$.

N.B. Duality is used to define **superharmonics**: $w \in \mathrm{LSC}(X)$ is *$\mathcal{F}$-superharmonic on X* if $-w \in \mathrm{USC}(X)$ is $\widetilde{\mathcal{F}}$-subharmonic on X, which in terms of *lower test jets* is equivalent to

$$J_x^{2,-} w \subset (\mathrm{Int}\ \mathcal{F}_x)^c, \quad \forall\, x \in X.$$

Monotonicity and fiberegularity

4. Monotonicity: is a unifying concept where $\mathcal{F}$ is *$\mathcal{M}$-monotone* for $\mathcal{M} \subset \mathcal{J}^2(X)$ if

$$\mathcal{F}_x + \mathcal{M}_x \subset \mathcal{F}_x \quad \text{for each } x \in X.$$

- The *minimal monotonicity cone* $\mathcal{M}_0 := \{(r, 0, A) \in \mathcal{J}^2 : r \leq 0 \text{ and } A \geq 0\}$ encodes properties (P) and (N) $\rightsquigarrow$ operators F which are *proper elliptic* (needed for comparison).
- Monotonicity combines with duality in the fundamental *jet addition formula*

$$\mathcal{F}_x + \mathcal{M}_x \subset \mathcal{F}_x \quad \Longrightarrow \quad \mathcal{F}_x + \widetilde{\mathcal{F}}_x \subset \widetilde{\mathcal{M}}_x, \quad \text{for each } x \in X,$$

5. Fiberegularity: is often sufficient to extend results from constant to variable coefficients.

- A subequation $\mathcal{F} \subset \mathcal{J}^2(X)$ is *fiberegular* if the fiber map is (Hausdorff) continuous; i.e. if

$$\Theta : (X, |\cdot|) \to (\mathcal{K}(\mathcal{J}^2), d_{\mathcal{H}}) \quad \text{with} \quad \Theta(x) := \mathcal{F}_x, \quad \forall\, x \in X$$

  is continuous, where $d_{\mathcal{H}}$ is the Hausdorff distance on the closed subsets of $\mathcal{J}^2$.
- Useful reformulation when $\mathcal{F}$ is $\mathcal{M}$-monotone (some monotonicity cone subequation $\mathcal{M}$): for each fixed $J_0 \in \mathrm{Int}\, \mathcal{M}$, $\Omega \Subset X$ and $\eta > 0$ there exists $\delta > 0$ such that

$$x, y \in \Omega, |x - y| < \delta \quad \Longrightarrow \quad \Theta(x) + \eta J_0 \subset \Theta(y).$$

- Ensures that "small perturbations of all short range translates of an $\mathcal{F}$-subharmonic remain $\mathcal{F}$-subharmonic".

2. Comparison by monotonicity-duality-fiberegularity

Theorem (General comparison theorem)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Suppose that a subequation $\mathcal{F} \subset \mathcal{J}^2(\Omega)$ is fiberegular and $\mathcal{M}$-monotone on Ω for some monotonicity cone subequation $\mathcal{M}$. If $\mathcal{M}$ admits a strict subharmonic $\psi \in C^2(\Omega) \cap C(\overline{\Omega})$ on Ω, then comparison holds for $\mathcal{F}$ on $\overline{\Omega}$; that is,

$$u \leq w \text{ on } \partial\Omega \quad \Longrightarrow \quad u \leq w \text{ on } \Omega \tag{CP}$$

for all $u \in \mathrm{USC}(\overline{\Omega})$, $\mathcal{F}$-subharmonic on Ω, and $w \in \mathrm{LSC}(\overline{\Omega})$, $\mathcal{F}$-superharmonic on Ω.

- $\mathcal{F}_x \equiv \mathcal{F} \subset \mathcal{S}(n)$ in [Harvey-Lawson, CPAM'09]: constant coefficient pure second order, $\mathcal{M} = \mathcal{P}$ and ψ exists for every $\mathcal{F}$.
- $\mathcal{F} \subset \Omega \times \mathcal{S}(n)$ in [Cirant-P., PM'17]: fiberegular pure second order, $\mathcal{M} = \mathcal{P}$ and ψ exists for every $\mathcal{F}$.
- $\mathcal{F} \subset \Omega \times (\mathbb{R} \times \mathcal{S}(n))$ in [Cirant-P., ME'21], fiberegular gradient-free, $\mathcal{M} = \mathcal{Q} = \mathcal{N} \times \mathcal{P}$ and ψ exists for every $\mathcal{F}$.
- $\mathcal{F}_x \equiv \mathcal{F} \subset \mathcal{J}^2(\Omega)$ in [Cirant-Harvey-Lawson-P, AoMS, to appear]: constant coefficients, complete study of which cones $\mathcal{M}$ admit ψ on Ω.
- General case in [Cirant-P.-Redaelli, preprint '22]; imports the class of admissible cones $\mathcal{M}$ from the constant coefficient case.

Outline of the proof

Step 1 (Duality reformulation): Use **duality** to reformulate (CP) as:

$$u + v \leq 0 \text{ on } \partial\Omega \quad \Longrightarrow \quad u + v \leq 0 \text{ on } \Omega \qquad \text{(CP')}$$

for all $u \in \mathrm{USC}(\overline{\Omega})$, $\mathcal{F}$-subharmonic on Ω, and $v \in \mathrm{USC}(\overline{\Omega})$, $\widetilde{\mathcal{F}}$-subharmonic on Ω.

- Just define $v := -w$ and use duality.
- (CP') is the *zero maximum principle* (ZMP) for the sum of $\mathcal{F}$ and $\widetilde{\mathcal{F}}$ subharmonics:

$$\forall\, z \in \mathrm{USC}(\overline{\Omega}) \cap (\mathcal{F}(\Omega) + \widetilde{\mathcal{F}}(\Omega)) : \qquad z \leq 0 \text{ on } \partial\Omega \quad \Longrightarrow \quad z \leq 0 \text{ on } \Omega \qquad \text{(ZMP)}$$

Step 2 (Jet Addition): Establish the fundamental *jet addition formula*

$$\mathcal{F}_x + \mathcal{M}_x \subset \mathcal{F}_x \quad \Longrightarrow \quad \mathcal{F}_x + \widetilde{\mathcal{F}}_x \subset \widetilde{\mathcal{M}}_x, \quad \text{for each } x \in X,$$

using elementary properties of **duality** and **monotonicity** (Harvey-Lawson, SDG'13).

- This is the key to duality.
- Very useful if $\mathcal{M}$ has constant coefficients.

Step 3 (Local quasi-convexity): For locally quasi-convex functions u, v, establish:

- the *Almost Everywhere Theorem*:

$$J^2_x u = (u(x), Du(x), D^2 u(x)) \in \mathcal{F}_x \ \text{ for } \ \mathcal{L}^n\text{-a.e. } x \in X \quad \Longleftrightarrow \quad u \in \mathcal{F}(X),$$

- the *Subharmonic Addition Theorem* (quasi-convex version): for subequations $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$

$$\mathcal{F}_x + \mathcal{G}_x \subset \mathcal{H}_x, \quad \text{for each } x \in X \qquad \text{(Jet addition)}$$

  implies

$$u + v \in \mathcal{H}(X), \quad \text{for all } u \in \mathcal{F}(X), v \in \mathcal{G}(X). \qquad \text{(Subharmonic addition)}$$

in order to conclude

$$z = u + v \in \widetilde{\mathcal{M}}(\Omega) \quad \text{if } u \in \mathcal{F}(\Omega) \text{ and } v \in \widetilde{\mathcal{F}}(\Omega) \text{ are locally quasi-convex.}$$

N.B. This difficult step relies on the Jensen [ARMA'88] or Slodkowski [ASNSP'84] Lemma, which control the measure of *upper contact points* near x for locally quasi-convex functions. These Lemmas and are equivalent ([Harvey-Lawson, arXiv'16, P.-Redaelli '22]).

Step 4: Use **fiberegularity** to prove the *Subharmonic Addition Theorem* ($\mathcal{M}$-monotone version):

$$u \in \mathcal{F}(\Omega), v \in \widetilde{\mathcal{F}}(\Omega) \implies u + v \in \widetilde{\mathcal{M}}(\Omega)$$

if $\mathcal{F}$ (and hence $\widetilde{\mathcal{F}}$) is fiberegular and $\mathcal{M}$-monotone for some constant coefficient mponotonicity subequation cone $\mathcal{M}$ which admits a C^2-strict subharmonic ψ on Ω.

- Use *sup-convolution approximations* $u^\varepsilon, v^\varepsilon$ of u, v:

$$u^\varepsilon(x) := \sup_{y \in X} \left(u(y) - \frac{1}{2\varepsilon}|y - x|^2 \right), \quad x \in X, \quad \text{which are } \frac{1}{\varepsilon}\text{-quasi-convex.}$$

- If $\mathcal{F}$ and (hence) $\widetilde{\mathcal{F}}$ have constant coefficients, then the approximations remain subharmonic and the extension holds [Cirant-Harvey-Lawson-P, AoMS, to appear].
- For **fiberegular** and $\mathcal{M}$-monotone subequations with ψ as above, one can prove a *uniform translation property*: *for each $\theta > 0$ there exist $\eta = \eta(\psi, \theta) > 0$ and $\delta = \delta(\psi, \theta) > 0$ such that*

$$u_{y,\theta} = \tau_y u + \theta\psi \ \text{ belongs to } \mathcal{F}(\Omega_\delta), \quad \forall\, y \in B_\delta(0),$$

  where $\tau_y u(\,\cdot\,) := u(\,\cdot - y)$.

Step 5: Apply the following constant coefficient result of [CHLP, AoMS, to appear]

Theorem (The Zero Maximum Principle for Dual Monotonicity Cones)

Suppose that $\mathcal{M}$ is a constant coefficient monotonicity cone subequation that admits a stict subharmonic $\psi \in C^2(\Omega) \cap C(\overline{\Omega})$ on a domain $\Omega \Subset \mathbb{R}^n$. Then the zero maximum principle holds for $\widetilde{\mathcal{M}}$ on $\overline{\Omega}$; that is,

$$z \leq 0 \text{ on } \partial\Omega \quad \Longrightarrow \quad z \leq 0 \text{ on } \Omega \tag{ZMP}$$

for all $z \in \mathrm{USC}(\overline{\Omega}) \cap \widetilde{\mathcal{M}}(\Omega)$.

- $\widetilde{\mathcal{M}}$ is a (constant coefficient) subequation and hence satisfies the *sliding property*

$$z - m \in \widetilde{\mathcal{M}}(\Omega) \quad \text{for each} \quad m \in [0, +\infty).$$

- Since $z - m < 0$ on $\partial\Omega$ compact

$$z - m + \varepsilon\psi \leq 0 \text{ on } \partial\Omega \quad \text{for each } \varepsilon \text{ sufficiently small.}$$

- Since $z - m \in \widetilde{\mathcal{M}}(\Omega)$ and since $\varepsilon\psi \in C(\overline{\Omega}) \cap C^2(\Omega)$ is strictly $\mathcal{M}$-subharmonic, by *definitional comparison* (with $\mathcal{F} = \widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{F}} = \widetilde{\widetilde{\mathcal{M}}} = \mathcal{M}$) one has

$$z - m + \varepsilon\psi \leq 0 \text{ on } \Omega \quad \text{for each } \varepsilon \text{ sufficiently small,}$$

and passes to the limit for $\varepsilon \to 0^+$.

Monotonicity cone subequations

Question

Given a constant coefficient monotonicity cone subequation $\mathcal{M}$, for which bounded domains $\Omega \subset \mathbb{R}^n$ do there exist the needed strictly $\mathcal{M}$-subharmonic $\psi \in C^2(\Omega) \cap C(\overline{\Omega})$? This ensures comparison in every potential theory determined by a fiberegular and $\mathcal{M}$-monotone $\mathcal{F}$.

- Detailed study of monotonicity cone subequations in [CHLP, AoMs].
- There is a three parameter *fundamental family* of monotonicity cone subequations:

$$\mathcal{M}(\gamma, \mathcal{D}, R) := \left\{ (r, p, A) \in \mathcal{J}^2 : \; r \leq -\gamma |p|, \; p \in \mathcal{D}, \; A \geq \frac{|p|}{R} I \right\} \quad \text{where}$$

$$\gamma \in [0, +\infty), R \in (0, +\infty] \text{ and } \mathcal{D} \subseteq \mathbb{R}^n,$$

  with $\mathcal{D}$ a *directional cone* (closed convex cone, vertex in 0, non-empty interior).
- "Fundamental" means that for any $\mathcal{M}$, there exists $\mathcal{M}(\gamma, \mathcal{D}, R)$ with $\mathcal{M}(\gamma, \mathcal{D}, R) \subset \mathcal{M}$. Hence every $\mathcal{M}$-monotone $\mathcal{F}$ is $\mathcal{M}(\gamma, \mathcal{D}, R)$-monotone for some triple $(\gamma, \mathcal{D}, R)$.
- There is a simple dichotomy; one has the needed ψ on Ω (and hence comparison):
  - $R = +\infty$: for every Ω
  - $R < +\infty$: for every Ω contained in a translate of the truncated cone $\mathcal{D}_R := \mathcal{D} \cap B_R(0)$.

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The class of PDEs amenable to the above consierations are determined by the following:

Proper elliptic operators: Any operator $F \in C(\mathcal{G})$ such that for each $x \in X$ and each $(r, p, A) \in \mathcal{G}_x$ one has

$$F(x, r, p, A) \leq F(x, r+s, p, A+P) \quad \forall\, s \leq 0 \text{ in } \mathbb{R} \text{ and } \forall\, P \geq 0 \text{ in } \mathcal{S}(n). \qquad \text{(PE)}$$

where either

$$\mathcal{G} = \mathcal{J}^2(X) \quad \textbf{(unconstrained case)}$$

or

$$\mathcal{G} \subsetneq \mathcal{J}^2(X) \text{ is a subequation constraint set} \quad \textbf{(constrained case)}$$

The pair $(F, \mathcal{G})$ will be called a *proper elliptic (operator-subequation) pair*.

- A given operator F must often be restricted to a suitable background constraint domain $\mathcal{G} \subset \mathcal{J}^2(X)$ in order to satisfy the minimal monotonicty (PE) (the constrained case).
- The historical example is the Monge-Ampère operator $F(D^2u) = \det(D^2u)$, where one restricts F to the convexity subequation $\mathcal{G} = \mathcal{P} := \{A \in \mathcal{S}(n) : A \geq 0\}$.
- This is the simplest example of an operator defined by a *Dirichlet-Gårding polynomial*, which illutrate best the constained case.

Viscosity solution notions

Definition (Admissible viscosity solutions)

Given $F \in C(\mathcal{G})$ with $\mathcal{G} = \mathcal{J}^2(X)$ or $\mathcal{G} \subsetneq \mathcal{J}^2(X)$ a subequation on an open subset $X \subset \mathbb{R}^n$:

(a) *$u \in \mathrm{USC}(X)$ is a ($\mathcal{G}$-admissible) viscosity subsolution of $F(J^2 u) = 0$ on X if for every $x \in X$ one has*

$$J \in J^{2,+}_x u \quad \Rightarrow \quad J \in \mathcal{G}_x \quad \text{and} \quad F(x, J) \geq 0; \qquad \text{(sub)}$$

(b) *$u \in \mathrm{LSC}(\Omega)$ is a ($\mathcal{G}$-admissible) viscosity supersolution of $F(J^2 u) = 0$ on X if for every $x \in X$ one has*

$$J \in J^{2,-}_x u \quad \Rightarrow \quad \text{either } [\, J \in \mathcal{G}_x \text{ and } F(x, J) \leq 0 \,] \text{ or } J \notin \mathcal{G}_x. \qquad \text{(super)}$$

- In the unconstrained case where $\mathcal{G} \equiv \mathcal{J}^2(X)$, the definitions are standard.
- In the constrained case where $\mathcal{G} \subsetneq \mathcal{J}^2(X)$, the definitions give a systematic way of doing of what is sometimes done in an ad-hoc way [Ishii-Lions, JDE'90], [Trudinger, ARMA'90].
- In the constrained case, (sub) says that subsolutions are also $\mathcal{G}$-subharmonic and (super) says that $F(x, J) \leq 0$ for the lower test jets which lie in the constraint $\mathcal{G}_x$.

4. The Correspondence principle

Question (1)

For a given operator-subequation pair $(F, \mathcal{F})$ on an open set X, determine conditions under which (u, w) is an $\mathcal{F}$-subharmonic/ $\mathcal{F}$-superharmonic pair if and only if (u, w) is a subsolution/ supersolution pair for $F(J^2 u) = 0$.

- For **subharmonics/subsolutions** u the equivalence asks that: for each $x \in X$ one has

$$J^{2,+}_x u \subset \mathcal{F}_x \iff \text{both} \quad J^{2,+}_x u \subset \mathcal{G}_x \quad \text{and} \quad F(x, J) \geq 0 \ \text{ for each } \ J \in J^{2,+}_x u. \qquad \text{(CSub)}$$

  This holds if and only if one has the **correspondence relation**

$$\mathcal{F} = \{(x, J) \in \mathcal{G} : \ F(x, J) \geq 0\}. \qquad (1)$$

- For **superharmonics/supersolutions** w the equivalence asks that: for each $x \in X$ one has

$$J^{2,+}_x(-w) \subset \widetilde{\mathcal{F}}_x \iff J \notin \mathcal{G}_x \text{ or } [J \in \mathcal{G}_x \text{and } F(x, J) \leq 0], \ \forall\, J \in J^{2,-}_x w. \qquad \text{(CSuper)}$$

  Using duality and $J^{2,+}_x(-w) = -J^{2,-}_x w$ one can see that that the equivalence (CSuper) holds if and only if one has **compatibility**

$$\mathrm{Int}\ \mathcal{F} = \{(x, J) \in \mathcal{G} : \ F(x, J) > 0\}. \qquad (2)$$

  which for subequations $\mathcal{F}$ defined by (1) is equivalent to

$$\partial \mathcal{F} = \{(x, J) \in \mathcal{F} : \ F(x, J) = 0\}. \qquad (3)$$

Theorem (Correspondence Principle)

Suppose that $F \in C(\mathcal{G})$ is proper elliptic and $\mathcal{F}$, defined by the correspondence relation (1), *is a subequation. If compatibility* (2) *is satisfied, then (u, w) is an $\mathcal{F}$-subharmonic/ $\mathcal{F}$-superharmonic pair if and only if (u, w) is a subsolution/ supersolution pair for $F(J^2 u) = 0$.*

- In particular, for every $\Omega \Subset X$, a function $u \in C(\overline{\Omega})$ is $\mathcal{F}$-harmonic on Ω if and only if u is a $\mathcal{G}$-admissible viscosity solution of $F(J^2 u) = 0$ on Ω and the potential theoretic and operator theoretic formulations of the Dirichet problem are equivalent.
- Given $(F, \mathcal{G})$ or given $\mathcal{F}$, finding the other so that both the correspondence relation (1) and compatibility (2) hold can be impossible, easy or in between requiring some work.

Question (2)

Given a proper elliptic operator F with domain $\mathcal{G} \subset \mathcal{J}^2(X)$, can we ensure that the constraint set $\mathcal{F}$ defined by the correspondence relation (1)

$$\mathcal{F} := \{(x, J) \in \mathcal{G} : \; F(x, J) \geq 0\}$$

is a subequation and satisfies compatibility (2)

$$\mathrm{Int}\, \mathcal{F} = \{(x, J) \in \mathcal{G} : \; F(x, J) > 0\}.$$

Structural conditions on F

When is $\mathcal{F} := \{(x,J) \in \mathcal{G} : \ F(x,J) \geq 0\}$ a subequation?

- One needs positivity (P), negativity (N) and topological stability (T).
- (P) and (N) are equivalent to the (fiberwise) monotnicity property that for each $x \in X$

$$(r,p,A) \in \mathcal{F}_x \quad \Rightarrow \quad (r+s,p,A+P) \in \mathcal{F}_x, \quad \forall\, s \leq 0 \text{ in } \mathbb{R}, P \geq 0 \text{ in } \mathcal{S}(n);$$

  this follows from (P) and (N) for the domain $\mathcal{G}$ and and the proper ellipticity of F on $\mathcal{G}$
- This leaves property (T).

Lemma (Cirant-P.-Redaelli'22)

Suppose that $(F,\mathcal{G})$ is an $\mathcal{M}$-monotone operator-subequation pair for some monotonicity cone subequation, with $\mathcal{G} = \mathcal{J}^2(X)$ or $\mathcal{G} \subsetneq \mathcal{J}^2(X)$ a fiberegular subequation. Suppose that $(F,\mathcal{G})$ satisfies the regularity condition: for some fixed $J_0 \in \mathrm{Int}\ \mathcal{M}$, given $\Omega \Subset X$ and $\eta > 0$, there exists $\delta = \delta(\eta,\Omega) > 0$ such that

$$F(y, J + \eta J_0) \geq F(x,J), \quad \forall\, x,y \in \Omega \text{ with } |x-y| < \delta.$$

Then the constraint set $\mathcal{F}$ defined by (1) is a (fiberegular $\mathcal{M}$-monotone) subequation.

Finally, with $(F, \mathcal{G})$ and $\mathcal{F}$ as in the Lemma, it reamins only to check compatibility (2)

$$\mathrm{Int}\ \mathcal{F} = \{(x, J) \in \mathcal{G} : \ F(x, J) > 0\}.$$

- In the fiberegular and $\mathcal{M}$-monotone context, it suffices to have the fiberwise condition

$$\mathrm{Int}\ \mathcal{F}_x = \{J \in \mathcal{G}_x : \ F(x, J) > 0\}, \quad \forall x \in X.$$

- This condition is often easily checked for a given pair $(F, \mathcal{G})$ which determines $\mathcal{F}$ by checking that $F(x, J) = 0$ for $J \in \partial \mathcal{F}_x$ and using some strict monotonicity such as: for each $x \in X$ with some fixed $J_0 \in \mathrm{Int}\ \mathcal{M}$ there exists $t_0 > 0$ such that

$$F(x, J + tJ_0) > F(x, J), \quad \forall t \in (0, t_0), \forall J \in \partial \mathcal{F}_x.$$

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