Bi-Kolmogorov operators and weighted Rellich's inequalities

Abdelaziz Rhandi
(University of Salerno)

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A joint work with D. Addona, F. Gregorio and C. Tacelli

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Assume that μ is the density of a probability measure and

$$(H1): \qquad \mu \in H^1_{loc}(\mathbb{R}^N),\ \frac{\nabla \mu}{\mu} \in L^r_{loc}(\mathbb{R}^N),\ r > N,$$
$$\inf_K \mu > 0,\ \forall K \text{ compact}.$$

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Problem 1: Find general conditions on μ s.t. $D(L) = H^2_\mu(\mathbb{R}^N)$.

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- $$(C_0 - 1)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^4} dx \le \int_{\mathbb{R}^N} |\Delta u|^2 dx, \quad N \ge 5,$$

  where $C_0 := (N-2)^2/4$, $u \in C_c^\infty(\mathbb{R}^N)$.

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For more details and applications see [Balinsky-Evans-Lewis: "The Analysis and Geometry of Hardy's Inequality", Springer 2015]

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Problem 2: Find general conditions on μ s.t.

$$(C_0 - 1)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^4} d\mu \le \int_{\mathbb{R}^N} |Lu|^2 d\mu + C\|u\|^2_{H^1_\mu}, \qquad u \in C_c^\infty(\mathbb{R}^N)$$

holds with $(C_0 - 1)^2$ optimal.

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a_L densely defined, nonnegative and continuous closed form.
So, the bi-Kolmogorov operator

$$\begin{aligned} D(A) &:= \{u \in D(L) : \exists f \in L^2_\mu \,/\, a(u,v) = \int_{\mathbb{R}^N} fv\, d\mu,\, \forall v \in D(L)\} \\ Au &:= f \end{aligned}$$

generates an analytic semigroup of contractions (e^{-tA}) in L^2_μ.

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Problem 4: Study asymptotic properties and eventual positivity of e^{-tA}.

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Then

Proposition 1: μ is the unique invariant measure for e^{-tA}.

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Integrate the "formule de tous les jours",

$$\int_{\mathbb{R}^N} (e^{-tA} f - f) \, d\mu = -\int_0^t e^{-sA} \left(\int_{\mathbb{R}^N} Af \, d\mu \right) ds = 0, \quad t \geq 0.$$

Since C_c^∞ is a core for L, $1\!\mathrm{I} \in D(L)$ and $L1\!\mathrm{I} = 0$, which give $\int_{\mathbb{R}^N} Af \, d\mu = \int_{\mathbb{R}^N} LfL1\!\mathrm{I} \, d\mu = 0$.

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Uniqueness follows from the ergodicity of μ with respect to (e^{-tA}), i.e.

$$L^2_\mu - \lim_{t \to +\infty} \frac{1}{t} \int_0^t e^{-sA} f \, ds = \int_{\mathbb{R}^N} f \, d\mu, \, \forall f \in L^2_\mu.$$

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Positivity:

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For $A = \frac{d^4}{dx^4}$ [Evgrafov-Postnikov, Math. USSR Sbornik 1970, Hochberg, Annals Prob. 1978] proved that

$$e^{-tA}f(x) = \int_{\mathbb{R}} k(t,x,y)f(y)\,dy$$

with

$$k(t,x,y) \approx Mt^{-\frac{1}{6}}|x-y|^{-\frac{1}{3}} \exp\left(-\frac{3}{8}\left(\frac{|x-y|^4}{4t}\right)^{\frac{1}{3}}\right) \cos\left(\frac{3\sqrt{3}}{8}\left(\frac{|x-y|^4}{4t}\right)^{\frac{1}{3}}\right)$$

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k has an oscillatory character $\Longrightarrow$ no positivity.

Eventual positivity of $e^{-t\Delta^2}$:

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- $\exists \tau > 0$ that depends on u_0 such that for any $t > \tau$, $\exists x_t \in \mathbb{R}^N$ s.t. $e^{-t\Delta^2}u_0(x_t) < 0$.

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- If one assumes that $D(A^n) \subset L^\infty_{loc}(\mathbb{R}^N)$, applying recent results by Arora and Glück, one deduces, "*local individual eventual positivity*", $\forall 0 \le f \in L^2_\mu$, $\forall K \subset \mathbb{R}^N$ compact s.t. $f(x) > 0$, $x \in B \in \mathcal{B}(\mathbb{R}^N)$, $(B \subset K, |B| > 0)$, $\exists c > 0,\ t_0 > 0$ s.t.

$$e^{-tA}(\chi_K f)(x) \ge c, \quad t \ge t_0, x \in K.$$

Assumptions

Set

$$U := \frac{1}{4}\left|\frac{\nabla\mu}{\mu}\right|^2 - \frac{1}{2}\frac{\Delta\mu}{\mu}.$$

Assume

$$(H2) \qquad \begin{array}{l} \sqrt{\mu} \in H^1_{\mathrm{loc}}(\mathbb{R}^N),\ \Delta\mu \in L^1_{\mathrm{loc}}(\mathbb{R}^N); \\ \limsup_{x\to 0} |x|^2 U = 0,\ U \le C \text{ on } \mathbb{R}^N \setminus B_R. \end{array}$$

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Theorem 1: Assume $N \geq 5$, $(H2)$. For $u \in C_c^\infty(\mathbb{R}^N)$,

- $\forall \varepsilon > 0$,

$$\left((C_0 - 1)^2 - \varepsilon\right) \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^4} d\mu \leq \int_{\mathbb{R}^N} |Lu(x)|^2 d\mu + \frac{C}{\varepsilon} \|u\|_{L^2_\mu}^2;$$

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- This and $(H1)-(H2) \Longrightarrow$
since $C_c^\infty(\mathbb{R}^N)$ is a core for L, hence the weighted Rellich's inequalities in **Theorem 1** hold for all $u\in D(L)$.

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Theorem 2: Assume $N \geq 5$, $(H1)$ and $\forall \varepsilon > 0$, $\exists C_\varepsilon > 0$ s. t.

$$\left|\frac{\nabla \mu}{\mu}\right|^2 \leq \frac{\varepsilon}{|x|^2} + C_\varepsilon \quad \text{in } B_{R_0}.$$

Then,

$$c \int_{\mathbb{R}^N} \frac{u^2}{|x|^4} d\mu \leq \int_{\mathbb{R}^N} |Lu|^2 d\mu + C\|u\|^2_{H^1_\mu}, \quad u \in H^2_\mu(\mathbb{R}^N)$$

does not hold if $c > (C_0 - 1)^2 = \left(\frac{N(N-4)}{4}\right)^2$.

Assumptions

$(H3)$

$$(i)\ \mu \in W^{2,1}_{\rm loc}(\mathbb{R}^N),\ \left|D_i\left(\frac{D_j\mu}{\mu}\right)\right| \le \frac{\varepsilon}{|x|^2} + C_\varepsilon\left|\frac{\nabla\mu}{\mu}\right|;$$

$$(ii)\ \mu \in W^{3,1}_{\rm loc}(\mathbb{R}^N),\ \left|D_{ij}\left(\frac{D_k\mu}{\mu}\right)\right| \le \frac{\varepsilon}{|x|^3} + C_\varepsilon\left|\frac{\nabla\mu}{\mu}\right|$$

for $i, j, k = 1, \ldots, N$.

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The weighted Rellich inequality $+ (H3)(i) \implies$
"weighted Landau-Kolmogorov inequality":

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and "weighted Calderon-Zygmund inequality":

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Thus,

Theorem 3: Assume $N \geq 5$, $(H1) - (H3)(i)$. Then

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Examples:

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  $(H3)$ is essentially needed for $m \in (0,1]$.
- $\mu(x) = C\frac{1+|x|^\alpha}{1+|x|^\beta}$, $\alpha, \beta > 0$, $\beta > \alpha + N$ satisfies $(H1)-(H3)$.

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Theorem 4 (and the above example) $\Longrightarrow\ D(A) = H^4_\mu(\mathbb{R}^N)$.
Moreover,

$$\begin{aligned} e^{-tA}f(x) &= \int_{\mathbb{R}^N} k(t,x,y) f(y)\, dy \\ &= \sqrt{2}(8\pi)^{-\frac{N+1}{2}} \int_0^\infty e^{-\frac{s^2}{4}} (\sin(s\sqrt{t}))^{-N/2} e^{-\frac{|e^{-is\sqrt{t}}x-y|^2}{8}} \\ &\qquad \cos\left(\frac{N}{2}(s\sqrt{t} - \frac{\pi}{2}) + \frac{|e^{-is\sqrt{t}}x - y|^2}{8\tan(s\sqrt{t})}\right) ds \end{aligned}$$

Many thanks and stay healthy